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A Remark on BMO and Bloch Functions on the Unit Ball of R^n

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Let B denote the unit ball in the n dimensional Euclidean space $\boldsymbol{R}^n$, where n is a fixed integer ≥ 2. A real valued function f harmonic in B is said to be Bloch if

$$\|f\|_\beta := \sup_{x\in B} \frac{1}{2}(1-|x|^2)|\nabla f(x)| < \infty .$$

A Borel function f, locally integrable on B, is said to be of bounded mean oscillation on B if

$$\|f\|_{\mathrm{BMO}} := \sup \left\{ \frac{1}{|D|} \int_D |f(x)-f_D|^2 dx \right\}^{1/2} < \infty ,$$

where the supremum is taken over the set of all balls D contained in B. Here $|D|$ stands for the n dimensional volume of D, and f_D is the mean value of f over D. Note that, for a harmonic function, f_D is the value of f at the center of D.

In [1] Muramoto proved that, for harmonic functions,

$$\|f\|_{\mathrm{BMO}} \leq \alpha(n)^{1/2} \|f\|_\beta , \tag{1}$$

where

$$\alpha(n) = 2n \int_0^1 r^{n-1} \log \frac{1}{1-r^2}\, dr ,$$

and, in other direction,

$$\|f\|_\beta \leq (n(n+2))^{1/2} \|f\|_{\mathrm{BMO}} . \tag{2}$$

In proving (1) Muramoto used the stochastic theory. In this note we use a simple variant of Green's formula to improve these inequalities.

THEOREM. *If f is harmonic in B, then*

$$\|f\|_{\mathrm{BMO}} \leq \sqrt{2}\, \|f\|_\beta \tag{3}$$

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and

(4) $$\|f\|_\beta \le (n+2)^{1/2}\|f\|_{\mathrm{BMO}}\,.$$

Inequality (3) improves (1) only for $n\ge 3$ because $\alpha(2)=2$. On the other hand, writting $\alpha(n)$ as

$$\alpha(n)=4\int_0^1 (1-r^n)r(1-r^2)^{-1}dr\,,$$

we see that $\alpha(n)$ increases to ∞ (as $n\to\infty$), and this shows that (3) actually improves (1) for $n\ge 3$.

LEMMA 1. *If g is a C^2 function on B, then*

$$\frac{d}{dr}\int_S g(ry)d\sigma(y)=\frac{1}{n}r^{1-n}\int_{rB}\Delta g(x)dv(x)\,,\qquad 0<r<1\,,$$

where $rB=\{x\colon |x|<r\}$; v is the normalized Lebesgue measure on B, and σ the normalized surface measure on $S=\partial B$.

Although this lemma is a special case of Green's formula, its proof is very simple. Namely, first we reduce the proof to considering radial functions. Then, assuming that $g(x)=u(|x|^2)$, where u is C^2 on $[0, 1)$, we calculate the Laplacian and use the integration in polar coordinates to verify the formula. See [2], for example.

LEMMA 2. *If f is harmonic on B, then*

$$\frac{dI}{dr}=\frac{2}{n}r^{1-n}\int_{rB}|\nabla f|^2dv\,,\qquad 0<r<1\,,$$

where

(5) $$I(r)=\int_S |f(ry)-f(0)|^2d\sigma(y)\,.$$

PROOF. If f is C^2 on B, then $\Delta(f^2)=2f\Delta f+2|\nabla f|^2$. Hence, if f is harmonic, $\Delta(|f-f(0)|^2)=2|\nabla f|^2$. Now the result follows from Lemma 1. □

LEMMA 3. *Let f be harmonic in B and $f_a(x)=f(a+(1-|a|)x)$, $a\in B$. Then*

(i) $$\|f\|_{\mathrm{BMO}}=\sup_{a\in B}\|f_a-f_a(0)\|_2\,,$$

where $\|\ \|_2$ stands for the norm in $L^2(B, v)$;

(ii) $$\|f_a\|_\beta\le\|f\|_\beta\le\sup_{a\in B}|\nabla f_a(0)|\,.$$

PROOF. (i) For a fixed $a\in B$ let D_ε denote the ball of radius ε centered at a,

$0<\varepsilon\leq 1-|a|$. In view of the subharmonicity of $|f-f(a)|^2$, its mean value over D_ε increases with ε and equals $\|f_a-f_a(0)\|_2^2$ for $\varepsilon=1-|a|$. This proves "$\|f\|_{\mathrm{BMO}}\leq$". The rest is trivial.

(ii) This is verified by elementary calculation. See [1], Lemma 1. □

Proof of Theorem. Let f be harmonic in B. By Lemma 3, proving (3) reduces to proving

$$\|f-f(0)\|_2\leq\sqrt{2}\,\|f\|_\beta\,. \tag{6}$$

To prove this, assume that $\|f\|_\beta\leq 1$, i.e., $|\nabla f(x)|^2\leq 4(1-|x|^2)^{-2}$. Then, by Lemma 2,

$$\frac{dI}{dr}\leq\frac{8}{n}r^{1-n}\int_{rB}(1-|x|^2)^{-2}dv(x)\,,$$

where I is defined by (5). Hence, by integration in polar coordinates,

$$\frac{dI}{dr}\leq 8r^{1-n}\int_0^r t^{n-1}(1-t^2)^{-2}dt$$

$$=8\sum_{j=0}^{\infty}\frac{j+1}{2j+n}r^{2j+1}\,,$$

whence

$$I(r)\leq 4\sum_{j=0}^{\infty}\frac{r^{2j+2}}{2j+n}\,.$$

It follows that

$$\int_B|f-f(0)|^2dv=n\int_0^1 r^{n-1}I(r)dr$$

$$\leq 2n\sum_{j=0}^{\infty}\left(\frac{1}{2j+n}-\frac{1}{2j+2+n}\right)=2\,,$$

which proves (6).

In proving (4) we use the fact that the function $|\nabla f|^2$ is subharmonic ([3], Ch. VII §3). Then it follows from Lemma 2 that

$$\frac{dI}{dr}\geq\frac{2}{n}r|\nabla f(0)|^2\,,$$

whence

$$I(r)\geq(1/n)r^2|\nabla f(0)|^2\,.$$

Multiplying this inequality by $nr^{n-1}dr$ and integrating from 0 to 1, we see that

$$\|f-f(0)\|_2^2 \geq (n+2)^{-1} |\nabla f(0)|^2 .$$

Applying this to f_a and using the right inequality of Lemma 3 (ii) we obtain (4). □

References

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