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On the Topology of Fermat Type Surface of Degree 5 and the Numerical Analysis of Algebraic Curves

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1. Introduction.

Let V_n be a Fermat type algebraic surface of degree n, that is,

$$V_n=\{[z_0:z_1:z_2:z_3]\in \boldsymbol{CP}^3 \mid z_0^n-z_1^n-z_2^n+z_3^n=0\}\,.$$

We consider a fibration $f: V_n\to \boldsymbol{CP}^1$ given by

$$f:[z_0:z_1:z_2:z_3]\mapsto\begin{cases}[z_2^{n-1}:z_0^{n-1}] & \text{if } z_0=z_1 \text{ and } z_2=z_3\\ [z_0-z_1:z_2-z_3] & \text{otherwise}\,.\end{cases}$$

A general fiber of f is a Riemannian surface of genus $(n-2)(n-3)/2$. If $n\leq 4$, a general fiber is a sphere or a torus and the singular fibers and their monodromies are known. (See [K].). In the case $n=5$ the genus of a general fiber is 3. Matsumoto calculates in his notes [M] the positions and homeomorphism-types of all singular fibers appearing in the fibration f: $V_n\to \boldsymbol{CP}^1$ for general n. From his results we know the conjugate class of the local monodromy for each singular fiber.

In this paper we suppose $n=5$ and we give an algorithm to calculate the global monodromy map

$$[\tilde{\rho}_*]:\pi_1(\boldsymbol{CP}^1-SF,\sigma_0)\to \mathfrak{M}_3=\mathrm{Aut}\,\pi_1\Sigma_3/\mathrm{Inn}\,\pi_1\Sigma_3$$

using numerical analysis of algebraic curves in $\boldsymbol{CP}^2$. Here

$$SF=\{\sigma \mid F_\sigma=f^{-1}(\sigma) \text{ is a singular fiber}\}\,.$$

First we define a branched covering map

$$h_\sigma: F_\sigma\to \boldsymbol{CP}^1$$

for each general fiber $F_\sigma=f^{-1}(\sigma)$. Its branch loci are obtained as solutions of certain

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equations $l_j(x)=0$. (Lemma 2.5.)

Let $c\colon [0, 1]\to \boldsymbol{CP}^1 - SF$ be a continuous path. We define a homeomorphism between fibers on the ends of this path as follows. Let h_c be a map defined by

$$h_c\colon \bigsqcup_t F_{c(t)} \to [0, 1]\times \boldsymbol{CP}^1$$
$$F_{c(t)} \ni x \mapsto (t, h_{c(t)}(x)) .$$

Let $\rho(c)\colon [0, 1]\times \boldsymbol{CP}^1 \to \boldsymbol{CP}^1$ be a homeotopy of the base space of $h_{c(t)}$ preserving its branch loci, that is, $\rho(c)$ is continuous, $\rho(c)(t, \cdot)$ is a homeomorphism, $\rho(c)(0, \cdot)=\mathrm{id}$, and $\rho(c)(t, D_{c(0)})=D_{c(t)}$, where $D_{c(t)}$ is a set of branch loci of $h_{c(t)}$. For any y in $\boldsymbol{CP}^1$ we consider a path γ_y on $[0, 1]\times \boldsymbol{CP}^1$ defined by

$$\gamma_y\colon [0, 1]\to [0, 1]\times \boldsymbol{CP}^1\colon \qquad t\mapsto (t, \rho(c)(t, y)) .$$

We define a map $\tilde\rho(c)\colon F_{c(0)}\to F_{c(1)}$ by $x\mapsto \tilde\gamma_x(1)$, where $y=h_{c(0)}(x)$ and $\tilde\gamma_x$ is a lifting of γ_y with its starting point x. $\tilde\rho(c)$ is a homeomorphism.

Secondly we show that there exists an injection p_* from $\pi_1(\boldsymbol{CP}^1 - SF, \sigma_0)$ to a free group denoted by $\pi_1(T, \sigma_0)$ which is generated by four elements $[\gamma_0]$, $[\gamma_1]$, $[\gamma_2]$, $[\gamma_3]$, where $T=(\boldsymbol{CP}^1 - SF)/\langle\omega\rangle$, and $\omega=\exp(2\pi\sqrt{-1}/5)$. (Lemma 5.2.) For any $[\gamma]$ in $\pi_1(T, \sigma_0)$, $[\gamma]$ is represented by a path γ connecting σ_0 and $\sigma_j=\omega^j\sigma_0$, and j is a length of γ, which is defined in 5.2. Fix γ_0, γ_1, γ_2, and γ_3 as in section 5, and let πT be a free group of loops on T generated by γ_j's. For any $[\gamma]\in\pi_1(T, \sigma_0)$ we take a realization γ of $[\gamma]$ in πT.

Finally we calculate $\tilde\rho(\gamma)$ for $[\gamma]\in\pi_1(T, \sigma_0)$ in the following way.

(1) Notice that $D_{\sigma_0}=D_{\sigma_j}$. (See 2.5.) $\rho(\gamma)$ determines a braid on $\boldsymbol{C}^\times=\boldsymbol{CP}^1-\{\infty, 0\}$, where we may perturb γ_j's such that any branch loci of $h_{\gamma(t)}$ are not 0 nor ∞. Now we define a homomorphism $\rho_T\colon \pi T\to B_8(\boldsymbol{C}^\times)$, where $B_8(\boldsymbol{C}^\times)$ is a braid group of degree 8 on $\boldsymbol{C}^\times$. (Proposition 5.3.) Remark that ∞ is a branch locus if and only if 0 is also a branch locus. (See Lemma 2.5.)

(2) For a braid β of degree 8 on $\boldsymbol{C}^\times$, it induces a homeomorphism on $\boldsymbol{C}^\times$. Suppose β is represented by a map

$$\beta\colon \{8 \text{ points}\}\times[0, 1]\to \boldsymbol{C}^\times\colon \qquad (x, t)\mapsto \beta(x, t) .$$

Let $\rho_\beta\colon \boldsymbol{C}\times[0, 1]\to\boldsymbol{C}$ be an extension of β, that is, ρ_β is continuous, $\rho_\beta(\cdot, t)$ is a homeomorphism, $\rho_\beta(0, t)=0$, and $\rho_\beta(x, t)=\beta(x, t)$ for $x\in\{8 \text{ points}\}$. The homotopy type of ρ_β depends only on a braid class of β. This extension induces an automorphism of $\pi_1(\boldsymbol{C}-\{8 \text{ points}\}, 0)$. This automorphism only depends on a braid β and we define a homomorphism

$$\rho_B\colon B_8\to \mathrm{Aut}\,\pi_1(\boldsymbol{C}-\{8 \text{ pts}\}, 0) .$$

If we take $\{8 \text{ pts}\}=D_{\sigma_0}$ then we have $\rho(\gamma)_*=\rho_B\circ\rho_T(\gamma)$. (Proposition 5.5.)

(3) We calculate $\tilde\rho(\gamma)\colon h_{\sigma_0}^{-1}(0)\to h_{\sigma_j}^{-1}(0)$. (Proposition 5.6.)

(4) For $*$ in $h_{\sigma_0}^{-1}(0)$, the diagram

$$\begin{array}{ccc} \pi_1(F_{\sigma_0}-h_{\sigma_0}^{-1}(D_{\sigma_0}), *) & \xrightarrow{\tilde{\rho}(\gamma)_*} & \pi_1(F_{\sigma_j}-h_{\sigma_j}^{-1}(D_{\sigma_j}), \tilde{\rho}(\gamma)(*)) \\ \downarrow h_{\sigma_0*} & & \downarrow h_{\sigma_j*} \\ \pi_1(CP^1-D_{\sigma_0}, 0) & \xrightarrow{\rho(\gamma)_*} & \pi_1(CP^1-D_{\sigma_j}, 0) \end{array}$$

commutes, where $\tilde{\rho}(\gamma)$ is a monodromy homeomorphism which will be given in section 2. And $\tilde{\rho}(\gamma)_*=h_{\sigma_j*}^{-1}\circ\rho_B\circ\rho_T\circ h_{\sigma_0*}$ induces a homomorphism

$$\tilde{\rho}(\gamma)_*: \pi_1(F_{\sigma_0}, *)\to\pi_1(F_{\sigma_j}, \tilde{\rho}(\gamma)(*)) ,$$

where $h_{\sigma_j*}^{-1}$ is a lifting with its starting point $\tilde{\rho}(\gamma)(*)$. Because this diagram commutes, $\rho_B\circ\rho_T\circ h_{\sigma_0*}(x)$ has a lifting for any $x\in\pi_1(F_{\sigma_0}, *)$. In section 4, we have an algorithm to get this lifting.

This algorithm gives a monodromy map of a fibration f. We have some remarks.

(Remark 1) $\gamma\mapsto\tilde{\rho}(\gamma)$ is not a homomorphism because $\tilde{\rho}(\gamma)$ is not an automorphism on $\pi_1(F_{\sigma_0}, *)$. $\tilde{\rho}(\gamma)$ only gives an element $[\tilde{\rho}(\gamma)]$ in $\mathrm{Aut}\,\pi_1(F_{\sigma_0})/\mathrm{Inn}\,\pi_1(F_{\sigma_0})$.

(Remark 2) $\rho_T(\gamma)$, $\rho(\gamma)$, $\tilde{\rho}(\gamma)$ depend on a choice of γ_j's. Hence for any $[\gamma]\in\pi_1(T, \sigma_0)$, $\tilde{\rho}(\gamma)$ depends on a choice of the realization γ. But on the other hand $[\gamma]\mapsto[\tilde{\rho}(\gamma)]$ is well-defined and it is sufficient to calculate $\tilde{\rho}$ for one realization γ.

(Remark 3) This algorithm does not depend on the degree n of the surface V_n. Generally for any fibration from a smooth surface to a smooth line we can determine its singular fibers and its global monodromy map using a similar algorithm.

(Remark 4) It is known that $\mathfrak{M}_g$ is generated by Dehn twists along simple closed curves on Σ_g. In our cases, for some loops on CP^1-SF we easily give monodromy map using a product of Dehn twists. See Proposition 5.8.

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2. Singular fibers and branched covering of general fibers.

In this section we quote some results from Matsumoto's notes [M] and prepare some fundamental properties.

In the sequel we regard CP^1 as $C\cup\infty$. For $\sigma\in CP^1$, let $F_\sigma=f^{-1}(\sigma)$ be a fiber. In this section we suppose $n\geq 4$.

PROPOSITION 2.1. *F_σ is a singular fiber if and only if $\sigma=0$ or $\sigma=\infty$ or for some $j, k=1, 2, \cdots, n-2$,*

$$\sigma^n=\frac{(1-\omega_{n-1}^k)^{n-1}}{(1-\omega_{n-1}^j)^{n-1}} ,$$

where $\omega_{n-1}=\exp(2\pi i/(n-1))$.

COROLLARY 2.2. *If* $n=5$ *and* F_σ *is a singular fiber then* $\sigma=0$ *or* $\sigma=\infty$ *or* $\sigma^5=-1/4$ *or* 1 *or* -4.

The proof is given by Matsumoto ([M]). Matsumoto determines the homeomorphism-type of all of singular fibers. In case $n=5$ we have

PROPOSITION 2.3. *Let* $\alpha=\sqrt[5]{-4}$.

(1) $F_0\approx F_\infty\approx$ *Bouquet of* 4 S^2*'s.*

(2) *For* $j=0, 1, 2, 3, 4$, $F_{\omega_5^j}$ *is homeomorphic to the singular fiber shown in Figure* 2.1.

(3) *For* $j=0, 1, 2, 3, 4$, $F_{\alpha\omega_5^j}\approx F_{\alpha^{-1}\omega_5^j}$ *is homeomorphic to the singular fiber shown in Figure* 2.2.

(4) *If* F_σ *is a general fiber then* $F_\sigma\approx\Sigma_3$ *and* F_σ *is smooth.*

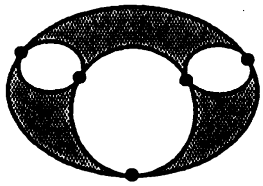

FIGURE 2.1

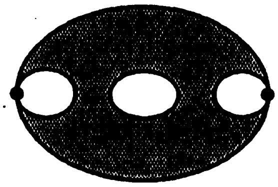

FIGURE 2.2

For σ such that $\sigma\neq 0$ nor $\sigma\neq\infty$ we define a branched covering map

$$h_\sigma\colon F_\sigma\to CP^1\colon\qquad [z_0:z_1:z_2:z_3]\mapsto[z_0:z_1]\,.$$

LEMMA 2.4. (1) *Suppose* $\sigma\neq 0$ *and* $\sigma\neq\infty$.

$$F_\sigma\cap\{z_0\neq 0\}=\{(x, y)\mid g_\sigma(x, y)=0,\ x\neq 1\}\cup\{(1, y)\mid y^{n-1}=\sigma\}\,,$$

where $g_\sigma(x, y)=1-x^n-y^n+(y+(x-1)/\sigma)^n$. *And* h_σ *is given by*

$$h_\sigma\colon (x, y)\in F_\sigma\cap\{z_0\neq 0\}\mapsto x=z_1/z_0\in C\,.$$

(2) $F_\sigma\cap\{z_0=0\}=h_\sigma^{-1}(\infty)$.

(3) *If* F_σ *is a general fiber then* h_σ *is a branched covering map.*

We leave the proof of this lemma to the readers.

LEMMA 2.5. (1) *If* F_σ *is a general fiber then the branch loci* $x=z_1/z_0$ *of* h_σ *is given by the solutions of* $l_j(x)=0$, $(j=1, 2, \cdots, n-2)$, *where*

$$l_j(x)=\frac{(x-1)^{n-1}}{(1-\omega_{n-1}^{-j})^{n-1}\sigma^n}+(-1)^n(x^{n-1}+\cdots+x+1)\,,$$

$$\omega_{n-1}=\exp\left(\frac{2\pi i}{n-1}\right).$$

(2) *If the degree of* $l_j(x)$ *is less than* $n-1$ *for some* j *then* $x=\infty$ *is one of the branch loci.*

(3) *For any branch locus, its monodromy group is* $\boldsymbol{Z}/2\boldsymbol{Z}$.

PROOF. (1) If x is a branch locus of h_σ and $x\neq 1$ and $x\neq\infty$ then

$$\frac{\partial}{\partial y}g_\sigma(x,y)=0\,,$$

$$-ny^{n-1}+n\left(y+\frac{x-1}{\sigma}\right)^{n-1}=0\,.$$

Therefore for some $j\in\{0,1,\cdots,n-2\}$ we obtain

$$y=\omega_{n-1}^{j}\left(y+\frac{x-1}{\sigma}\right).$$

From the assumption $x\neq 1$, we have $j\neq 0$ and

$$y=\frac{x-1}{(\omega_{n-1}^{-j}-1)\sigma}\,.$$

Substituting this for $g_\sigma(x,y)=0$,

$$1-x^n-\left(\frac{x-1}{(\omega_{n-1}^{-j}-1)\sigma}\right)^n(1-\omega_{n-1}^{-j})=0\,,$$

$$(-1)^{n-1}(x-1)\left\{\frac{(x-1)^{n-1}}{(1-\omega_{n-1}^{-j})^{n-1}\sigma^n}+(-1)^n(x^{n-1}+\cdots+x+1)\right\}=0\,,$$

$$l_j(x)=0\,.$$

The condition $\sigma\neq 0$ implies that $h_\sigma([z_0:z_1:z_2:z_3])=1$ if and only if $x=1$, $y^{n-1}=\sigma$, and hence $x=1$ is not a branch locus. It is easy to check that $l_j(1)\neq 0$.

(2) Since $\sigma\neq 0$ and $\sigma\neq\infty$,

$$h_\sigma^{-1}(\infty)=\{(x,y)\mid y=x+1/\sigma,\ -1-x^n+(x+1/\sigma)^n=0\}$$

and it is easy to show that the degree of $l_j(x)$ is less than $n-1$ for some j if and only if $-1-x^n+(x+1/\sigma)^n=0$ has multiple solutions.

(3) It is sufficient to show that for any given x such that $x\neq 1$ the equation $g_\sigma(x,y)=0$ for y does not have any triple solutions. In fact,

$$\frac{\partial}{\partial y}g_\sigma(x,y)=0\ \Rightarrow\ y=\omega_{n-1}^{j}\left(y+\frac{x-1}{\sigma}\right),$$

$$\frac{\partial^2}{\partial y^2}g_\sigma(x,y)=0\ \Rightarrow\ y=\omega_{n-2}^{j'}\left(y+\frac{x-1}{\sigma}\right).$$

The condition $n\geq 4$ and $x\neq 1$ implies that they have no common solutions for x, y.

LEMMA 2.6. (1) *F_σ $(\sigma\neq 0, \sigma\neq\infty)$ is a singular fiber if and only if $l_j(x)=0$ has multiple solutions for some j.*

(2) *If n is odd, $l_j(x)=l_{n-j-1}(x)$. If $l_j(x)=0$ and $l_{j'}(x)=0$ have a common solution then* (i) $j=j'$ *or* (ii) $j'=n-j-1$ *with odd n.*

PROOF. Let $y_j(x)$ be defined by

$$y_j(x)=\frac{x-1}{(\omega_{n-1}^{-j}-1)\sigma}\,.$$

From Lemma 2.5, $(-1)^{n-1}(x-1)l_j(x)=g_\sigma(x, y_j(x))$ holds. If F_σ has a singular point $(x_0, y_0)\in F_\sigma\cap\{z_0\neq 0\}$, then $g_\sigma=\partial g_\sigma/\partial x=\partial g_\sigma/\partial y=0$ at (x_0, y_0). It follows that $y_0=y_j(x_0)$ for some j and

$$g_\sigma(x, y)=\alpha(x-x_0)^2+\beta(x-x_0)(y-y_0)+\gamma(y-y_0)^2+(\text{higher terms})$$

holds for some constant α, β, γ. Hence

$$\begin{aligned} g_\sigma(x, y_j(x))&=\alpha(x-x_0)^2+\beta(x-x_0)(y_j(x)-y_j(x_0))+\gamma(y_j(x)-y_j(x_0))^2\\ &\quad+(\text{higher terms}) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}g_\sigma(x, y_j(x))\Big|_{x=x_0}&=\Big\{2\alpha(x-x_0)+\beta\{(y_j(x)-y_j(x_0))+(x-x_0)y_j'(x)\}\\ &\qquad+\gamma(y_j(x)-y_j(x_0))y_j'(x)\Big\}\Big|_{x=x_0}\\ &=0\,. \end{aligned}$$

Hence we have

$$\frac{d}{dx}l_j(x)\Big|_{x=x_0}=0$$

and x_0 is a multiple solution of $l_j(x)=0$.

Suppose x_0 is a multiple solution of $l_j(x_0)=0$, then $l_j(x_0)=l_j'(x_0)=0$ holds. If $L_j(x)=(x-1)l_j(x)$ then $L_j(x_0)=L_j'(x_0)=0$.

$$L_j(x_0)=\frac{(x_0-1)^n}{(1-\omega_{n-1}^{-j})^{n-1}\sigma^n}+(-1)^n(x_0^n-1)=0\,,$$

$$L_j'(x_0)=\frac{n(x_0-1)^{n-1}}{(1-\omega_{n-1}^{-j})^{n-1}\sigma^n}+(-1)^n n x_0^{n-1}=0\,,$$

$$x_0-1=\frac{x_0-1}{x_0^{n-1}}\,,$$

$$x_0^n - x_0^{n-1} = x_0^n - 1\,,$$
$$x_0^{n-1} = 1\,,$$
$$x_0 = \omega_{n-1}^k$$

for some k. $L'_j(x_0)=0$ implies that

$$\frac{(1-\omega_{n-1}^k)^{n-1}}{(1-\omega_{n-1}^{-j})^{n-1}} = \sigma^n$$

and hence F_σ is a singular fiber.

(2) If n is odd then $(1-\omega_{n-1}^{-j})^{n-1} = (1-\omega_{n-1}^{-(n-j-1)})^{n-1}$ and $l_{n-j-1}(x) = l_j(x)$.

Assume that x is a common solution of $l_j(x)=0$ and $l_{j'}(x)=0$ and $j \neq j'$. Easily we have

$$\frac{(x-1)^{n-1}}{(1-\omega_{n-1}^{-j})^{n-1}\sigma^n} = \frac{(x-1)^{n-1}}{(1-\omega_{n-1}^{-j'})^{n-1}\sigma^n}\,.$$

Remark that since $l_j(1) \neq 0$, $x \neq 1$ and we have

$$(1-\omega_{n-1}^{-j})^{n-1} = (1-\omega_{n-1}^{-j'})^{n-1}\,,$$
$$1-\omega_{n-1}^{-j} = \omega_{n-1}^k(1-\omega_{n-1}^{-j'})$$

for some k. Now it is easy to show that

$$1-\omega_{n-1}^{-j} = 2\sin\frac{\pi j}{n-1}\exp\left\{\left(\frac{1}{2}-\frac{j}{n-1}\right)\pi i\right\}.$$

And we have

$$\begin{cases} \sin\dfrac{\pi j}{n-1} = \sin\dfrac{\pi j'}{n-1} \\ \exp\dfrac{-j\pi i}{n-1} = \exp\dfrac{(-j'+2k)\pi i}{n-1}\,. \end{cases}$$

From the assumption $j \neq j'$,

$$\begin{cases} j+j' = n-1 \\ -j \equiv -j'+2k \pmod{n-1}\,. \end{cases}$$

We conclude that $j+j'=n-1$ and n is odd. □

We characterize the monodromy map as follows. Let $c\colon [0,1] \to \boldsymbol{CP}^1 - SF$ be a continuous path. We define a homeomorphism between $F_{c(0)}$ and $F_{c(1)}$. Let $\rho(c)\colon [0,1] \times \boldsymbol{CP}^1 \to \boldsymbol{CP}^1$ be a homeotopy of the base space of $h_{c(t)}$ preserving its branch loci, that is, $\rho(c)$ is continuous, $\rho(c)(t,\cdot)\colon \boldsymbol{CP}^1 \to \boldsymbol{CP}^1$ is a homeomorphism, $\rho(c)(0,\cdot)=\mathrm{id}$, and $\rho(c)(t, D_{c(0)})=D_{c(t)}$, where $D_{c(t)}$ is a set of branch loci of $h_{c(t)}$. Remark that the homotopy type of $\rho(c)$ is uniquely determined. In fact,

PROPOSITION 2.7. (1) *If $\rho_0(c)$ and $\rho_1(c)$ are homeotopy of the base space of $h_{c(t)}$ preserving their branch loci, then there exists a homotopy r_s: $CP^1 \to CP^1$, $s \in [0, 1]$ such that $r_0=\rho_0(c)(1, \cdot)$ and $r_1=\rho_1(c)(1, \cdot)$.*

(2) *Let c_1 and c_2 be mutually homotopic paths on CP^1-SF. Then there exists a homotopy r_s: $CP^1 \to CP^1$, $s \in [0, 1]$ such that $r_0=\rho_0(c_0)(1, \cdot)$ and $r_1=\rho_1(c_1)(1, \cdot)$ for homeotopies $\rho_0(c_0)$, $\rho_1(c_1)$.*

PROOF. (1) $r_s=\rho_0(c)(1, \cdot)\circ\rho_0(c)(s, \cdot)^{-1}\circ\rho_1(c)(s, \cdot)$.

(2) Let c_s be a homotopy between c_0 and c_1. It is easy to construct a family of homeotopies $\rho_s'(c_s)$ along c_s. From (1), $\rho_s'(c_s)(1, \cdot)$ and $\rho_s(c_s)(1, \cdot)$, are homotopic for $s=0, 1$. This completes the proof.

For any $y \in CP^1$, let a path γ_y on $[0, 1]\times CP^1$ be given by

$$\gamma_y: [0, 1]\to[0, 1]\times CP^1: \qquad t\mapsto(t, \rho(c)(t, y)) .$$

Let h_c be a branched covering on $[0, 1]\times CP^1$ defined by

$$h_c: \sqcup_t F_{c(t)}\to[0, 1]\times CP^1: \qquad x\in F_{c(t)}\mapsto(t, h_{c(t)}(x)) ,$$

and $\tilde{\gamma}_x$: $[0, 1]\to\sqcup_t F_{c(t)}$ be a lifting of γ_y with its starting point $x\in h_{c(0)}^{-1}(y)$. We define $\tilde{\rho}(c)$ by

$$\tilde{\rho}(c): F_{c(0)}\to F_{c(1)}: \qquad x\mapsto\tilde{\gamma}_x(1) .$$

Clearly the following proposition holds.

PROPOSITION 2.8. (1) *$\tilde{\rho}(c)$ is a homeomorphsim.*

(2) *If $c(0)=c(1)$ then $\tilde{\rho}(c)$ gives a monodromy of $F_{c(0)}$ along a loop c.*

3. Numerical analysis of 1-parameter equation.

Let $f_t(x)$ be a continuous family of polynomials of x with one parameter t. Assume that for any t the degree of $f_t(x)$ is n. We define D_f by

$$D_f=\{t \mid f_t(x)=0 \text{ has a multiple root}\}$$

and let $C_f=CP^1-D_f$.

Fix $t_0\in C_f$ as a base point. Let γ be a loop in C_f with a base point t_0 and $[\gamma]$ be a homotopy class of γ in $\pi_1(C_f, t_0)$. Let $\{x_1(c), \cdots, x_n(c)\}$, $(c\in[0, 1])$ be a set of continuous functions such that $x_1(c), \cdots, x_n(c)$ are the solutions of $f_{\gamma(c)}(x)=0$. (See (3.2).) Because $\gamma(0)=\gamma(1)=t_0$, $\{x_1(1), \cdots, x_n(1)\}$ is a permutation $\rho_f(\gamma)$ of $\{x_1(0), \cdots, x_n(0)\}$, where ρ_f is a monodromy map $\rho_f: \pi_1(C_f, t_0)\to\mathfrak{S}_n$. The problem we want to solve in a special case is as follows.

PROBLEM 3.1. *For a given $f_t(x)$, calculate the monodromy map ρ_f.*

In this section we will give a partial answer using Newton approximation. The following lemma is well-known.

LEMMA 3.2. *If $\gamma: [0, 1] \to C_f$ is continuous then the solutions of $f_{\gamma(t)}(x)=0$ move continuously.*

For a polynomial $f(x)$, let $E_f: CP^1 \to CP^1$ be a Newton approximation, that is,

$$E_f(x)=\begin{cases} x-\dfrac{f(x)}{df(x)/dx} & \text{if } x\neq\infty \text{ and } df(x)/dx\neq 0 \\ \infty & \text{otherwise}. \end{cases}$$

For a map $E: CP^1 \to CP^1$ we define attracting points in CP^1 with respect to E.

DEFINITION 3.3. (1) $p \in CP^1$ $(p\neq\infty)$ is an attracting point with respect to E if and only if there exist $\varepsilon>0$ and $0<\kappa<1$ such that

$$\frac{|E(x)-p|}{|x-p|}<\kappa$$

if x satisfies $|x-p|<\varepsilon$.

(2) $\infty \in CP^1$ is an attracting point with respect to E if and only if there exist $\varepsilon>0$ and $1<\kappa$ such that

$$\frac{|E(x)|}{|x|}>\kappa$$

for any x satisfying $|x|>1/\varepsilon$.

From Definition 3.3 it is easy to show the following lemma.

LEMMA 3.4. *Suppose that $p\in CP^1$ $(p\neq\infty)$ is an attracting point (if it exists). For x such that $|x-p|<\varepsilon$, we define a sequence $\{x_i\}_{i=0,1,2,\dots}$ by $x_0=x$ and $x_{i+1}=E(x_i)$. Then the sequence $\{x_i\}$ converges to p.*

LEMMA 3.5. *Let $f(x)$ be a polynomial of degree n and p is one of solutions of $f(x)=0$. Then p is an attracting point of E_f.*

PROOF. Replace x with $x-p$. If

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x\,,$$

then

$$E_f(x)=x-\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x}{na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+2a_2x+a_1}\,,$$

$$\left|\frac{E_f(x)}{x}\right|=\left|\frac{(n-1)a_nx^{n-1}+(n-2)a_{n-1}x^{n-2}+\cdots+a_2x}{na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+2a_2x+a_1}\right|.$$

The condition $a_n\neq 0$ implies that for x such that $|x|$ is small

$$\left|\frac{E_f(x)}{x}\right|<1-\frac{1}{2n}<1.$$

Hence 0 is an attracting point with respect to E_f and we complete the proof.

From Lemma 3.2, Lemma 3.4 and Lemma 3.5 we have

PROPOSITION 3.6. *For a given loop γ in C_f there exists a partition of the interval*

$$0=c_0<c_1<\cdots<c_{k-1}<c_k=1$$

such that for any $0\leq l<k$ if we define a sequence $\{y_i\}_{i=0,1,2,\cdots}$ by $y_0=x_j(\gamma(c_l))$ and $y_{i+1}=E_{f_{\gamma(c_{l+1})}}(y_i)$ for $i=0,1,2,\cdots$, then the sequence $\{y_i\}$ converges to $x_j(\gamma(c_{l+1}))$.

Using this proposition for given γ we make a partition of [0, 1] and we calculate $x_j(\gamma(c_l))$ and the monodromy map ρ_f.

4. General fiber F_{σ_0}.

Let $\sigma_0=1.1=11/10$. In this section we consider a branched covering $h_{\sigma_0}: F_{\sigma_0}\to CP^1$, which is given in Lemma 2.5, and characterize $F_{\sigma_0}\approx\Sigma_3$.

The branch loci of h_{σ_0} are given by solutions of

$$\left\{\frac{(x-1)^4}{16\sigma_0^5}-(x^4+x^3+x^2+x+1)\right\}\left\{\frac{(x-1)^4}{4\sigma_0^5}-(x^4+x^3+x^2+x+1)\right\}=0.$$

In fact the solutions are $\{A, \bar{A}, B, \bar{B}, C, \bar{C}, D, \bar{D}\}$, where $A\doteqdot -0.9256+0.3786i$, $B\doteqdot -0.3800+0.9250i$, $C\doteqdot 0.2159+0.9764i$, $D\doteqdot 0.3246+0.9458i$.

If $G_t(x)=g_{\sigma_0}(t, x)$ then we have $C_G=CP^1-\{A, \bar{A}, B, \bar{B}, C, \bar{C}, D, \bar{D}\}$. Let a, b, c, d, $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ in Figure 4.1 be generators of $\pi_1(C_G, 0)$. We number the solutions of $G_0(x)=0$ as in Figure 4.2. Using Proposition 3.6, we calculate numerically movement

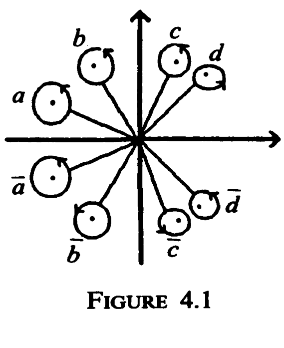

FIGURE 4.1

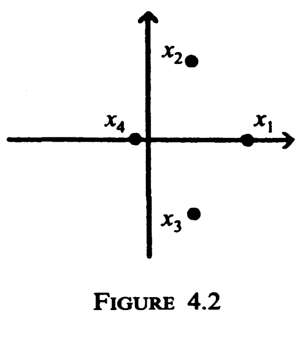

FIGURE 4.2

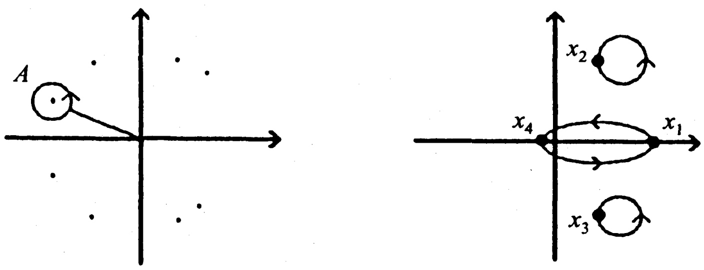

FIGURE 4.3.1

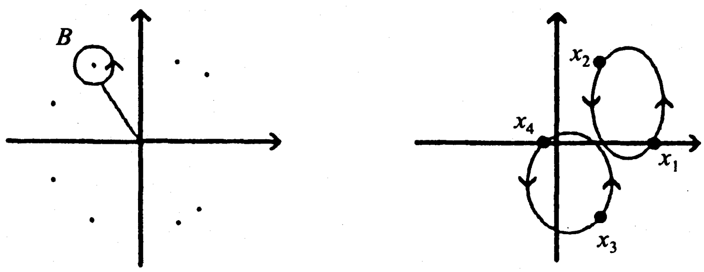

FIGURE 4.3.2

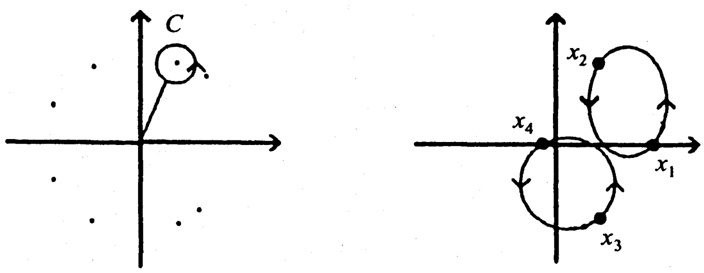

FIGURE 4.3.3

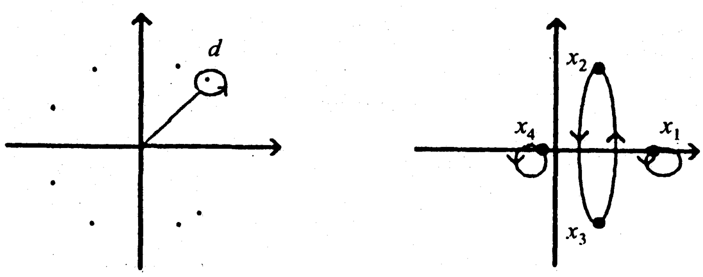

FIGURE 4.3.4

of solutions of $G_t(x)=0$ when t runs along $a, b, \cdots$, and so on. Figures 4.3.1, 4.3.2, 4.3.3, and 4.3.4 give some examples of movement of solutions of $G_t(x)=0$.

REMARK. Since $g_{\sigma_0}(\bar{t}, x)=\overline{g_{\bar{\sigma}_0}(t, \bar{x})}=\overline{g_{\sigma_0}(t, \bar{x})}$, $G_t(x)=0$ if and only if $G_{\bar{t}}(\bar{x})=0$. Hence the movements of solutions for $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are conjugations of those of a, b, c, d

respectively.

And we have

$$\rho_G : \pi_1(C_G, 0) \to \mathfrak{S}_4$$

$$\begin{array}{ll} a \mapsto (14)(2)(3) & \bar{a} \mapsto (14)(2)(3) \\ b \mapsto (12)(34) & \bar{b} \mapsto (13)(24) \\ c \mapsto (12)(34) & \bar{c} \mapsto (13)(24) \\ d \mapsto (1)(23)(4) & \bar{d} \mapsto (1)(23)(4)\,. \end{array}$$

Here for example (1)(23)(4) is an element of $\mathfrak{S}_4$ given by $x_1 \mapsto x_1$, $x_2 \mapsto x_3$, $x_3 \mapsto x_2$, $x_4 \mapsto x_4$.

Let e_1, e_2, e_3, e_4 be mutually disjoint simple closed curves on C_G such that they are homotopic to $a\bar{a}$, bc, $\bar{c}\bar{b}$, $\bar{d}d$, respectively in G_G. (Figure 4.4.) Let D_1, D_2, D_3, D_4

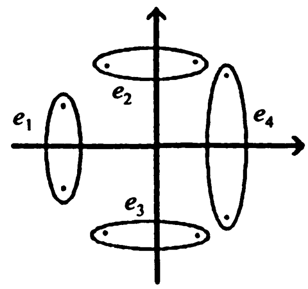

FIGURE 4.4

be mutually disjoint open disks in CP^1 such that they bound e_1, e_2, e_3, e_4 respectively and let $\tilde{D}_i := h_{\sigma_0}^{-1}(D_i)$ for $i = 1, 2, 3, 4$. Suppose $CP^1 - \bigcup_i D_i$ contains 0. Observing the monodromy map ρ_G we have the following proposition.

PROPOSITION 4.1. (1) $\rho_G(a\bar{a}) = \rho_G(bc) = \rho_G(\bar{c}\bar{b}) = \rho_G(\bar{d}d) = (1)(2)(3)(4)$.

(2) $h_{\sigma_0}^{-1}(CP^1 - \bigcup_i D_i) \approx (CP^1 - \bigcup_i D_i) \times \{x_1, x_2, x_3, x_4\}$.

(3) *Let $\partial\tilde{D}_i^j$ ($j = 1, 2, 3, 4$) be components of $\partial(h_{\sigma_0}^{-1}(D_i))$ defined by*

$$\partial\tilde{D}_i^j = \partial(h_{\sigma_0}^{-1}(D_i)) \cap ((CP^1 - \textstyle\bigcup_i D_i) \times \{x_j\})\,.$$

Then

$$\begin{aligned} \tilde{D}_1 &:= h_{\sigma_0}^{-1}(D_1) = D_1^2 \amalg N_1^{14} \amalg D_1^3 \\ \tilde{D}_2 &:= h_{\sigma_0}^{-1}(D_2) = N_2^{12} \amalg N_2^{34} \\ \tilde{D}_3 &:= h_{\sigma_0}^{-1}(D_3) = N_3^{13} \amalg N_3^{24} \\ \tilde{D}_4 &:= h_{\sigma_0}^{-1}(D_4) = N_4^{23} \amalg D_4^1 \amalg D_4^4\,, \end{aligned}$$

where (a) *D_i^j is a disk such that $h_{\sigma_0}(D_i^j) = D_i$ and $\partial D_i^j = \partial\tilde{D}_i^j$,* (b) *$N_i^{jk}$ is an annulus*

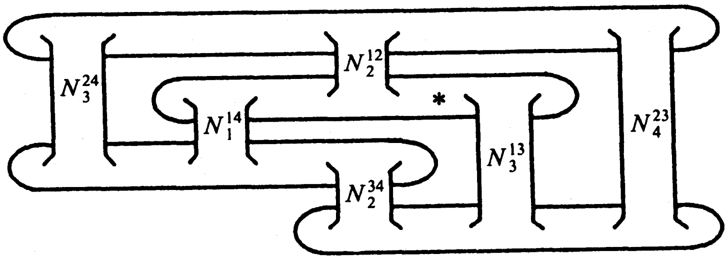

FIGURE 4.5

such that $h_{\sigma_0}(N_i^{jk})=D_i$ *and* $\partial N_i^{jk}=\partial\tilde{D}_i^j\coprod\partial\tilde{D}_i^k$.

(4) $F_{\sigma_0}\approx(\boldsymbol{CP}^1-\bigcup_i D_i)\times\{4\text{ pts}\}\coprod(\coprod_i\tilde{D}_i)\approx\Sigma_3$. (*See Figure* 4.5.)

Next we fix generators of $\pi_1(F_{\sigma_0}, *_1)$, where $*_j$ is given by $*_j=0\times x_j\in(\boldsymbol{CP}^1-\bigcup_i D_i)\times\{x_j\}$ $(j=1, 2, 3, 4)$. We define $\pi_1(\boldsymbol{C}_G, 0)_j$ by

$$\pi_1(\boldsymbol{C}_G, 0)_j=\{\gamma\in\pi_1(\boldsymbol{C}_G, 0)\mid\rho_G(\gamma)(x_j)=x_j\}\ .$$

$\gamma\in\pi_1(\boldsymbol{C}_G, 0)_j$ if and only if γ can be lifted to $\pi_1(F_{\sigma_0}, *_j)$. We denote the lifting map by

$$h_{\sigma_0*}^{-1}\colon\pi_1(\boldsymbol{C}_G, 0)_j\to\pi_1(F_{\sigma_0}, *_j)\ .$$

PROPOSITION 4.2. *If* $l_1=b\bar{c}a$, $l_2=ab\bar{b}$, $l_3=\bar{c}dc$, $m_1=a\bar{c}\bar{b}a$, $m_2=\bar{b}bc\bar{b}$, $m_3=cd\bar{d}c$ *then*

(1) $l_1, l_2, l_3, m_1, m_2, m_3$ *are contained in* $\pi_1(\boldsymbol{C}_G, 0)_1$.

(2) *Their liftings to* $\pi_1(F_{\sigma_0}, *_1)$ *are generators of* $\pi_1(F_{\sigma_0}, *_1)$.

PROOF. (1) For example, $\rho_G(l_1)=\rho_G(b\bar{c}a)=(12)(34)(13)(24)(14)=(1)(23)(4)$ hence $l_1\in\pi_1(\boldsymbol{C}_G, 0)_1$. We can prove the proposition similarly in the other cases.

(2) To prove 4.2(2) we prepare some notations and Lemma 4.3.

For $\gamma\in\pi_1(\boldsymbol{C}_G, 0)$ and for $j=1, 2, 3, 4$ we denote a lifting of γ with its starting point $0\times x_j\in(\boldsymbol{CP}^1-\bigcup_i D_i)\times\{x_j\}$ and with its end point $0\times x_{j'}$ by $(\gamma)_j^{j'}$. (We may denote $(\gamma)_j^{j'}$ by $(\gamma)_j$.) Then we have following lemma.

LEMMA 4.3. (1) $(a)_2=(0)_2$, $(a)_3=(0)_3$, $(\bar{a})_2=(0)_2$, $(\bar{a})_3=(0)_3$.

(2) $(d)_1=(0)_1$, $(d)_4=(0)_4$, $(\bar{d})_1=(0)_1$, $(\bar{d})_4=(0)_4$.

(3) $(a^2)_j=(b^2)_j=(c^2)_j=(d^2)_j=(\bar{a}^2)_j=(\bar{b}^2)_j=(\bar{c}^2)_j=(\bar{d}^2)_j=(0)_j$, *for* $j=1, 2, 3, 4$.

(4) $(dcba\bar{a}\bar{b}\bar{c}\bar{d})_j=(0)_j$ for $j=1, 2, 3, 4$.

PROOF. (1) (2) For instance, $(a)_2$ and $(\bar{a})_2$ are homotopic to zero because they are represented by loops contained in D_1^2. (3) Since any monodromy of $a, \cdots, \bar{d}$ is $\boldsymbol{Z}/2\boldsymbol{Z}$, any lifting of $a^2, \cdots, \bar{d}^2$ bounds a disk. (4) $dcba\bar{a}\bar{b}\bar{c}\bar{d}$ bounds a neighborhood of ∞ in $\boldsymbol{C}_G$ then its lifting is also zero.

To show Proposition 4.2(2) it is sufficient to show that for any $\gamma\in\pi_1(\boldsymbol{C}_G, 0)_1$, $(\gamma)_1$ can be written as a product of $(l_k)_1$, $(m_k)_1$ $(k=1, 2, 3)$. We leave the proof to the readers.

PROPOSITION 4.4. (1) $(bc)_1=(l_1m_1^{-1}l_1^{-1}m_3)_1$.
(2) $(\bar{a}a)_1=(l_2m_2^{-1}l_2^{-1}m_1)_1$.
(3) $(\bar{c}\bar{b})_1=(l_3m_3^{-1}l_3^{-1}m_2)_1$.
(4) *If* $n_1=bc$, $n_2=\bar{a}a$, $n_3=\bar{c}\bar{b}$ *then* $(n_3n_2n_1)_1=(0)_1$.

PROOF. We only prove (1) and leave (2) (3) (4) to the readers.

$$\begin{aligned}(l_1m_1^{-1}l_1^{-1}m_3)_1(cb)_1&=((b\bar{c}a)(a\bar{b}\bar{c}a)(a\bar{c}b)(cd\bar{d}c)(cb))_1\\&=(b\bar{c}\bar{b}bcd\bar{d}b)_1\\&=(b)_1^2(\bar{c}\bar{b})_2^2(bcd\bar{d})_2^2(b)_2^1\\&=(b)_1^2(\bar{c}\bar{b}\bar{a}abcd\bar{d})_2^2(b)_2^1\\&=(b)_1^2(0)_2^2(b)_2^1=(0)_1 .\end{aligned}$$

□

5. Monodromy action on $\mathfrak{M}_3$.

In this section we calculate monodromy of $f\colon V_5\mapsto CP^1$. Let SF be

$$SF=\{\sigma\in CP^1 \mid F_\sigma \text{ is a singular fiber}\} .$$

We denote a monodromy map by

$$[\tilde{\rho}_*]\colon \pi_1(CP^1-SF,\sigma_0)\to\mathfrak{M}_3 ,$$

where $\mathfrak{M}_3$ is a mapping class group of $\Sigma_3\approx F_{\sigma_0}$, that is,

$$\begin{aligned}\mathfrak{M}_3&=\mathrm{Homeo}_+\Sigma_3/\mathrm{Isotopy}\\&\simeq\mathrm{Aut}(\pi_1\Sigma_3)/\mathrm{Inn}(\pi_1\Sigma_3) .\end{aligned}$$

The latter equivalent is proved by Nielsen [N].

We calculate the monodromy map in the following 5 steps.
Step 1: We fix generators $[\gamma_j]$ of $\pi_1(T,\sigma_0)$.
Step 2: We calculate $\rho_T\colon \pi_T\to B_8(C^\times)$.
Step 3: We calculate $\rho_B\colon B_8(C^\times)\to\mathrm{Aut}\,\pi_1(C-D_{\sigma_0},0)$ and show $\rho(\gamma)_*=\rho_B\circ\rho_T(\gamma)$.
Step 4: We calculate $\rho_S\colon \pi T\to\tilde{\rho}(\gamma)\in\mathfrak{S}_4(h_{\sigma_0}^{-1}(0))$.
Step 5: We calculate $\tilde{\rho}(\gamma)_*\colon \pi_1(F_{\sigma_0},*)\to\pi_1(F_{\sigma_j},\tilde{\rho}(\gamma)(*))$.

Step 1: We determine $\pi_1(CP^1-SF,\sigma_0)$. If $\omega=\exp(2\pi i/5)$ then $\langle\omega\rangle=Z/5Z$ acts on CP^1-SF freely. Let $T=(CP^1-SF)/\langle\omega\rangle$. We define four loops $\gamma_0,\gamma_1,\gamma_2,\gamma_3$ on T.

$$\gamma_0\colon t\in[0,5]\mapsto\begin{cases}1+(1/10)\exp(t\pi i) & 0\le t\le 1\\(9/10)(2-t)+(1/\alpha-1/10)(t-1) & 1\le t\le 2\\(1/\alpha-1/10)\exp(2(t-2)\pi i/5) & 2\le t\le 3\\(1/\alpha-1/10)\omega(4-t)+(9/10)\omega(t-3) & 3\le t\le 4\\\omega+(1/10)\exp((7/5-(t-4))\pi i) & 4\le t\le 5 .\end{cases}$$

$$\gamma_1 : t \in [0, 3] \mapsto \begin{cases} 1+(1/10)\exp(t\pi i) & 0 \le t \le 1 \\ (9/10)\exp(2(t-1)\pi i/5) & 1 \le t \le 2 \\ \omega+(1/10)\exp((7/5-(t-2))\pi i) & 2 \le t \le 3 . \end{cases}$$

$$\gamma_2 : t \in [0, 1] \mapsto (11/10)\exp(2t\pi i/5) .$$

$$\gamma_3 : t \in [0, 3] \mapsto \begin{cases} (11/10)(1-t)+(\alpha+1/10)t & 0 \le t \le 1 \\ (\alpha+1/10)\exp(2(t-1)\pi i/5) & 1 \le t \le 2 \\ (\alpha+1/10)\omega(3-t)+(11/10)\omega(t-2) & 2 \le t \le 3 . \end{cases}$$

Here $\alpha = \sqrt[5]{4}$.

LEMMA 5.1. *$\pi_1(T, \sigma_0)$ is a free group generated by $[\gamma_0]$, $[\gamma_1]$, $[\gamma_2]$, $[\gamma_3]$.*

PROOF. T is homeomorphic to $\boldsymbol{C}-\{0, 1, -1/4, -4\}$ by $z \mapsto z^5 : T \to \boldsymbol{C}-\{0, 1, -1/4, -4\}$. T is homotopic to a bouquet of four S^1's and hence $\pi_1(T)$ is a free group.

Let $p : \boldsymbol{CP}^1 - SF \to T$ be a projection. Since p is a covering map, we have the following lemma.

LEMMA 5.2. (1) *$p_* : \pi_1(\boldsymbol{CP}^1 - SF, \sigma_0) \to \pi_1(T, \sigma_0)$ is injective.*

(2) *Let len: $\pi_1(T, \sigma_0) \to \boldsymbol{Z}/5\boldsymbol{Z}$: $[\prod_i \gamma_i^{\varepsilon_i}] \mapsto \sum_i \varepsilon_i$ mod 5 be a length map. Then the sequence*

$$0 \longrightarrow \pi_1(\boldsymbol{CP}^1 - SF, \sigma_0) \xrightarrow{p_*} \pi_1(T, \sigma_0) \xrightarrow{len} \boldsymbol{Z}/5\boldsymbol{Z} \longrightarrow 0$$

is exact.

Let πT be a free group of loops on T generated by γ_0, γ_1, γ_2, and γ_3. For any $[\gamma] \in \pi_1(t, \sigma_0)$ we take realization γ of $[\gamma]$ in πT.

Step 2: Consider the following equation.

$$H_t(x) = \left\{ \frac{(x-1)^4}{16t^5} - (x^4+x^3+x^2+x+1) \right\} \left\{ \frac{(x-1)^4}{4t^5} - (x^4+x^3+x^2+x+1) \right\} .$$

From Lemma 2.5, the solutions of $H_\sigma(x)=0$ give all of branch loci of $h_\sigma : F_\sigma \to \boldsymbol{CP}^1$. From Lemma 2.6, $H_\sigma(x)$ has multiple solutions only if F_σ is a singular fiber. Hence

$$C_H = \{\sigma \mid F_\sigma \text{ is a general fiber}\} = \boldsymbol{CP}^1 - SF .$$

For $[\gamma] \in \pi_1(C_H, \sigma_0)$ we denote $\{x_j(t)\}_{j=1,2,\cdots,8}$ by the solutions of $H_{\gamma(t)}(x)=0$, where $\{x_j(0)\} = \{\bar{D}, \bar{C}, \bar{B}, \bar{A}, A, B, C, D\}$.

We regard $\{x_j(c)\}$ as a braid in $\boldsymbol{C}^\times = \boldsymbol{CP}^1 - \{0, \infty\}$. Notice that for any $[\gamma]$ we take a representation γ from πT such that neither 0 nor ∞ are solutions of $H_{\gamma(t)}(x)=0$. Thus we define $\rho_T : \pi T \to B_8(\boldsymbol{C}^\times)$, where $B_8(\boldsymbol{C}^\times)$ is a braid group of degree 8 on $\boldsymbol{C}^\times$.

We illustrate it in the following way. For example if $\gamma=\gamma_1^{-1}\gamma_2$ then $\{x_j(c)\}$ is given by Figure 5.1.1 and we illustrate the braid by Figure 5.1.2. Let $\beta_1, \cdots, \beta_8$ be braids on $C^\times$ given by Figure 5.2.

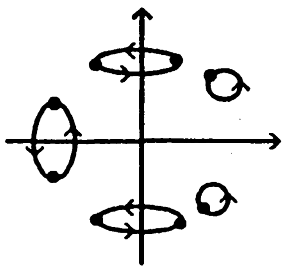

FIGURE 5.1.1

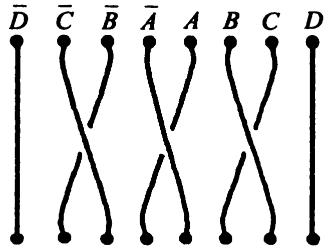

FIGURE 5.1.2

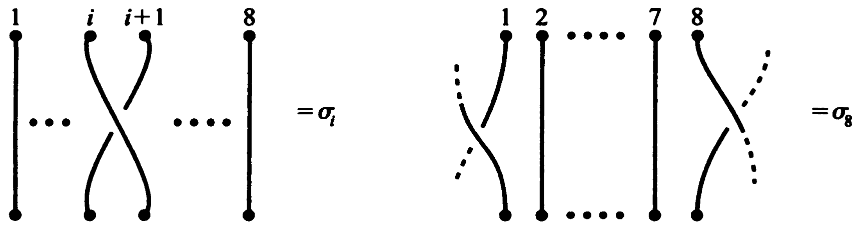

FIGURE 5.2

We have numerical solutions of $H_t(x)$ when t runs along $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ and we obtain the following proposition.

PROPOSITION 5.3. *If $\rho_T: \pi_1(T, \sigma_0)\to B_8$ is a braid representation then ρ_T is a homomorphism and*

$$\begin{aligned}\rho_T:\ &\gamma_0\mapsto\beta_1^{-1}\beta_2\beta_1\beta_3^{-1}\beta_2^{-1}\beta_7^{-1}\beta_6\beta_7\beta_5^{-1}\beta_6^{-1}\beta_8\beta_4^{-1}\beta_1\beta_7\\ &\gamma_1\mapsto\beta_1^{-2}\beta_3^{-1}\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_5^{-1}\beta_7^{-2}\beta_2^{-1}\beta_4^{-1}\beta_6^{-1}\\ &\gamma_2\mapsto\beta_1^{-2}\beta_3^{-1}\beta_5^{-1}\beta_4^{-1}\beta_3^{-1}\beta_5^{-1}\beta_7^{-2}\\ &\gamma_3\mapsto(\beta_1^{-1}\beta_3^{-1}\beta_5^{-1}\beta_7^{-1})^2\,.\end{aligned}$$

For example, see Figure 5.3.

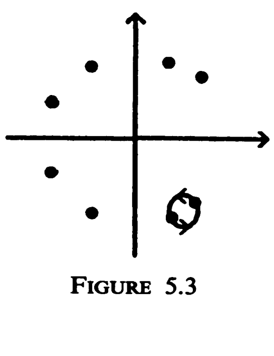

FIGURE 5.3

Step 3: We define a homomorphism $\rho_B: B_8(\boldsymbol{C}^\times)\to \mathrm{Aut}\,\pi_1(\boldsymbol{C}-D_{\sigma_0}, 0)$. Suppose a braid $\beta\in B_8$ is represented by a map β: $\{8 \text{ points}\}\times[0, 1]\to \boldsymbol{C}^\times$. Let ρ_β: $\boldsymbol{C}\times[0, 1]\to\boldsymbol{C}$ be an extension of β, that is, ρ_β is continuous, $\rho_\beta(\cdot, t)$ is a homeomorphism, $\rho_\beta(0, t)=0$, and $\rho_\beta(\beta(x, 0), t)=\beta(x, t)$ for $x\in\{8 \text{ points}\}$. ρ_β gives a map $\boldsymbol{C}\to\boldsymbol{C}$: $y\mapsto\rho_\beta(y, 1)$. $\rho_B(\beta)\in\mathrm{Aut}(\boldsymbol{C}-\{8 \text{ points}\}, 0)$ denotes an induced map of this map. Similarly in Proposition 2.7, it is shown that this induced map $\rho_B(\beta)$ is well-defined and depends only on a braid class of β.

In our case it is sufficient to calculate ρ_B for $\beta_i\in B_8$. (Notice that $\beta_1, \cdots, \beta_8$ do not generate B_8.)

PROPOSITION 5.4. *ρ_B is a homomorphism and*

$$\begin{aligned}
\rho_B(\beta_1)\colon\ & a\mapsto a,\ b\mapsto b,\ c\mapsto c,\ d\mapsto d,\\
& \bar a\mapsto\bar a,\ \bar b\mapsto\bar b,\ \bar c\mapsto\bar c\bar d\bar c^{-1},\ \bar d\mapsto\bar c,\\
\rho_B(\beta_2)\colon\ & a\mapsto a,\ b\mapsto b,\ c\mapsto c,\ d\mapsto d,\\
& \bar a\mapsto\bar a,\ \bar b\mapsto\bar b\bar c\bar b^{-1},\ \bar c\mapsto\bar b,\ \bar d\mapsto\bar d,\\
\rho_B(\beta_3)\colon\ & a\mapsto a,\ b\mapsto b,\ c\mapsto c,\ d\mapsto d,\\
& \bar a\mapsto\bar a\bar b\bar a^{-1},\ \bar b\mapsto\bar a,\ \bar c\mapsto\bar c,\ \bar d\mapsto\bar d,\\
\rho_B(\beta_4)\colon\ & a\mapsto a\bar a a^{-1},\ b\mapsto b,\ c\mapsto c,\ d\mapsto d,\\
& \bar a\mapsto a,\ \bar b\mapsto\bar b,\ \bar c\mapsto\bar c,\ \bar d\mapsto\bar d,\\
\rho_B(\beta_5)\colon\ & a\mapsto b,\ b\mapsto bab^{-1},\ c\mapsto c,\ d\mapsto d,\\
& \bar a\mapsto\bar a,\ \bar b\mapsto\bar b,\ \bar c\mapsto\bar c,\ \bar d\mapsto\bar d,\\
\rho_B(\beta_6)\colon\ & a\mapsto a,\ b\mapsto c,\ c\mapsto cbc^{-1},\ d\mapsto d,\\
& \bar a\mapsto\bar a,\ \bar b\mapsto\bar b,\ \bar c\mapsto\bar c,\ \bar d\mapsto\bar d,\\
\rho_B(\beta_7)\colon\ & a\mapsto a,\ b\mapsto b,\ c\mapsto d,\ d\mapsto dcd^{-1},\\
& \bar a\mapsto\bar a,\ \bar b\mapsto\bar b,\ \bar c\mapsto\bar c,\ \bar d\mapsto\bar d,\\
\rho_B(\beta_8)\colon\ & a\mapsto a,\ b\mapsto b,\ c\mapsto c,\ d\mapsto\bar d,\\
& \bar a\mapsto\bar a,\ \bar b\mapsto\bar b,\ \bar c\mapsto\bar c,\ \bar d\mapsto\bar d d\bar d^{-1}.
\end{aligned}$$

PROOF. The action of β_1 on $\boldsymbol{CP}^1$ is given by Figure 5.3, so a, b, c, d, $\bar a$, $\bar b$ do not move. $\bar d$ becomes $\bar c$. $\bar c$ becomes $\bar c\bar d\bar c^{-1}$. See Figure 5.4. □

From the definition of ρ_T and ρ_B, we have

LEMMA 5.5. *For $\gamma\in\pi T$, $\rho(\gamma)_*=\rho_B\circ\rho_T(\gamma)$.*

Step 4: We calculate a monodromy map ρ_S of $h_{\sigma_0}^{-1}(0)=\{*_1, *_2, *_3, *_4\}$. (See Figure 4.2.)

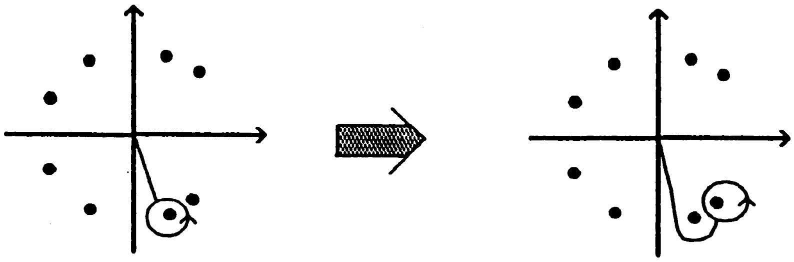

FIGURE 5.4

PROPOSITION 5.6. *Let $\rho_S: \pi_1(T, \sigma_0) \to \mathfrak{S}_4(h_{\sigma_0}^{-1}(0))$ be a monodromy map of $h_{\sigma_0}^{-1}(0)$, that is, ρ_S is a monodromy map of*

$$f^o: V_5^o \to (CP^1 - SF)/(Z/5Z),$$

where $V_5^o = \{x \mid \sigma \in CP^1 - SF, x \in F_\sigma, h_\sigma(x) = [1:0]\}/\sim$, $[z_0 : z_1 : z_2 : z_3] \sim [\omega z_0 : \omega z_1 : z_2 : z_3]$, and $f^o([x]) = [f(x)]$. Then $\rho_S(\gamma_0) = (14)(2)(3)$, $\rho_S(\gamma_1) = \rho_S(\gamma_2) = \rho_S(\gamma_3) = (1243)$.

PROOF. We prove this proposition by the numerical analysis. Let $I_t(x) = g_t(0, x)$. If t runs along γ_j $(j = 1, 2, 3)$ then the permutations of the solutions of $I_t(x) = 0$ are given by (1243) in $C/\langle\omega\rangle$, where the numbers of the solutions are given in Figure 4.2. In case of $\rho_S(\gamma_0)$ we can show this lemma similarly. □

Step 5: For $*$ in $h_{\sigma_0}^{-1}(0)$, the diagram

$$\begin{array}{ccc} \pi_1(F_{\sigma_0} - h_{\sigma_0}^{-1}(D_{\sigma_0}), *) & \xrightarrow{\tilde{\rho}(\gamma)_*} & \pi_1(F_{\sigma_j} - h_{\sigma_j}^{-1}(D_{\sigma_j}), \tilde{\rho}(\gamma)(*)) \\ \downarrow h_{\sigma_0 *} & & \downarrow h_{\sigma_j *} \\ \pi_1(CP^1 - D_{\sigma_0}, 0) & \xrightarrow{\rho(\gamma)_*} & \pi_1(CP^1 - D_{\sigma_j}, 0) \end{array}$$

commutes. And $\tilde{\rho}(\gamma)_* = h_{\sigma_j *}^{-1} \circ \rho_B \circ \rho_T \circ h_{\sigma_0 *}$ induces a homomorphism

$$\tilde{\rho}(\gamma)_* : \pi_1(F_{\sigma_0}, *) \to \pi_1(F_{\sigma_j}, \tilde{\rho}(\gamma)(*)) .$$

Immediately we have the following proposition.

PROPOSITION 5.7. *For $\gamma \in \pi T$, $\rho_B \circ \rho_T(\gamma)(l_k) \in \pi_1(C_G, 0)_{\rho_S(\gamma)(1)}$, $\rho_B \circ \rho_T(\gamma)(m_k) \in \pi_1(C_G, 0)_{\rho_S(\gamma)(1)}$.*

EXAMPLE. We calculate $\tilde{\rho}(\gamma_0)_*(l_1)$ in the following way. From Propositions 5.3, 5.4,

$$\begin{aligned} \rho_B\rho_T(\gamma_0):\ & a \mapsto \bar{a}^{-1}b^{-1}db\bar{a}, & \bar{a} &\mapsto \bar{c}d\bar{c}^{-1} \\ & b \mapsto \bar{c}, & \bar{b} &\mapsto \bar{c}\bar{d}^{-1}\bar{c}^{-1}\bar{a}^{-1}b^{-1}cb\bar{a}\bar{c}\bar{d}\bar{c}^{-1} \\ & c \mapsto \bar{c}\bar{a}^{-1}b\bar{a}\bar{c}^{-1}, & \bar{c} &\mapsto \bar{c}\bar{d}^{-1}\bar{c}^{-1}\bar{b}\bar{c}\bar{d}\bar{c}^{-1} \\ & d \mapsto \bar{c}a\bar{c}^{-1}, & \bar{d} &\mapsto \bar{c}dcbab^{-1}c^{-1}d^{-1}\bar{c}^{-1} . \end{aligned}$$

Notice that $\rho_S(\gamma_0)(*_1)=(*_4)$. Then, for $\beta=\rho_T(\gamma_0)$, $l_1=b\bar{c}a$, we have

$$\rho_B(\beta)(l_1)=\bar{c}(\bar{c}\bar{d}^{-1}\bar{c}^{-1}\bar{b}\bar{c}\bar{d}\bar{c}^{-1})(\bar{a}^{-1}b^{-1}db\bar{a}),$$

$$\rho_G(\rho_B(\beta)(l_1))(*_4)=*_4.$$

Hence $\rho_B(\beta)(l_1)$ can be lifted to $\pi_1(F_{\sigma_0}, *_4)$ and we have

$$\begin{aligned}\tilde{\rho}(\gamma_0)_*(l_1)&=(\bar{c}\bar{c}\bar{d}^{-1}\bar{c}^{-1}\bar{b}\bar{c}\bar{d}\bar{c}^{-1}\bar{a}^{-1}b^{-1}db\bar{a})_4^4\\&\sim(a)_1^4(\bar{d}^{-1}\bar{c}^{-1}\bar{b}\bar{c}\bar{d}\bar{c}^{-1}\bar{a}^{-1}b^{-1}db\bar{a})_4^4(a^{-1})_4^1\\&\sim(m_1l_1^{-1}n_1m_3^{-1}l_3^{-1}n_2l_2n_3^{-1}l_3n_1^{-1})_1.\end{aligned}$$

As mentioned in Lemma 5.2,

$$\begin{aligned}\pi_1(C_H,\sigma_0)&\cong\ker(len:\pi_1(T,\sigma_0)\to\boldsymbol{Z}/5\boldsymbol{Z})\\&\subset\pi_1(T,\sigma_0)\cong\pi T.\end{aligned}$$

Then we determine the monodromy map

$$[\tilde{\rho}_*]:\pi_1(C_H,\sigma_0)\to\mathrm{Aut}\,\pi_1F_{\sigma_0}/\mathrm{Inn}\,\pi_1F_{\sigma_0}\cong\mathfrak{M}_3.$$

EXAMPLE. Let $\gamma=\gamma_1^{-1}\gamma_2$, then $\rho_T(\gamma)=\beta_2\beta_4\beta_6$ and $\rho_S(\gamma)=(1)(2)(3)(4)$.

$$\begin{aligned}\rho_B\rho_T(\gamma):\quad &a\mapsto a\bar{a}a^{-1}, &&\bar{a}\mapsto a\\&b\mapsto c, &&\bar{b}\mapsto\bar{b}\bar{c}\bar{b}^{-1}\\&c\mapsto cbc^{-1}, &&\bar{c}\mapsto\bar{b}\\&d\mapsto d, &&\bar{d}\mapsto\bar{d}.\end{aligned}$$

$$\begin{aligned}\tilde{\rho}(\gamma)_*:\quad(l_1)_1=(b\bar{c}a)_1&\mapsto(c\bar{b}a\bar{a}a^{-1})_1\\&=(c\bar{b}a\bar{a}a)_1\\&=(cb)_1(b\bar{c}a)_1(a\bar{c}\bar{b}a)_1(\bar{a}a)_1\\&=(n_1^{-1}l_1m_1n_2)_1.\end{aligned}$$

Similarly we have

$$\begin{aligned}\tilde{\rho}(\gamma)_*:\quad&(l_2)_1\mapsto(n_2^{-1}l_2m_2n_3)_1\\&(l_3)_1\mapsto(n_3^{-1}l_3n_1)_1\\&(m_1)_1\mapsto(n_2^{-1}m_1n_2)_1\\&(m_2)_1\mapsto(n_3^{-1}m_2n_3)_1\\&(m_3)_1\mapsto(n_1^{-1}m_3n_1)_1.\end{aligned}$$

REMARK. It is known that $\mathfrak{M}_g$ is generated by Dehn twists along simple closed curves on Σ_g. In our cases for some loops in $\pi_1(T, \sigma_0)$ we easily give their monodromy maps as products of Dehn twists. It is easy to check the following proposition.

PROPOSITION 5.8. *Let N be an annulus, $D=\{z\in C \mid |z|<3\}$ be a disk. Let $h: N\to D$ be a branched double covering map with two branch points ± 1. If $\varphi_t: D\to D$, $t\in[0, 1]$ is a homeotopy on D such that*

(0) $\varphi_0=\mathrm{id}$,

(1) $\varphi_t|\partial D=\mathrm{id}$ *for* $t\in[0, 1]$,

(2) $\varphi_t|\{|z|<2\}(z)=\exp(t\pi\sqrt{-1})z$,

then there exists a lifting $\tilde{\varphi}_1$ of φ_1 and $\tilde{\varphi}_1$ coincides with Dehn twist τ_δ along δ, where δ is a simple closed curve on N which is parallel to a boundary ∂N of N.

EXAMPLE. If $\gamma=\gamma_1^{-1}\gamma_2$ then $\rho_T(\gamma)=\beta_2\beta_4\beta_6$. We apply 5.8 to D_i's in 4.1, (see Figures 4.4 and 5.1.1), and we conclude that

$$[\tilde{\rho}(\gamma)]=\tau_{m_1}\tau_{m_2}\tau_{n_1}\tau_{n_2}\tau_{n_3}\,.$$

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