TOKYO J. MATH.
VOL. 17, NO. 1, 1994

0-Dimensional Moduli Space of Stable Rank 2 Bundles and Differentiable Structures on Regular Elliptic Surfaces

Yukio KAMETANI and Yoshihisa SATO

Tokyo Metropolitan University

1. Introduction.

The study of moduli spaces of stable rank 2 vector bundles over smooth complex algebraic surfaces has contributed to the study of differentiable structures on their underlying smooth 4-manifolds through Donaldson's result in [Do1]: the moduli space of irreducible Hermitian-Einstein connections on a $U(2)$-bundle P coincides with the moduli space of stable holomorphic structures on P. By Friedman, Morgan, Okonek, Van de Ven and others, a lot of work on moduli spaces of stable bundles over certain algebraic surfaces (for example, Dolgachev surfaces) has been done ([Fri], [FrM1], [LO], [OV]), and they showed that their underlying topological 4-manifolds admit distinct differentiable structures. The bundles which they considered have trivial first Chern classes and moduli spaces of nonzero dimension. In [Kot] Kotschick studied the 0-dimensional moduli space of stable bundles $\mathscr{E}$ with $c_1(\mathscr{E})=K_B$ and $c_2(\mathscr{E})=1$ over the Barlow surface B, and he showed that the Barlow surface B is homeomorphic but not diffeomorphic to $CP^2\#8\overline{CP^2}$. Here, K_B is the canonical divisor of B.

In this paper, we consider 0-dimensional moduli spaces of stable bundles with nontrivial first Chern classes. Our target surfaces are certain generic regular elliptic surfaces S, that is, S is a compact complex surface with a certain type of holomorphic map $\pi: S\to CP^1$ called an elliptic fibration and with a section. We show that for a certain ample divisor L, the moduli space of L-stable rank 2 bundles over S with $c_1=\Sigma-K_S$ and $c_2=1$ consists of exactly one point. Here Σ is a section of π. Moreover, by using a result about the effect of surgery on the simple invariants obtained by Gompf and Mrowka [GoM], we show the following, which was announced by Friedman and Morgan in [FrM2]:

THEOREM. *Let S be a minimal elliptic surface with strictly positive geometric genus. Then the underlying topological 4-manifold of S admits infinitely many differentiable structures.*

Received April 1, 1993

Smooth 4-manifolds obtained from elliptic surfaces with at most two multiple fibers by twisted constructions give such exotic differentiable structures on S. In [FrM2], Friedman and Morgan considered the diffeomorphism types of elliptic surfaces by using the Donaldson's polynomial invariant, but we use the simple invariant. The diffeomorphism types of elliptic surfaces with $p_g=0$, i.e., Dolgachev surfaces, are considered in [FrM1], [LO] and [OV].

ACKNOWLEDGMENTS. The authors would like to thank Professor M. Ue for helpful conversations.

2. Moduli spaces of stable bundles.

We begin with some definitions. Let X be a smooth complex algebraic surface and let $\mathscr{V}$ be a holomorphic rank 2 vector bundle over X. We fix an ample divisor L on X. A rank 2 bundle $\mathscr{V}$ over X is L-stable (resp. L-semistable) if and only if for all sub-line bundles $\varphi: \mathscr{F} \to \mathscr{V}$ we have

$$c_1(F) \cdot L < \tfrac{1}{2} c_1(\mathscr{V}) \cdot L \qquad (\text{resp. } \leq \tfrac{1}{2} c_1(\mathscr{V}) \cdot L) .$$

For an ample divisor L on X, let $\mathscr{M}_X(c_1, c_2; L)$ be the moduli space of L-stable holomorphic rank 2 vector bundles $\mathscr{V}$ with $c_1(\mathscr{V})=c_1$ and $c_2(\mathscr{V})=c_2$ over X.

We want to study $\mathscr{M}_S(c_1, c_2; L)$ of formal dimension 0 on a regular elliptic surface S. A *regular elliptic surface* S is a compact complex surface admitting the following type of onto holomorphic map $\pi: S \to CP^1$ called an *elliptic fibration*. The map π has only finitely many critical values, and away from these values it is smoothly a bundle projection whose fibers are nonsingular elliptic curves. These fibers are called the *general* fibers of π, and the preimage of each critical value is called a *singular* fibers and has ordinary double points as singularities. In particular, we call S *generic* if all fibers of π are irreducible. We will always assume that S is *relatively minimal*, i.e., no fiber in S contains an exceptional curve of the first kind. Since an elliptic surface S has $c_1(S)^2=0$, the Noether's formula implies that its Euler characteristic $\chi(S)$ is divided by 12. For elliptic surfaces, we may smoothly change π so that all singular fibers of π are *cusp fibers* or *multiple fibers*. A cusp fibers is a rational curve with one cusp, i.e., a 2-sphere with one singular point which is locally a cone on a right-handed trefoil knot. In a general fiber F near a cusp fiber $\pi^{-1}(p)$, we can take two essential circles C_1 and C_2 called *vanishing cycles*. The cusp fiber is obtained from F by collapsing $C_1 \cup C_2$ to a point.

We fix an integer k. Let $\pi: S_k \to CP^1$ be the elliptic fibration of a regular elliptic surface S_k with $\chi(S_k)=12k$ and with a section. Throughout this section, we assume that S_k is *generic* and S_k *does not have multiple fibers* and $k \geq 1$. Let f be a general fiber and let Σ be a section of π. By the canonical bundle formula for elliptic fibrations [BPV], $K_{S_k}=(k-2)f$. Our guide to study the moduli spaces is a method by Qin [Qi]. He has studied the moduli spaces of stable rank 2 bundles over S_2, that is $K3$ surface with generic elliptic fibration. We begin with a criterion of ampleness of a class of divisors.

LEMMA 2.1. (1) $f \cdot f = 0$, $f \cdot \Sigma = 1$ *and* $\Sigma \cdot \Sigma = -k$.
(2) *For an integer* r, *we put* $L_r = \Sigma + rf$. *A divisor* L_r *is ample if and only if* $r > k$.

PROOF. (1). Since the genus of Σ is 0, the adjunction formula implies that $\Sigma \cdot \Sigma = -k$. (2). If L_r is ample, then $0 < L_r \cdot \Sigma = -k + r$ by the Nakai-Moishezon criterion for ample divisors. Hence $r > k$. Suppose that $r > k$. Then $L_r^2 = (\Sigma + rf)^2 = -k + 2r > 0$. Let C be any irreducible curve on S. If $C = \Sigma$, then $L_r \cdot C = L_r \cdot \Sigma = r - k > 0$. Next we consider the case where C is different from Σ. Since C, f and Σ are irreducible curves and $f^2 = 0$, we have $\Sigma \cdot C \geq 0$ and $f \cdot C \geq 0$. If $f \cdot C > 0$, then $L_r \cdot C = \Sigma \cdot C + rf \cdot C > 0$. If $f \cdot C = 0$, then $f' \cdot C = 0$ for any general fiber f'. Hence, $\pi(C)$ consists of finite points in CP^1. Then $Supp(C)$ consists of fibers of π. Because of the irreducibility of C, C is a fiber of π and so $\Sigma \cdot C = 1$. Thus $L_r \cdot C = 1$. Therefore, L_r is ample by the Nakai-Moishezon criterion. q.e.d.

Now we study the moduli space of stable holomorphic rank 2 bundles $\mathscr{V}$ with $c_1(\mathscr{V}) = \Sigma - (k-2)f$ and $c_2(\mathscr{V}) = 1$ over a regular elliptic surface. If Z is a 0-dimensional subscheme of an algebraic surface X and $\mathscr{I}_Z$ is its ideal sheaf, then we have

$$c_1(\mathscr{I}_Z) = 0 \quad \text{and} \quad c_2(\mathscr{I}_Z) = \ell(Z)\,, \text{ the length of } Z\,,$$

where by definition $\ell(Z) = \dim H^0(\mathcal{O}_Z) = \dim H^0(\mathcal{O}_X/\mathscr{I}_Z)$. We can describe an L_{2k+1}-stable bundle as an extension of sheaves of rank 1 as follows.

PROPOSITION 2.2. *If a bundle* $\mathscr{V}$ *is* L_{2k+1}*-stable, then there exists an extension as* $\mathcal{O}$*-modules of the form,*

$$0 \to \mathcal{O}(f) \to \mathscr{V} \to \mathcal{O}(\Sigma - (k-1)f) \to 0\,. \tag{2.1}$$

PROOF. Since $c_1(\det \mathscr{V}) = c_1(\mathscr{V}) = \Sigma - (k-2)f$ and the irregularity $q(S) = 0$, there is a canonical isomorphism $\mathscr{V}^* \cong \mathscr{V} \otimes (\det \mathscr{V})^* \cong \mathscr{V} \otimes \mathcal{O}(-\Sigma + (k-2)f)$, where $\mathscr{V}^*$ denotes the dual of $\mathscr{V}$. So, by the Serre duality we have $h^2(\mathscr{V} \otimes \mathcal{O}(-f)) = h_0((\mathscr{V} \otimes \mathcal{O}(-f))^* \otimes \mathcal{O}((k-2)f)) = h^0(\mathscr{V} \otimes \mathcal{O}(-\Sigma(2k-3)f))$. On the other hand, from the Riemann-Roch formula we obtain

$$\begin{aligned} &h^0(\mathscr{V} \otimes \mathcal{O}(-f)) - h^1(\mathscr{V} \otimes \mathcal{O}(-f)) + h^2(\mathscr{V} \otimes \mathcal{O}(-f)) \\ &\quad = \deg(ch(\mathscr{V} \otimes \mathcal{O}(-f)) \cdot tod(\mathscr{T}_{S_k}))_2 = 1 > 0\,. \end{aligned}$$

Hence we have either $h^0(\mathscr{V} \otimes \mathcal{O}(-f)) > 0$ or $h^0(\mathscr{V} \otimes \mathcal{O}(-\Sigma + (2k-3)f)) = h^2(\mathscr{V} \otimes \mathcal{O}(-f)) > 0$.

Case (A). $h^0(\mathscr{V} \otimes \mathcal{O}(-f)) > 0$. Then, there is a sub-line bundle

$$\varphi : \mathcal{O}(f) \to \mathscr{V}\,.$$

Possibly φ does not have torsion-free cokernel. If we note that $\det \mathscr{V} = \mathcal{O}(\Sigma - (k-2)f)$

and use Schwarzenberger's argument ([Sc], [FrM1]), that is, we enlarge the divisor f by an effective divisor E on S so that φ has a torsion-free cokernel and $\mathscr{V}\otimes\mathcal{O}(E)$ has a section with a 0-dimensional subscheme Z of zeroes, then we see that $\mathscr{V}$ is given as an extension

$$0\longrightarrow\mathcal{O}(f+E)\xrightarrow{\hat{\varphi}}\mathscr{V}\xrightarrow{\hat{\psi}}\mathcal{O}(\Sigma-(k-1)f-E)\otimes\mathscr{I}_Z\longrightarrow 0\,. \tag{2.2}$$

By the L_{2k+1}-stability of $\mathscr{V}$, it satisfies that $(f+E)\cdot L_{2k+1}<(1/2)c_1(\mathscr{V})\cdot L_{2k+1}=3/2$, and so $E\cdot L_{2k+1}<1/2$. Since E is effective, $E\cdot L_{2k+1}=0$, and so $E=0$. Furthermore, the exact sequence (2.2) gives

$$\begin{aligned}ch(\mathscr{V})&=ch(\mathcal{O}(f))+ch(\mathcal{O}(\Sigma-(k-1)f)\otimes\mathscr{I}_Z)\\&=ch(\mathcal{O}(f))+ch(\mathcal{O}(\Sigma-(k-1)f))ch(\mathscr{I}_Z)\,,\end{aligned}$$

and so

$$\ell(Z)=c_2(\mathscr{I}_Z)=c_2(\mathscr{V})+f^2-f\cdot c_1(\mathscr{V})=0\,.$$

Hence, an L_{2k+1}-stable bundle $\mathscr{V}$ is described as

$$0\to\mathcal{O}(f)\to\mathscr{V}\to\mathcal{O}(\Sigma-(k-1)f)\to 0\,.$$

Case (B). $h^0(\mathscr{V}\otimes\mathcal{O}(-\Sigma+(2k-3)f))>0$. There is a sub-line bundle $\varphi:\mathcal{O}(\Sigma-(2k-3)f)\to\mathscr{V}$. In the same manner as Case (A), we have an exact sequence

$$0\to\mathcal{O}(\Sigma-(2k-3)f+E)\to\mathscr{V}\to\mathcal{O}((k-1)f-E)\otimes\mathscr{I}_Z\to 0\,, \tag{2.3}$$

where $\mathscr{I}_Z$ is the ideal sheaf of a 0-dimensional subscheme Z. Because of the L_{2k+1}-stability of $\mathscr{V}$, it must satisfy that $(\Sigma-(2k-3)f+E)\cdot L_{2k+1}<3/2$, and so we have

$$4-k+E\cdot L_{k+1}+kE\cdot f<3/2\,. \tag{2.4}$$

Since L_{k+1} is ample, we have $E\cdot L_{k+1}\geq 0$, and so $E\cdot f<1-5/(2k)$. This implies $E\cdot f\leq 0$. Since E is effective and $f\cdot f=0$, $Supp(E)$ consists of fibers of π. So by the irreducibility of fibers we can write $E=mf$ for some integer $m\geq 0$. Then the inequality (2.4) gives

$$4-k+m<3/2\,. \tag{2.5}$$

On the other hand, the length $\ell(Z)$ of Z in the exact sequence (2.3) is given by

$$\begin{aligned}\ell(Z)&=c_2(\mathscr{V})+(\Sigma-(2k-3)f+E)^2-(\Sigma-(2k-3)f+E)\cdot(\Sigma-(k-2)f)\\&=2-k+m\,.\end{aligned}$$

Thus, from the inequality (2.5) we obtain $\ell(Z)=2-k+m<-1/2<0$, inducing a contradiction. Therefore, Case (B) dose not occur for any integer $k\geq 1$. This completes the proof. q.e.d.

PROPOSITION 2.3. *There exist locally free extensions of the form*

$$0 \to \mathcal{O}(f) \to \mathscr{V} \to \mathcal{O}(\Sigma-(k-1)f) \to 0\,, \tag{2.1}$$

and the set of all isomorphism classes of such extensions consists of two points essentially.

PROOF. The set of all extensions of $\mathcal{O}(\Sigma-(k-1)f)$ by $\mathcal{O}(f)$ is classified by $Ext^1(\mathcal{O}(\Sigma-(k-1)f), \mathcal{O}(f))$ [GrH, p. 725]. From the Serre duality, we obtain a canonical isomorphism

$$Ext^1(\mathcal{O}(\Sigma-(k-1)f), \mathcal{O}(f)) \cong H^1(\mathcal{O}(-\Sigma+kf))\,.$$

For an ample divisor L_s, $(-\Sigma+kf)\cdot L_s=2k-s$. If $s>2k$, then $(-\Sigma+kf)\cdot L_s<0$ and $-\Sigma+kf$ is not effective. Hence $H^0(\mathcal{O}(-\Sigma+kf))=0$. Again, from the Serre duality we obtain an isomorphism $H^2(\mathcal{O}(-\Sigma+kf)) \cong H^0(\mathcal{O}(\Sigma-2f))$. Since $(\Sigma-2f)\cdot L_{k+1}=-1<0$ for the ample divisor L_{k+1}, $H^0(\mathcal{O}(\Sigma-2f))=0$. From the Riemann-Roch formula, we obtain

$$\begin{aligned}-h^1(\mathcal{O}(-\Sigma+kf)) &= h^0(\mathcal{O}(-\Sigma+kf))-h^1(\mathcal{O}(-\Sigma+kf))+h^2(\mathcal{O}(-\Sigma+kf))\\ &= \chi(\mathcal{O}(-\Sigma+kf))=-1\,.\end{aligned}$$

Hence $h^1(\mathcal{O}(-\Sigma+kf))=1$. Therefore, $Ext^1(\mathcal{O}(\Sigma-(k-1)f), \mathcal{O}(f)) \cong \boldsymbol{C}$ and a nonzero class of $Ext^1(\mathcal{O}(\Sigma-(k-1)f), \mathcal{O}(f))$ defines a locally free extension. Furthermore, the corresponding extension class is unique mod $\boldsymbol{C}^*$. q.e.d.

PROPOSITION 2.4. *Let $\mathscr{V}$ be a rank 2 vector bundle over S given by an exact sequence of type* (2.1). *Then one of the followings holds*:

(1) *$\mathscr{V}$ is L_{2k+1}-stable, or*

(2) *We have a canonical isomorphism $\mathscr{V} \cong \mathcal{O}(f) \oplus \mathcal{O}(\Sigma-(k-1)f)$ and $\mathscr{V}$ is not L_{2k+1}-semistable.*

PROOF. Note that the bundle $\mathscr{V}$ corresponds to either a nonzero class of $Ext^1(\mathcal{O}(\Sigma-(k-1)f), \mathcal{O}(f))$ or the zero class. We consider the case where $\mathscr{V}$ corresponds to a nonzero class, that is, the exact sequence giving $\mathscr{V}$,

$$0 \longrightarrow \mathcal{O}(f) \xrightarrow{\ \varphi\ } \mathscr{V} \xrightarrow{\ \psi\ } \mathcal{O}(\Sigma-(k-1)f) \longrightarrow 0\,,$$

does not split. Suppose that there is a destabilizing sub-line bundle $\mathscr{F} \to \mathscr{V}$ with $c_1(F)\cdot L_{2k+1} \geq (1/2)c_1(\mathscr{V})\cdot L_{2k+1}$. If we tensor the exact sequence (2.1) by the dual sheaf $\mathscr{F}^*$, then we obtain the exact sequence

$$0 \to \mathcal{O}(f)\otimes\mathscr{F}^* \to \mathscr{V}\otimes\mathscr{F}^* \to \mathcal{O}(\Sigma-(k-1)f)\otimes\mathscr{F}^* \to 0\,,$$

and this yields the exact sequence

$$0 \to H^0(\mathcal{O}(f)\otimes\mathscr{F}^*) \to H^0(\mathscr{V}\otimes\mathscr{F}^*) \to H^0(\mathcal{O}(\Sigma-(k-1)f)\otimes\mathscr{F}^*) \to \cdots\,.$$

Since $H^0(\mathscr{V}\otimes\mathscr{F}^*) \neq 0$, we have either (a) $H^0(\mathcal{O}(f)\otimes\mathscr{F}^*) \neq 0$ or (b) $H^0(\mathcal{O}(\Sigma-$

$(k-1)f)\otimes\mathscr{F}^*)\neq 0$. We consider the case (a). Then the line bundle $\mathcal{O}(f)\otimes\mathscr{F}^*$ is given by an effective divisor and so $(f-c_1(F))\cdot L_{2k+1}\geq 0$. Hence $1=f\cdot L_{2k+1}\geq c_1(F)\cdot L_{2k+1}\geq(1/2)c_1(\mathscr{V})\cdot L_{2k+1}=3/2$. It is impossible. Next we consider the case (b). There is a nonzero section σ of $\mathcal{O}(\Sigma-(k-1)f)\otimes\mathscr{F}^*$. If the zero set of σ, $\sigma^{-1}(0)$, is empty, then the section σ gives an isomorphism $\sigma_*: \mathcal{O}(\Sigma-(k-1)f)\to\mathscr{F}$. Composing σ_* with the destabilizing map, we have a nonzero map

$$\tau: \mathcal{O}(\Sigma-(k-1)f)\to\mathscr{V}.$$

Since $H^0(\mathscr{H}om(\mathcal{O}(\Sigma-(k-1)f), \mathcal{O}(\Sigma-(k-1)f)))\cong H^0(\mathcal{O})\cong \boldsymbol{C}$, the composite $\psi\circ\tau$ is αid for some $\alpha\in\boldsymbol{C}$. Then α is not 0. In fact, if $\alpha=0$, i.e., $\psi\circ\tau$ were the zero map, there would be $\rho: \mathcal{O}(\Sigma-(k-1)f)\to\mathcal{O}(f)$ so that $\varphi\circ\rho=\tau$. However, since $(kf-\Sigma)\cdot L_{2k+1}=-1<0$, we have $H^0(\mathscr{H}om(\mathcal{O}(\Sigma-(k-1)f), \mathcal{O}(f)))\cong H^0(\mathcal{O}(kf-\Sigma))=0$ and there is not such a map ρ. Thus we may assume that $\psi\circ\tau$ is the identity on $\mathcal{O}(\Sigma-(k-1)f)$ by normalizing it suitably, and so the exact sequence

$$0\longrightarrow\mathcal{O}(f)\xrightarrow{\varphi}\mathscr{V}\xrightarrow{\psi}\mathcal{O}(\Sigma-(k-1)f)\longrightarrow 0$$

must split, inducing a contradiction. If $\sigma^{-1}(0)$ is not empty, then the line bundle $\mathcal{O}(\Sigma-(k-1)f)\otimes\mathscr{F}^*$ is given by a nonzero effective divisor, and so we have $\{(\Sigma-(k-1)f)-c_1(F)\}\cdot L_{2k+1}>0$. Thus $3/2\leq c_1(F)\cdot L_{2k+1}<2$. But there is no integer $c_1(F)\cdot L_{2k+1}$ satisfying these inequalities, inducing a contradiction. Therefore, there is not a destabilizing sub-line bundle and $\mathscr{V}$ is L_{2k+1}-stable.

We consider the case where $\mathscr{V}$ corresponds to the zero class. Then $\mathscr{V}$ is isomorphic to $\mathcal{O}(f)\oplus\mathcal{O}(\Sigma-(k-1)f)$, and so we have the inclusion $\iota: \mathcal{O}(\Sigma-(k-1)f)\to\mathscr{V}$. Since $(\Sigma-(k-1)f)\cdot L_{2k+1}=2>3/2$, $\mathscr{V}$ is not L_{2k+1}-semistable. q.e.d.

REMARK. Let $\mathscr{F}\to\mathscr{U}$ be a sub-line bundle and L an ample divisor with odd $c_1(U)\cdot L$. If $c_1(F)\cdot L\leq(1/2)c_1(U)\cdot L$, then it must satisfy that $c_1(F)\cdot L<(1/2)c_1(U)\cdot L$. Hence, there is no L_{2k+1}-semistable but not L_{2k+1}-stable rank 2 bundle U with $c_1(U)=\Sigma-(k-2)f$.

The following is a consequence of Propositions 2.2, 2.3 and 2.4.

THEOREM 2.5. *Let $\mathscr{M}_{S_k}(\Sigma-K_{S_k}, 1; L_{2k+1})$ be the moduli space on a regular elliptic surface which has no multiple fiber and the Euler characteristic* $12k$. *Then for any integer $k\geq 1$, $\mathscr{M}_{S_k}(\Sigma-K_{S_k}, 1; L_{2k+1})$ consists of exactly one point.*

We now show the smoothness of the one point for $k=1, 2$, though we do not need it here.

DEFINITION 2.6. When E is a holomorphic vector bundle over an algebraic surface X, we let $\mathscr{E}nd\,\mathscr{E}$ be the sheaf of endomorphisms of E: $\mathscr{E}nd\,\mathscr{E}=\mathscr{E}\otimes\mathscr{E}^*$. Let $\mathscr{a}d\,\mathscr{E}$ be the sheaf defined by the exact sequence

$$0 \longrightarrow \mathcal{ad}\,\mathcal{E} \longrightarrow \mathcal{End}\,\mathcal{E} \xrightarrow{Tr} \mathcal{O}_X \longrightarrow 0\,,$$

where Tr is the naturally defined trace map. We call a bundle E good if $H^2(\mathcal{ad}\,\mathcal{E})=0$.

It is known that if a point e in a moduli space $\mathcal{M}$ is represented by a bundle E and E is good, then $\mathcal{M}$ is smooth and reduced at the point e.

PROPOSITION 2.7. *Let $k=1$ or 2. Then, the one point in $\mathcal{M}_{L_{2k+1}}(\Sigma-K_S, 1)$ is smooth.*

PROOF. First we consider the case of $k=1$. Then, we have $K_S=-f$ and $\Sigma-K_S-f=\Sigma$, and so the L_3-stable bundle V is given by an exact sequence

$$0\to\mathcal{O}_S(f)\to\mathcal{V}\to\mathcal{O}_S(\Sigma)\to 0\,.$$

Tensoring this sequence by $\mathcal{V}\otimes\mathcal{O}_S(-2f-\Sigma)$, we obtain the exact sequence

$$0\to\mathcal{V}\otimes\mathcal{O}_S(-f-\Sigma)\to\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{O}_S(-2f-\Sigma)\to\mathcal{V}\otimes\mathcal{O}_S(-2f)\to 0\,.$$

We consider the cohomology exact sequence given by this sequence,

$$0\to H^0(\mathcal{V}\otimes\mathcal{O}_S(-f-\Sigma))\to H^0(\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{O}_S(-2f-\Sigma))$$
$$\to H^0(\mathcal{V}\otimes\mathcal{O}_S(-2f))\to H^1(\mathcal{V}\otimes\mathcal{O}_S(-f-\Sigma))\to\cdots\,.$$

Suppose that there is a nonzero section $\mathcal{O}_S(f+\Sigma)\to\mathcal{V}$. Then we have that $(f+\Sigma)\cdot L_3=3>3/2=(1/2)c_1(V)\cdot L_3$, contradicting the L_3-stability of V. Thus, $H^0(\mathcal{V}\otimes\mathcal{O}_S(-f-\Sigma))=0$. In the same manner, the L_3-stability of V gives $H^0(\mathcal{V}\otimes\mathcal{O}_S(-2f))=0$. Hence, $H^2(\mathcal{End}\,\mathcal{V})\cong H^0(\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{O}_S(-2f-\Sigma))=0$, where the first isomorphism comes from the Serre duality. Noting that the irregularity q and the geometric genus p_g are 0, the exact sequence defining $\mathcal{ad}\,\mathcal{V}$ yields the isomorphism $H^2(\mathcal{ad}\,\mathcal{V})\cong H^2(\mathcal{End}\,\mathcal{V})$. Therefore, V is good.

Next we consider the case of $k=2$, i.e. a $K3$-surface. In general, for any vector bundle V over a $K3$-surface S, we have $H^2(\mathcal{ad}\,\mathcal{V})\cong H^0(\mathcal{ad}\,\mathcal{V})$ from the Serre duality, because $\mathcal{ad}\,\mathcal{V}$ is self-dual and $K_S=0$. It is known [OSS, p. 172] that $H^0(\mathcal{ad}\,\mathcal{V})=0$ if V is stable with respect to an ample divisor. Therefore if the moduli space $\mathcal{M}$ on a $K3$-surface with respect to an ample divisor is not empty, then every point in $\mathcal{M}$ is smooth. q.e.d.

3. An application to the differentiable topology of elliptic surfaces.

We first recall a simple invariant γ defined from the moduli spaces of anti-self-dual (ASD) connections on certain principal bundles by Donaldson [Do2], [DoK, Chapter 9]. Let X be an oriented closed smooth 4-manifold with the following properties:

A1. $\pi_1(X)$ is cyclic of odd order.

A2. $b^2_+(X)\geq 3$ and $b^2_+(X)$ is odd, where b_+^2 denotes the dimension of the maximal positive subspace for the form on H^2. Recall that there is a well defined lift of mod 2 cup square called the Pontrjagin square $H^2(X; \boldsymbol{Z}_2) \to H^4(X; \boldsymbol{Z}_4)$. Composing this map with evaluation on the fundamental class defines a quadratic map $\mathfrak{p}: H^2(X; \boldsymbol{Z}_2) \to \boldsymbol{Z}_4$.

Let $E \to X$ be a principal $SO(3)$-bundle with $w_2(E)=\eta$ and $p_1(E)=l$. These characteristic classes satisfy $w_2(E)^2 \equiv p_1(E) \pmod 4$. A theorem of Dold and Whitney [OW] tells us that $SO(3)$-bundles over a 4-manifold are classified completely by their second Stiefel-Whitney classes w_2 and first Pontrjagin classes p_1. For a Riemannian metric g on X, let $\mathscr{M}_X(\eta, l; g)$ denote the moduli space of ASD connections on E. Then the Atiyah-Singer index theorem [AHS] gives the formal dimension of $\mathscr{M}_X(\eta, l; g)$,

$$\dim \mathscr{M}_X(\eta, l; g) = -2l - 3(1 + b_+^2(X)) .$$

Throughout this section we choose l so that $\dim \mathscr{M}_X(\eta, l; g)=0$. Now we define $\mathscr{C}_X$ to be the set of $\eta \in H^2(X; \boldsymbol{Z}_2)$ satisfying the following conditions:

(1) $\mathfrak{p}(\eta) \equiv l \pmod 4$, and

(2) $\eta \neq 0$.

Let $\eta \in \mathscr{C}_X$. Then for a generic metric g on X we have the following properties ([DoK], [FrU]):

(1) The moduli space $\mathscr{M}_X(\eta, l; g)$ contains only irreducible connections.

(2) $\mathscr{M}_X(\eta, l; g)$ is a 0-dimensional smooth manifold and a finite set of points.

(3) Once an orientation of $H_+^2(X)$ and an integral lift c of η are chosen, they induce an orientation on $\mathscr{M}_X(\eta, l; g)$.

Thus by counting the signed number of points in moduli spaces, we define a function $\gamma_X : \mathscr{C}_X \to \boldsymbol{Z}$.

THEOREM I (Donaldson [Do2], [DoK]). *This function γ_X is independent of a choice of generic metrics, that is, γ_X satisfies the following: If $\phi: X \to X'$ is an orientation preserving diffeomorphism, then*

$$\gamma_X(\phi^*(\eta)) = \varepsilon(\phi)\gamma_{X'}(\eta) \qquad \textit{for any} \quad \eta \in \mathscr{C}_{X'} .$$

Here $\varepsilon(\phi)$ is -1 if either ϕ^* is an orientation reversing map from $H_+^2(X')$ onto $H_+^2(X)$ or $((c-\phi^*(c'))/2)^2 \equiv 1 \pmod 2$ and is 1 otherwise. (Here c' denotes the integral lift of η and c is the integral lift of $\phi^*(\eta)$ used in orienting their respective moduli spaces.)

We assume that X is a simply-connected complex algebraic surface and L an ample divisor on X. We take the Hodge metric g_L defined by L. Let $E \to X$ be an $SO(3)$-bundle with $w_2(E)$ nonzero. Then any integral lift c of $w_2(E)$ defines a lifting of E to a $U(2)$-bundle P with $c_1(P)$ and $c_2(P)$ related by

$$c_1(P) = c , \quad \text{and}$$

$$c_1(P)^2 - 4c_2(P) = p_1(E) .$$

Donaldson showed in [Do1] that the moduli space of irreducible Hermitian-Einstein

(HE) connections on P coincides with the moduli space of stable holomorphic rank 2 bundles $\mathscr{V}$ with $c_1(\mathscr{V})=c_1(P)$ and $c_2(\mathscr{V})=c_2(P)$. Since X is simply-connected, any ASD connection on E can be lifted uniquely to an HE connection on E [Kot]. Thus, ASD connections on E and stable holomorphic structures on P are related as follows:

THEOREM II (Donaldson). *There exists a natural one-to-one correspondence between the moduli space $\mathscr{M}_X^*(w_2(E), p_1(E); g_L)$ of irreducible ASD connections on E and the moduli space $\mathscr{M}_X(c_1(P), c_2(P); L)$ of L-stable rank 2 bundles over X. Furthermore, each reducible ASD connection on E corresponds to an L-semistable but not L-stable rank 2 bundle U over X with $c_1(U)=c_1(P)$ and $c_2(U)=c_2(P)$.*

Hence, if a metric g_L is generic, then in order to evaluate $\gamma_X(w_2(E))$ it is sufficient to determine $\mathscr{M}_X(c_1(P), c_2(P); g_L)$. Since moduli spaces of stable bundles are defined as complex spaces, they have natural orientations. This fact implies that the signs at different points agree and there is no cancellation. Therefore if g_L is generic, then

$$|\gamma_X(w_2(E))| = \#\mathscr{M}_X(c_1(P), c_2(P); L)\,,$$

where $\# A$ denotes the cardinal number of a set A. If g_L is not generic, then the moduli space $\mathscr{M}_X(c_1(P), c_2(P); L)$ is not reduced. However, if $\mathscr{M}_X(c_1(P), c_2(P); L)$ consists of finite points, then

$$|\gamma_X(w_2(E))| \geq \#\mathscr{M}_X(c_1(P), c_2(P); L)\,.$$

We can translate Theorem 2.5 in terms of ASD connections as follows:

COROLLARY 3.1. *Let $PD_2[\Sigma]$ denote the Poincaré dual of $[\Sigma]\in H_2(S_k; \boldsymbol{Z})$. Then, we have*

$$\begin{aligned} &\gamma_{S_k}(PD_2[\Sigma])\neq 0 && \text{if } k \text{ is even}\,,\\ &\gamma_{S_k}(PD_2([\Sigma]+[f]))\neq 0 && \text{if } k \text{ is odd}\,. \end{aligned}$$

REMARK. The first author proved that the value of $\gamma_{S_k}(PD_2[\Sigma])$ (k is even) and $\gamma_{S_k}(PD_2([\Sigma]+[f]))$ (k is odd) are both 1 differential-geometrically [Ka].

Now we discuss relevant aspects of the topology of elliptic surfaces. We deform the elliptic fibration S_k so that its singular fibers consists of exactly $6k$ cusp fibers. Then the section Σ deform to an embedded $\boldsymbol{CP}^1$, which we also denote by Σ. By $S_k(p, q)$ we denote the resultant 4-manifold obtained from an elliptic surface S_k by logarithmic transforms of multiplicities p and q. Note that $S_k(1, 1)$ is fiber preserving diffeomorphic to the original S_k. Since S_k has a section diffeomorphic to 2-sphere, the fundamental group $\pi_1(S_k(p, q))$ is isomorphic to the cyclic group of order $\gcd(p, q)$.

From now on, we assume $k\geq 2$. Then, up to fiber preserving diffeomorphism we can reinterpret this surface as a fiber sum [Mo] as follows: Given S_1 and S_{k-1}, identify a tubular neighborhood of a general fiber in each with $T^2\times D^2$ so that the fibrations correspond to projection onto D^2. Remove the interiors of tubular neighborhoods from

S_1 and S_{k-1}, and glue the two remaining manifolds together by the map $\varphi = id_{T^2} \times \rho$ on the boundary 3-torus $T^2 \times S^1$. Here, $\rho\colon S^1 \to S^1$ is a complex conjugation, and so we can give the resultant 4-manifold the orientation inherited from ones of S_1 and S_{k-1}. Then the resultant manifold is independent of our choices of identification of fibers with T^2, because of the monodromy associated to a cusp fiber in S_1 or S_{k-1}. The fibration fit together into a fibration $\pi\colon S_k \to CP^1$. In this section, however, we take a different construction of S_k as follows: As before, we start with S_1 and S_{k-1} and identify a tubular neighborhood of a general fiber in each with $T^2 \times D^2$. Then we choose our general fiber to be near a cusp fiber in each and identify it with $T^2 = S^1 \times S^1$ so that the vanishing cycles correspond to $S^1 \times \{1\}$ and $\{1\} \times S^1$. We remove each $T^2 \times intD^2$ and we let L_1 and L_{k-1} be the remaining manifolds. We write ∂L_1 as $S^1 \times S^1 \times S^1$ and we let σ be a cyclic permutation of the three factors. For the fiber preserving map φ, we compose φ with σ and glue the boundaries of L_1 and L_{k-1} by the composition $\varphi \circ \sigma$. We denote the resultant manifold by S_k^*. Then S_k^* is diffeomorphic to S_k. In fact, a tubular neighborhood of a cusp fiber F is obtained from a tubular neighborhood of a nearby general fiber by ambiently adding two 2-handles along two vanishing cycles with framing -1. Let $Q \subset L_1$ be a collar of ∂L_1. Let h_1 and h_2 be 2-handles in L_1 attached to the inner boundary of Q along the two vanishing cycles with framing -1. There is also a (-1)-framing 2-handle h_3 attached to Q along $\{1\} \times \{1\} \times S^1$, which is determined by the section Σ_1, restricted to L_1. Set $Q' = Q \cup h_1 \cup h_2 \cup h_3$. Then the given diffeomorphism σ of ∂L_1 extends over Q' by cyclically permuting the handles. The inner boundary of Q' is a Poincaré homology sphere, and so any self-diffeomorphism of this boundary is isotopic to the identity [BO]. Thus we can extend the given diffeomorphism of Q' over L_1 and we also denote it by σ. Now we have a diffeomorphism $\Phi_k = id_{L_{k-1}} \cup \sigma^{-1}\colon S_k \to S_k^*$.

We take two general fibers F_1 and F_2 in L_1 of S_k^* so that

$$\Phi_k(\Sigma) \cap F_1 = \varnothing \quad \text{and} \quad \Phi_k(\Sigma) \cap F_2 = \varnothing \,.$$

Logarithmic transforms of multiplicities p and q on a pair of F_1 and F_2 yield a new manifold $S_k^*(p, q)$ and multiple fibers f_p, f_q in $S_k^*(p, q)$ of multiplicities p and q. These multiple fibers satisfy

$$\Phi_k(\Sigma) \cap f_p = \varnothing \quad \text{and} \quad \Phi_k(\Sigma) \cap f_q = \varnothing \,. \tag{3.1}$$

Moreover, we take a general fiber f in L_{k-1} of S_k^*. Then

$$\Phi_k(\Sigma \cup f) \cap f_p = \varnothing \quad \text{and} \quad \Phi_k(\Sigma \cup f) \cap f_q = \varnothing \,. \tag{3.2}$$

Note that $S_k^*(1, 1)$ is diffeomorphic to S_k and $S_k^*(p, q)$ is homeomorphic to an elliptic surface $S_k(p, q)$.

REMARK. The twisted manifold $S_2^*(p, q)$ is diffeomorphic to $S_2(p, q)$ because a permutation σ on $S^1 \times S^1 \times S^1$ extends over another piece L_1 without multiple fibers.

Ue shows in [Ue] that if k is odd, then $S_k^*(p, q)$ is not always diffeomorphic to $S_k(p, q)$.

Now we use the results obtained by Gompf and Mrowka [GoM]. They defined the simple invariant over 4-manifolds with cylindrical end $T^3 \times [0, \infty)$ and obtained a gluing formula with the solid torus $T^2 \times D^2$.

PROPOSITION 3.2. *Put $Y^* = S_k^*(p, q) - \operatorname{int} N(f_p)$, where $N(f_p)$ is a tubular neighborhood of f_p. Let $j^*: H^2(S_k^*(p, q); \mathbf{Z}_2) \to H^2(Y^*; \mathbf{Z}_2)$ be a homomorphism induced from the inclusion $j: Y^* \to S_k^*(p, q)$. Let η be an element of $\mathscr{C}_{S_k^*(p,q)}$.*

(a) *If p is odd and $\langle \eta, f_p \rangle \neq 0$, then $|\gamma_{S_k^*(p,q)}(\eta)| = |\gamma_{S_k^*(1,q)}(\eta)|$.*

(b) *If $\langle \eta, f_p \rangle = 0$ and $j^*(\eta) \neq 0$, then $|\gamma_{S_k^*(p,q)}(\eta)| = p |\gamma_{S_k^*(1,q)}(\eta)|$.*

We can prove the above theorem in the same manner as [GoM, Theorem IV. 0.6]. We have only to notice that Proposition IV. 1.3 in [GoM] also remain true for $k \geq 3$.

Now we can find infinitely differentiable structures of S_k in $S_k^*(p, q)$, since for odd pair (p, q) with $\gcd(p, q) = 1$, all 4-manifolds $S_k^*(p, q)$ are homotopy equivalent to each other, and the set $\mathscr{C}_{S_k^*(p,q)}$ is finite.

In the following, we shall distinguish the diffeomorphism type $S_k^*(p, q)$ more precisely. Note that for $S_k^*(p, q)$ the set $\mathscr{C}_{S_k^*(p,q)}$ is defined by

$$\mathscr{C}_{S_k^*(p,q)} = \{\eta \in H^2(S_k^*(p, q); \mathbf{Z}_2) \mid \eta \neq 0,\ \mathfrak{p}(\eta) \equiv -3k \pmod 4\}\,.$$

If q is odd, then $PD_2 f_p$ is a nonzero element in $H^2(S_k^*(p, q); \mathbf{Z}_2)$. In addition, if $k \equiv 0 \pmod 4$, then $0 = PD_2 f_p \cdot PD_2 f_p \equiv -3k \pmod 4$ and so $PD_2 f_p \in \mathscr{C}_{S_k^*(p,q)}$. Then $\langle PD_2 f_p, f_p \rangle = 0$ but $j^*(PD_2 f_p) = 0$. Hence $j^*(PD_2 f_p) \notin \mathscr{C}_{Y^*}$ and we can not estimate $\gamma_{S_k^*(p,q)}(PD_2 f_p)$ by a value of $\gamma_{S_k^*(1,q)}$ from Proposition 3.2 (c). However we have the following, which is easily seen from the exact sequence

$$\cdots \longrightarrow H^2(S_k^*(p, q), Y^*; \mathbf{Z}_2) \longrightarrow H^2(S_k^*(p, q); \mathbf{Z}_2) \xrightarrow{j^*} H^2(Y^*; \mathbf{Z}_2) \longrightarrow \cdots\,.$$

LEMMA 3.3.

$$\operatorname{Ker} j^* = \begin{cases} \mathbf{Z}_2, \text{ which is generated by } PD_2 f_p\,, & \text{if } q \text{ is odd} \\ 0\,, & \text{if } q \text{ is even}\,. \end{cases}$$

If $k \not\equiv 0 \pmod 4$, then $PD_2 f_p \notin \mathscr{C}_{S_k^*(p,q)}$ and so when $k \not\equiv 0 \pmod 4$, it follows from Proposition 3.2 (c) that if $\eta \in \mathscr{C}_{S_k^*(p,q)}$ satisfies $\langle \eta, f_p \rangle = 0$ then $|\gamma_{S_k^*(p,q)}(\eta)| = p |\gamma_{S_k^*(1,q)}(\eta')|$ for some $\eta \in \mathscr{C}_{S_k^*(1,q)}$. From Proposition 3.2 and Lemma 3.3 we have the following.

THEOREM 3.4. *Let k be an integer with $k \not\equiv 0 \pmod 4$ and $k \geq 2$. Suppose that p, q and p', q' are odd. If $S_k^*(p, q)$ is diffeomorphic to $S_k^*(p', q')$, then there exist positive integers n, $n' \geq 1$ and nonnegative integers a, a', b and b' such that*

$$(pq)^n = (p'q')^{n'}\,.$$

In particular, if two products pq and $p'q'$ do not have a common prime factor, then $S_k^*(p, q)$ is not diffeomorphic to $S_k^*(p', q')$.

PROOF. Recall that the fundamental group of $S_k^*(p, q)$ is isomorphic to the cyclic group of order $\gcd(p, q)$. Let $\mathscr{D}_{S_k^*(p,q)}$ be the set of $\eta \in \mathscr{C}_{S_k^*(p,q)}$ so that $\gamma_{S_k^*(p,q)}(\eta) \neq 0$. Let $n(p, q)$ denote the number of $\eta \in \mathscr{D}_{S_k^*(p,q)}$ so that $\langle \eta, f_p \rangle = 0$ and $\langle \eta, f_q \rangle = 0$. By Theorem I, $\mathscr{D}_{S_k^*(p,q)}$ is a diffeomorphism invariant set. Moreover it is non-empty and $n(p, q)$ is non-zero, since, by Proposition 3.2,

$$|\gamma_{S_k^*(p,q)}(PD_2(\Phi_{k*}([\Sigma])))| = pq|\gamma_{S_k}(PD_2[\Sigma])| \neq 0 \qquad \text{if} \quad k \text{ is even},$$

$$|\gamma_{S_k^*(p,q)}(PD_2(\Phi_{k*}([\Sigma]+[f])))| = pq|\gamma_{S_k}(PD_2([\Sigma]+[f]))| \neq 0 \qquad \text{if} \quad k \text{ is odd}.$$

We put $\mathscr{D}_{S_k^*(p,q)} = \{\eta_1, \cdots, \eta_{l(p,q)}\}$ and $\mathscr{D}_{S_k^*(p',q')} = \{\eta'_1, \cdots, \eta'_{l(p',q')}\}$ for $S_k^*(p, q)$ and $S_k^*(p', q')$ respectively. Then from Proposition 3.2 we have

$$\begin{aligned} &|\gamma_{S_k^*(p,q)}(\eta_1) \cdots \gamma_{S_k^*(p,q)}(\eta_{l(p,q)})| \\ &\qquad = (pq)^{n(p,q)} |\gamma_{S_k^*(1,1)}(\eta_1) \cdots \gamma_{S_k^*(1,1)}(\eta_{l(p,q)})|, \\ &|\gamma_{S_k^*(p',q')}(\eta'_1) \cdots \gamma_{S_k^*(p',q')}(\eta'_{l(p',q')})| \\ &\qquad = (p'q')^{n(p',q')} |\gamma_{S_k^*(1,1)}(\eta'_1) \cdots \gamma_{S_k^*(1,1)}(\eta'_{l(p',q')})|. \end{aligned}$$

Assume that there is a diffeomorphism $\phi: S_k^*(p, q) \to S_k^*(p', q')$. (Since the signature of $S_k^*(p, q)$ is $-8k$ and nonzero, ϕ is orientation-preserving.) Then we have $l(p, q) = l(p', q')$ and we may write $\phi^*(\eta'_i) = \eta_{\sigma(i)}$ $(1 \leq i \leq l(p, q))$. Here σ is a permutation of $l(p, q)$ elements. Then from Theorem I we have

$$|\gamma_{S_k^*(p,q)}(\eta_1) \cdots \gamma_{S_k^*(p,q)}(\eta_{l(p,q)})| = |\gamma_{S_k^*(p',q')}(\eta'_1) \cdots \gamma_{S_k^*(p',q')}(\eta'_{l(p',q')})|.$$

This implies that $(pq)^{n(p,q)} = (p'q')^{n(p',q')}$. q.e.d.

Now we have the following as a consequence of Theorem 3.4.

COROLLARY 3.5. *Let k be an integer with $k \not\equiv 0$ (mod 4) and $k \geq 2$. For any odd integer $r \geq 1$, we can find infinitely many homeomorphic, but not diffeomorphic closed smooth 4-manifolds with π_1 isomorphic to the cyclic group of order r, $\boldsymbol{Z}_r$ concretely.*

PROOF. For example, we consider the family $\mathscr{A}$,

$$\mathscr{A} = \{S_k^*(1, p) \mid p \text{ is } 1 \text{ or any odd prime integer}\}.$$

Then it follows from [Fre] that all manifolds in $\mathscr{A}$ are homeomorphic to S_k. But, by Theorem 3.4, any two manifolds in $\mathscr{A}$ are not diffeomorphic.

For any odd integer $r \geq 1$, we consider the family $\mathscr{A}_r^*$,

$$\mathscr{A}_r^* = \{S_k^*(r, rp) \mid p \text{ is } 1 \text{ or any odd prime integer with } \gcd(r, p) = 1\}.$$

From [HK] it follows that all manifolds in $\mathscr{A}_r^*$ have π_1 isomorphic to $\boldsymbol{Z}_r$ and are

homeomorphic. By Theorem 3.4, however, any two manifolds in $\mathscr{A}_r^*$ are not diffeomorphic.
q.e.d.

REMARKS. (1) In the case $k=1$, the diffeomorphism types of Dolgachev surfaces, $S_1(p, q)$, are investigated by using the Donaldson's Γ-invariant ([FrM1], [LO], [OV]).

(2) In the case $k \geq 2$, Friedman and Morgan considered the diffeomorphism types of elliptic surfaces, $S_k(p, q)$, by using the Donaldson's polynomial invariant and stated that the product pq is a diffeomorphism invariant [FrM2].

Next we extend our result slightly. First we choose disjoint tubular neighborhoods of $(k-1)$ general fibers in S_1 and identify them with $\coprod_{i=1}^{k-1}(T^2 \times D^2)_i$ so that the fibration corresponds to projection onto D^2. Remove the interior of tubular neighborhoods from S_1 and denote the remaining manifold by L'_1. We write $\partial L'_1$ as $\coprod_{i=1}^{k-1}(S^1 \times S^1 \times S^1)_i$. Then to the boundary of L'_1 we attach the boundaries of $(k-1)$ copies of L_1 as above, $L_1^{(1)}, \cdots, L_1^{(k-1)}$ by the composition $\varphi \circ \sigma$ as before. We denote the resultant manifold by S_k^{**}, which is diffeomorphic to S_k. Now we take two general fibers $F_1^{(i)}$, $F_2^{(i)}$ in $L_1^{(i)}$ of S_k^{**} $(i=1, \cdots, k-1)$. Logarithmic transforms of multiplicities p_i and q_i on a pair of $F_1^{(i)}$ and $F_2^{(i)}$ for each i yield a new manifold $S_k^{**}(p_1, q_1; \cdots; p_{k-1}, q_{k-1})$. Note that $S_k^{**}(p, q; 1, 1; \cdots; 1, 1)$, $\cdots$, $S_k^{**}(1, 1; \cdots; 1, 1; p, q)$ are diffeomorphic to $S_k^*(p, q)$. It is easy to see that if each pair $\{p_j, q_j\}$ $(j=1, \cdots, i-1, i+1, \cdots, k-1)$ except for one pair $\{p_i, q_i\}$ is relatively prime, then the fundamental group of $S_k^{**}(p_1, q_1; \cdots; p_{k-1}, q_{k-1})$ is isomorphic to the cyclic group of order $\gcd(p_i, q_i)$. Then by noting (3.1) and (3.2), we can prove the following in the same manner as the proof of Theorem 3.4.

THEOREM 3.6. *Let k be an integer with $k \not\equiv 0$ (mod 4) and $k \geq 2$. Suppose that $\{p_l, q_l\}$ $(l=1, \cdots, i-1, i+1, \cdots, k-1)$ and $\{p'_l, q'_l\}$ $(l=1, \cdots, j-1, j+1, \cdots, k-1)$ are relatively prime. Suppose that p_i, q_i and p'_j, q'_j are odd. If $S_k^{**}(p_1, q_1; \cdots; p_{k-1}, q_{k-1})$ is diffeomorphic to $S_k^{**}(p'_1, q'_1; \cdots; p'_{k-1}, q'_{k-1})$, then there exist nonnegative integers n_l, n'_l, a_l, a'_l, b_l and b'_l $(l=1, \cdots, k-1)$ such that*

$$\prod_{l=1}^{k-1}(p_l q_l)^{n_l} = \prod_{l=1}^{k-1}(p'_l q'_l)^{n'_l}.$$

In particular, if two products $p_1 q_1 \cdots p_{k-1} q_{k-1}$ and $p'_1 q'_1 \cdots p'_{k-1} q'_{k-1}$ do not have a common prime factor, then $S_k^{**}(p_1, q_1; \cdots; p_{k-1}, q_{k-1})$ is not diffeomorphic to $S_k^{**}(p'_1, q'_1; \cdots; p'_{k-1}, q'_{k-1})$.

REMARK. Noting the number of hyperbolic summands of the intersection form, we can take maximally $2k-1$ (resp. $2(k-1)$) nuclei in S_k [Go] and insert $4k-2$ (resp. $4(k-1)$) multiple fibers into S_k if k is even (resp. odd) so that we can derive the same result as Theorem 3.6.

In the same manner, Theorem 3.4 and Theorem 3.6 hold for the resultant 4-manifolds obtained from $S_k^*(p, q)$ and $S_k^{**}(p_1, q_1; \cdots; p_{k-1}, q_{k-1})$ by connected sums with $\overline{CP^2}$s.

References

[AHS] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four dimensional Riemannian geometry, Proc. Royal Soc. London Ser. A **362** (1978), 425–461.

[BPV] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Springer (1984).

[BO] M. Boileau and J. Otal, Groupe des difféotopies de certaines variétés de Seifert, C. R. Acad. Sci. Paris Sér. I **303** (1986), 19–22.

[OW] A. Dold and H. Whitney, Classification of oriented sphere bundles over a 4-complex, Annals of Math. **69** (1959), 667–677.

[Do1] S. K. Donaldson, Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. **50** (1985), 1–26.

[Do2] S. K. Donaldson, Polynomial invariants for smooth four-manifolds, Topology **29** (1990), 257–315.

[DoK] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Math. Monographs, Oxford Univ. Press (1990).

[FrU] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds*, M.S.R.I. Publications, Springer (1984).

[Fre] M. H. Freedman, The topology of four-dimensional manifolds, J. Differential Geom. **17** (1982), 357–453.

[Fri] R. Friedman, Rank two vector bundles over regular elliptic surfaces, Invent. Math. **96** (1989), 283–332.

[FrM1] R. Friedman and J. W. Morgan, On the diffeomorphism types of certain algebraic surfaces. I, II, J. Differential Geom. **27** (1988), 297–398.

[FrM2] R. Friedman and J. W. Morgan, Complex versus differentiable classification of algebraic surfaces, Topology Appli. **32** (1989), 135–139.

[Go] R. E. Gompf, Nuclei of elliptic surfaces, Topology **30** (1991), 479–511.

[GoM] R. E. Gompf and T. S. Mrowka, Irreducible four manifolds need not be complex, preprint (1991).

[GrH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978).

[HK] I. Hambleton and M. Kreck, Smooth structures on algebraic surfaces with cyclic fundamental group, Invent. Math. **91** (1988), 53–59.

[Ka] Y. Kametani, Torus sum formula of simple invariants for 4-manifolds, Kodai Math. J. **16** (1993), 138–170.

[Kod] K. Kodaira, On compact analytic surfaces: II, Annals of Math. **77** (1963), 563–626.

[Kot] D. Kotschick, On manifolds homeomorphic to $CP^2\#8\overline{CP^2}$, Invent. Math. **95** (1989), 591–600.

[LO] M. Lübke and C. Okonek, Differentiable structures of elliptic surfaces with cyclic fundamental group, Compositio Math. **63** (1987), 217–222.

[Mo] B. Moishezon, *Complex Surfaces and Connected Sums of Complex Projective Planes*, Lecture Notes in Math. **603** (1977), Springer.

[OSS] C. Okonek, M. Schneider and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Progress in Math. **3** (1980), Birkhäuser.

[OV] C. Okonek and A. Van de Ven, Stable bundles and differentiable structures on certain elliptic surfaces, Invent. Math. **86** (1986), 357–370.

[Qi] Z. Qin, Chamber structures of algebraic surfaces with Kodaira dimension zero and moduli spaces of stable rank two bundles, Math. Z. **207** (1991), 121–136.

[Sc] R. L. E. Schwarzenberger, Vector bundles on algebraic surfaces, Proc. London Math. Soc. (3) **11** (1961), 601–622.

[Ue] M. Ue, A remark on the simple invariants for elliptic surfaces and their exotic structures not coming from complex surfaces, preprint (1991).

Present Addresses:
YUKIO KAMETANI
DEPARTMENT OF MATHEMATICS, Faculty of SCIENCE AND TECHNOLOGY, SAGA UNIVERSITY, SAGA, 840 JAPAN.

YOSHIHISA SATO
DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, YAMAGUCHI UNIVERSITY, YAMAGUCHI, 753 JAPAN.