TOKYO J. MATH.
VOL. 17, NO. 1, 1994

On Connectedness of the Space of Harmonic 2-Spheres in Quaternionic Projective Spaces

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(Communicated by T. Nagano)

Introduction.

The research on the spaces of harmonic maps of compact Riemann surfaces is one of important areas in harmonic map theory. Recently the connectedness of the spaces of harmonic 2-spheres in specific Reimannian manifolds has been investigated by several mathematicians. Kotani [Ko] showed that the space of harmonic 2-spheres in the n-dimensional standard sphere S^n with fixed energy is path-connected if $n \geq 3$. In case $n=4$, this result was previously proved by Loo [Lo] and Verdier [Ve]. Furthermore Guest and Ohnita [GO] investigated group actions on harmonic maps into symmetric spaces and used Morse-Bott theoretic deformations for harmonic maps to show some results on the connectedness of the space of harmonic 2-spheres in the unitary group, the sphere and the complex projective space. Moreover the fundamental group of the space of harmonic 2-spheres in the n-sphere was determined by [FGKO]. It is natural to study the connectedness of the space of harmonic 2-spheres in the quaternionic projective space.

Let $\boldsymbol{H}P^n$ be an n-dimensional quaternionic projective space. It is known that there are two natural twistor spaces $\mathscr{T}_n$ and $\boldsymbol{C}P^{2n+1}$ over $\boldsymbol{H}P^n$ (see Section 1). A harmonic map $\varphi: \Sigma \to \boldsymbol{H}P^n$ is *strongly isotropic* if and only if φ can be lifted to a horizontal holomorphic map into $\mathscr{T}_n$ (see [G1]). According to [BED-W], $\varphi: \Sigma \to \boldsymbol{H}P^n$ is called *a quaternionic mixed pair* if φ can be lifted to a horizontal holomorphic map into $\boldsymbol{C}P^{2n+1}$.

Denote by c the maximum of the sectional curvatures of $\boldsymbol{H}P^n$. Let $\varphi: \Sigma \to \boldsymbol{H}P^n(c)$ be a harmonic map of a compact Riemann surface. If φ is strongly isotropic or a quaternionic mixed pair, then φ has energy $4\pi d/c$, for some nonnegative integer d (see Section 2).

The purpose of this paper is to prove the following theorem, by virtue of the method of [GO] applied to the twistor spaces $\mathscr{T}_n$ and $\boldsymbol{C}P^{2n+1}$.

THEOREM A. *The space of harmonic 2-spheres in $\boldsymbol{H}P^n(c)$ with fixed energy $4\pi d/c$*

Received March 19, 1993

which are strongly isotropic or quaternionic mixed pairs is path-connected for all $n \geq 1$ *and* $d \in \boldsymbol{Z}$.

We can also restate it as Theorem B. Let $\varphi: \Sigma \to Gr_k(\boldsymbol{C}^N)$ be a harmonic map of a Riemann surface. In [BW], the *∂′-Gauss bundle* $G'(\varphi)$ of φ is defined as a complex subbundle of $\underline{\varphi}^{\perp}$ generated by the image of the ∂'-second fundamental form $A'_{\varphi} = \pi_{\varphi}^{\perp}\partial' : \underline{\varphi} \to \underline{\varphi}^{\perp}$. Then $G'(\varphi)$ corresponds to a harmonic map $\Sigma \to Gr_l(\boldsymbol{C}^N)$. φ is called *∂′-irreducible* if the rank of $G'(\varphi)$ is equal to the rank of $\underline{\varphi}$, and *∂′-reducible* otherwise. We say that φ has *infinite isotropy order* or is *strongly isotropic* if, for all $i \geq 1$, $G^{(i)}(\varphi)$ is orthogonal to $\underline{\varphi}$ with respect to the Hermitian inner product, and *finite isotropy order* otherwise.

We shall regard $\boldsymbol{H}P^n$ as a totally geodesic submanifold of $Gr_2(\boldsymbol{C}^{2(n+1)})$. According to the classification theory of [BED-W], there are four classes of harmonic 2-spheres in $\boldsymbol{H}P^n$ as follows; (I) strongly isotropic and ∂'-reducible, (II) strongly isotropic and ∂'-irreducible, (III) finite isotropy order and ∂'-reducible, (IV) finite isotropy order and ∂'-irreducible.

Bahy-El-Dien and Wood [BED-W] showed that if a harmonic map $\varphi: S^2 \to \boldsymbol{H}P^n$ is of class (III), then φ is a quaternionic mixed pair. If a harmonic map $\varphi: S^2 \to \boldsymbol{H}P^n$ is of class (IV), then φ can be lifted to a horizontal holomorphic map into neither $\mathscr{T}_n$ nor $\boldsymbol{C}P^{2n+1}$. However they showed that φ of class (IV) can be transformed to a map of class (III) after a finite number of forward and backward replacements.

Then Theorem A implies the following theorem.

THEOREM B. *The space of harmonic 2-spheres in* $\boldsymbol{H}P^n(c)$ *with fixed energy* $4\pi d/c$ *which are of class* (I), (II) *or* (III) *is path-connected for all* $n \geq 1$ *and* $d \in \boldsymbol{Z}$.

ACKNOWLEDGEMENTS. The author would like to thank Professors Y. Ohnita and K. Tsukada for valuable discussions and useful suggestion and Miss H. Sakagawa for constant encouragement.

1. Twistor spaces over $\boldsymbol{H}P^n$ and harmonic 2-spheres.

We denote by $Gr_2(\boldsymbol{C}^{2(n+1)})$ the complex Grassmann manifold of all complex 2-dimensional subspaces of $\boldsymbol{C}^{2(n+1)}$ with the standard Kähler structure. Let $\boldsymbol{H}P^n$ be the set of all quaternionic 1-dimensional subspaces of $\boldsymbol{H}^{n+1}$, namely, an n-dimensional quaternionic projective space. We define the conjugate linear map $J: \boldsymbol{C}^{2(n+1)} \to \boldsymbol{C}^{2(n+1)}$ as follows:

$$J(z_1, \cdots, z_{n+1}, z_{n+2}, \cdots, z_{2n+2}) = (-\bar{z}_{n+2}, \cdots, -\bar{z}_{2n+2}, \bar{z}_1, \cdots, \bar{z}_{n+1}) . \tag{1.1}$$

Since we have an identification $\boldsymbol{H}P^n = \{V \in Gr_2(\boldsymbol{C}^{2(n+1)}) \mid V = JV\}$, we can regard $\boldsymbol{H}P^n$ as a totally geodesic submanifold of $Gr_2(\boldsymbol{C}^{2(n+1)})$.

For $v = (v_1, \cdots, v_{2(n+1)})$, $w = (w_1, \cdots, w_{2(n+1)}) \in \boldsymbol{C}^{2(n+1)}$, we define the standard

Hermitian inner product $\langle\ ,\ \rangle$ as $\langle v, w\rangle=\sum_{i=1}^{2(n+1)} v_i\bar{w}_i$, where $\bar{}$ denotes complex conjugation. Set $(v, w)^a=\langle v, Jw\rangle$, then $(\ ,\ )^a$ is an antisymmetric bilinear form.

DEFINITION. We call a space E *J-isotropic* if $(v, w)^a=0$ for $v, w\in E$, namely, $E\perp JE$ relative to $\langle\ ,\ \rangle$.

Now we introduce two twistor spaces $\mathscr{T}_n$ and CP^{2n+1} over HP^n (cf. [G1], [Br2]). Let G and G^C denote the symplectic group and its complexification, namely,

$$(1.2)\qquad \begin{aligned} G&=Sp(n+1)\\ &=\{A\in GL(2n+2, C) \mid A^{-1}JA=J\}\,, \end{aligned}$$

$$(1.3)\qquad \begin{aligned} G^C&=Sp(n+1, C)\\ &=\{A\in GL(2n+2, C) \mid A^*JA=J\}\\ &=\{A\in GL(2n+2, C) \mid (Av, Aw)^a=(v, w)^a \text{ for } v, w\in C^{2(n+1)}\}\,. \end{aligned}$$

Note that $HP^n=Sp(n+1)/(Sp(1)\times Sp(n))$.

First, we define the twistor space $\mathscr{T}_n$ as

$$(1.4)\qquad \mathscr{T}_n=\{E\in Gr_n(C^{2(n+1)}) \mid E \text{ is } J\text{-isotropic}\}\,.$$

The group G acts transitively on $\mathscr{T}_n$, and then we have $\mathscr{T}_n=Sp(n+1)/(Sp(1)\times U(n))$. The complex dimension of $\mathscr{T}_n$ is $n(n+5)/2$. We define the projection $\pi_1: \mathscr{T}_n\ni E\mapsto (E\oplus JE)^\perp\in HP^n$. Here $\oplus$ denotes a Hermitian orthogonal direct sum with respect to $\langle\ ,\ \rangle$.

We define three tautological complex vector bundles $\mathscr{E}$, $J\mathscr{E}$ and $\mathscr{W}$ over $\mathscr{T}_n$ for $E\in\mathscr{T}_n$ as follows; $\mathscr{E}_E=E$, $(J\mathscr{E})_E=JE$ and $\mathscr{W}_E=W$, where $C^{2(n+1)}=E\oplus JE\oplus W$. From a natural inclusion map $\mathscr{T}_n\hookrightarrow Gr_n(C^{2(n+1)})$, we have the holomorphic tangent bundle of $\mathscr{T}_n$

$$(T\mathscr{T}_n)^{1,0}=\mathrm{Hom}(\mathscr{E}, J\mathscr{E})^a\oplus \mathrm{Hom}(\mathscr{E}, \mathscr{W})\,,$$

where $\mathrm{Hom}(\mathscr{E}, J\mathscr{E})^a=\{S\in \mathrm{Hom}(\mathscr{E}, J\mathscr{E}) \mid (Sv, v)^a=0 \text{ for } v\in\mathscr{E}\}$ corresponds to *the vertical subspaces* of π_1 and we remark that $\mathrm{Hom}(\mathscr{E}, \mathscr{W})$ corresponds to *the horizontal subspaces* of π_1. The map $f: \Sigma\to\mathscr{T}_n$ is said to be *a horizontal holomorphic map* if $df(T\Sigma^{1,0})\subset \mathrm{Hom}(\mathscr{E}, \mathscr{W})$. The holomorphicity and horizontality conditions are written respectively as

$$(1.5)\qquad \partial'' C^\infty(f^{-1}\mathscr{E})\subset C^\infty(f^{-1}\mathscr{E})\,,$$

$$(1.6)\qquad \partial' C^\infty(f^{-1}\mathscr{E})\perp C^\infty(f^{-1}(J\mathscr{E}))\,.$$

We know that if a map $\varphi: \Sigma\to HP^n$ is of the form $\varphi=\pi_1\circ f$, where $f: \Sigma\to\mathscr{T}_n$ is a horizontal holomorphic map, then φ is harmonic. The group $G^C=Sp(n+1, C)$ acts transitively on $\mathscr{T}_n$ in the natural way; for $A\in G^C$ and $E\in\mathscr{T}_n$, $A(E)\in\mathscr{T}_n$. Then the following facts hold.

LEMMA 1.1. (1) *This action of G^C on $\mathcal{T}_n$ preserves the complex structure of $\mathcal{T}_n$ and the horizontal subspaces with respect to $\pi_1 : \mathcal{T}_n \to HP^n$.*

(2) *Let $A \in G^C$ and $f : \Sigma \to \mathcal{T}_n$ be a horizontal holomorphic map. Then $A \circ f : \Sigma \to \mathcal{T}_n$ is a horizontal holomoprhic map.*

PROOF. (1) Let $A \in G^C$ and $T \in \mathrm{Hom}(\mathscr{E}, \mathscr{W})$. For any sections $s, s' \in C^\infty(\mathscr{E})$, then we have

$$\begin{aligned}\langle (AT)s, Js' \rangle &= ((AT)s, s')^a = (A(T(A^{-1}s)), s')^a \\ &= (T(A^{-1}s), A^{-1}s')^a = 0 ,\end{aligned}$$

because $T(A^{-1}s) \in \mathscr{W}$, and $A^{-1}s' \in \mathscr{E}$. Since $(AT)s \perp Js'$ and $Js' \in J\mathscr{E}$, we obtain $AT \in \mathrm{Hom}(\mathscr{E}, \mathscr{W})$.

(2) For any sections $s, s' \in C^\infty(f^{-1}\mathscr{E})$, $A(s)$ is a section of $(A \circ f)^{-1}\mathscr{E}$. Note that $(A \circ f)^{-1}\mathscr{E} = A(f^{-1}\mathscr{E})$. From (1.5) and (1.6), we have

$$\partial'' A(s) = A(\partial'' s) \in AC^\infty(f^{-1}\mathscr{E}) = C^\infty((A \circ f)^{-1}\mathscr{E}) .$$

Hence $A \circ f$ is holomorphic. Also we have

$$\begin{aligned}\langle \partial' A(s), JA(s') \rangle &= (\partial' A(s), A(s'))^a = (A(\partial' s), A(s'))^a \\ &= (\partial' s, s')^a = \langle \partial' s, Js' \rangle \\ &\in \langle \partial' C^\infty(f^{-1}\mathscr{E}), C^\infty(f^{-1}(J\mathscr{E})) \rangle = 0 .\end{aligned}$$

Hence $A \circ f$ is horizontal. □

Secondly let us consider another twistor space CP^{2n+1}. The space CP^{2n+1} is a set of complex 1-dimensional subspaces of $C^{2(n+1)}$. The group $G = Sp(n+1)$ acts transitively on CP^{2n+1}, and then we have $CP^{2n+1} = Sp(n+1)/(Sp(n) \times U(1))$. We define the projection $\pi_2 : CP^{2n+1} \ni L \mapsto L \oplus JL \in HP^n$. We define three tautological complex vector bundles over CP^{2n+1} for $L \in CP^{2n+1}$ as follows; $\mathscr{L}_L = L$, $(J\mathscr{L})_L = JL$ and $\mathscr{V}_L = V$, where $C^{2(n+1)} = L \oplus JL \oplus V$. The holomorphic tangent bundle of CP^{2n+1} is given by

$$(TCP^{2n+1})^{1,0} = \mathrm{Hom}(\mathscr{L}, J\mathscr{L}) \oplus \mathrm{Hom}(\mathscr{L}, \mathscr{V}) .$$

Here $\mathrm{Hom}(\mathscr{L}, J\mathscr{V})$ corresponds to *the vertical subspaces* of π_2 and $\mathrm{Hom}(\mathscr{L}, \mathscr{V})$ corresponds to *the horizontal subspaces* of π_2. The map $g : \Sigma \to CP^{2n+1}$ is said to be *a horizontal holomorphic map* if $dg(T\Sigma^{1,0}) \subset \mathrm{Hom}(\mathscr{L}, \mathscr{V})$. The holomorphicity and horizontality conditions are written respectively as

$$\partial'' C^\infty(g^{-1}\mathscr{L}) \subset C^\infty(g^{-1}\mathscr{L}) , \tag{1.7}$$

$$\partial' C^\infty(g^{-1}\mathscr{L}) \perp C^\infty(g^{-1}(J\mathscr{L})) . \tag{1.8}$$

The following facts hold.

LEMMA 1.2. (1) *This action of G^C on CP^{2n+1} preserves the complex structure of CP^{2n+1} and the horizontal subspaces with respect to $\pi_2: CP^{2n+1} \to HP^n$.*

(2) *Let $A \in G^C$ and $g: \Sigma \to CP^{2n+1}$ be a horizontal holomorphic map. Then $A \circ g: \Sigma \to CP^{2n+1}$ is a horizontal holomoprhic map.*

DEFINITION. A map $\varphi: \Sigma \to HP^n$ is *a quaternionic mixed pair* if and only if there exists a horizontal holomoprhic map $g: \Sigma \to CP^{2n+1}$ such that $\varphi = \pi_2 \circ g$.

Now we mention the relation between the classification of harmonic maps $\Sigma \to HP^n$ and the lift to the twistor space $\mathcal{T}_n$ over HP^n.

PROPOSITION 1.3 ([G1], [BED-W]). *Let $\varphi: \Sigma \to HP^n$ be a harmonic map. φ is strongly isotropic if and only if there exists a horizontal holomoprhic map $f: \Sigma \to \mathcal{T}_n$ such that $\varphi = \pi_1 \circ f$.*

REMARK. (1) $\varphi: S^2 \to CP^n \subset HP^n$ is a holomorphic map if and only if φ is both strongly isotropic and a quaternionic mixed pair, namely, φ can be lifted to horizontal holomorphic maps into both $\mathcal{T}_n$ and CP^{2n+1}.

(2) If $\varphi: S^2 \to HP^1$ is harmonic, then φ is strongly isotropic or a quaternionic mixed pair.

2. Energy of the harmonic 2-spheres into $HP^n(c)$.

It is known that an isotropic harmonic map of a compact Riemann surface into the n-dimensional unit standard sphere has the energy $4\pi d$ for a nonnegative integer d ([Ba]).

Let $HP^n(c)$ be the quaternionic projective space with the maximum c of the sectional curvatures. We now consider the energy of harmonic 2-spheres into $HP^n(c)$ which are strongly isotropic or quaternionic mixed pairs.

PROPOSITION 2.1. *Let $\varphi: \Sigma \to HP^n(c)$ be a harmonic map of a compact Riemann surface. If φ is strongly isotropic or a quaternionic mixed pair, then the energy $\mathscr{E}(\varphi)$ is $4\pi d/c$ for some nonnegagive integer d.*

PROOF. *Case* 1. Suppose that φ is strongly isotropic. There is a natural inclusion $\mathcal{T}_n \hookrightarrow Gr_n(C^{2(n+1)})$. We denote by $\mathcal{U}_n$ a tautological bundle over $Gr_n(C^{2(n+1)})$. Let g_{G_r} be the the Kähler metric induced from the standard Hermitian inner product $\langle\, , \rangle$ through $TGr_n(C^{2(n+1)})^{1,0} \cong \mathrm{Hom}(\mathcal{U}_n, \mathcal{U}_n^{\perp})$. Set $\omega_{G_r}(X, Y) = g_{G_r}(JX, Y)$. We note that the inclusion map $HP^1(c) \hookrightarrow HP^n(c)$ is totally geodesic and quaternionic. Then the Kähler metric g_{G_r} induces a Kähler metric on $\mathcal{T}_1$. We also denote by ω_{G_r} the Kähler form induced on $\mathcal{T}_1$.

We remark $\mathcal{T}_1 = Sp(2)/(Sp(1) \times U(1)) \cong CP^3$ and $Gr_1(C^4) = CP^3$, hence we have an identification $\mathcal{T}_1 \cong Gr_1(C^4)$. Let $\mathcal{U}_1$ be the tautological bundle over $\mathcal{T}_1$. It is known that the first Chern class of $\mathcal{U}_1$ is given by

(2.1)
$$c_1(\mathcal{U}_1)=\left[-\frac{1}{2\pi}\omega_{Gr}\right].$$

Suppose that the twistor fibration $\psi:\mathcal{T}_n\to \boldsymbol{H}P^n$ is a Riemannian submersion, and the maximum of sectional curvature of $\boldsymbol{H}P^n$ is c'. In particular, that of $\boldsymbol{H}P^1$ is the same value c' as the maximum of sectional curvature of $\boldsymbol{C}P^3$. Then it is known that the first Chern class of $\mathcal{U}_1$ is

(2.2)
$$c_1(\mathcal{U}_1)=\left[-\frac{c'}{4\pi}\omega_{c'}\right],$$

where we denote by $\omega_{c'}$ the Kähler form of constant holomorphic sectional curvature c'.

As ω_{Gr} is equal to $\omega_{c'}$ on $\mathcal{T}_1$ and both are the harmonic forms, we have

$$-\frac{1}{2\pi}\omega_{Gr}=-\frac{c'}{4\pi}\omega_{c'}.$$

Then we get $c'=2$.

Since $f:\Sigma\to\mathcal{T}_n$ is horizontal and holomorphic, we obtain the energy of φ,

(2.3)
$$\begin{aligned}\mathcal{E}(\varphi)=\mathcal{E}(f)&=\frac{2}{c}\int_\Sigma f^*\omega_{Gr}\\&=\frac{2}{c}(-2\pi(f^*c_1(\mathcal{U}_1)))(\Sigma)\\&=-\frac{4\pi}{c}c_1(f^{-1}\mathcal{U}_1)(\Sigma)\in\frac{4\pi}{c}\boldsymbol{Z}.\end{aligned}$$

Case 2. Suppose that φ is a quaternionic mixed pair. Assume that the twistor fibration $\psi':\boldsymbol{C}P^{2n+1}\to\boldsymbol{H}P^n$ is a Riemannian submersion, and the maximum of sectional curvature of $\boldsymbol{C}P^{2n+1}$ is c''. Let $\mathcal{U}$ be a tautological bundle over $\boldsymbol{C}P^{2n+1}$ $(=Gr_1(\boldsymbol{C}^{2(n+1)}))$. Then we have

$$c_1(\mathcal{U})=\left[-\frac{1}{2\pi}\omega_{Gr}\right]=\left[-\frac{c''}{4\pi}\omega_{c''}\right].$$

Hence we get $c''=2$. Similarly we obtain the energy of φ,

(2.4)
$$\mathcal{E}(\varphi)\in\frac{4\pi}{c}\boldsymbol{Z}.$$
□

3. Deformations of strongly isotropic harmonic maps into $\boldsymbol{H}P^n$.

(A) Morse-Bott theory over twistor space $\mathcal{T}_n$. Let G and $\mathfrak{g}$ denote the symplectic group and its Lie algebra. Then we can regard $\mathscr{T}_n$ as an orbit of the adjoint representation of G as follows: If we let E_0 a fixed element of $\mathscr{T}_n$ and set $\xi=\sqrt{-1}\pi_{E_0}-\sqrt{-1}\pi_{JE_0}$, then we have $\mathscr{T}_n\cong Ad(G)\xi$. Here π_{E_0} denotes the Hermitian projection in $C^{2(n+1)}$ onto E_0.

Fix an element $L\in CP^{2n+1}$ and put $P=\sqrt{-1}\pi_L-\sqrt{-1}\pi_{JL}\in\mathfrak{g}$. For $X=\sqrt{-1}\pi_E-\sqrt{-1}\pi_{JE}\in Ad(G)\xi$, where $E\in\mathscr{T}_n$, we define the height function $h^P\colon Ad(G)\xi\to\boldsymbol{R}$ by

$$h^P(X)=(X,P)\,. \tag{3.1}$$

Here (,) is an $Ad(G)$-invariant inner product on $\mathfrak{g}$. Then it is known that h^P is a Morse-Bott function. Let gradh^P be a gradient of h^P with respect to the Kähler metric. The following fact is due to Frankel; the flow of $-(\mathrm{grad}\,h^P)$ is given by the action of $\{\exp\sqrt{-1}tP\}$.

We shall describe non-degenerate critical manifolds of h^P. It is known that a point $X\in Ad(G)\xi$ is a critical point of h^P if and only if $[X,P]=0$, i.e.

$$[\sqrt{-1}\pi_E-\sqrt{-1}\pi_{JE},\sqrt{-1}\pi_L-\sqrt{-1}\pi_{JL}]=0\,. \tag{3.2}$$

Then a critical point X of h^P is characterized by $E=E_1\oplus E_2\oplus E_3$ with $E_1\subseteq L$, $E_2\subseteq JL$, $E_3\subseteq(L\oplus JL)^\perp$, where $C^{2(n+1)}=L\oplus JL\oplus(L\oplus JL)^\perp$. We obtain the following lemma.

LEMMA 3.1. *There are three connected non-degenerate critical manifolds of* h^P;

$$\mathscr{T}_+=\{E\in\mathscr{T}_n\mid L\subset E\}\cong\mathscr{T}_{n-1}\,,$$

$$\mathscr{T}_0=\{E\in\mathscr{T}_n\mid E\subset(L\oplus JL)^\perp\}\cong Sp(n)/U(n)\,,$$

$$\mathscr{T}_-=\{E\in\mathscr{T}_n\mid JL\subset E\}\cong\mathscr{T}_{n-1}\,.$$

PROOF. *Case* $E_1\neq\{0\}$. Then $E_1=L$ and we have $E_2=\{0\}$. Hence we get the critical manifold $\mathscr{T}_+$ of h^P. It is easy to show that $\mathscr{T}_+\cong\mathscr{T}_{n-1}$.

Case $E_1=\{0\}$. Then $E=E_2\oplus E_3$. If we let $E_2=\{0\}$, then we get the critical manifold $\mathscr{T}_0$. We show that $\mathscr{T}_0\cong Sp(n)/U(n)=\{V\subset C^{2n}\mid V\oplus JV=C^{2n}\}$. For $E\in\mathscr{T}_0$, E is an n-dimensional J-isotropic subspace of $(L\oplus JL)^\perp$ whose dimension is $2n$. Hence we have the above identification. Now if we let $E_2\neq\{0\}$, then we get the critical manifold $\mathscr{T}_-\cong\mathscr{T}_{n-1}$ in the same way as $\mathscr{T}_+$. □

We set $G_P=\{A\in G^C\mid A(L)=L\}$. In general, we know that the stable manifold for a non-degenerate critical manifold N is given by $S^P(N)=G_PX$ for $X\in N$. In our case we shall determine the corresponding stable manifolds.

LEMMA 3.2. *For three non-degenerate critical manifolds in Lemma* 3.1, *the corresponding stable manifolds* $S^P(\mathscr{T}_+)$, $S^P(\mathscr{T}_0)$, $S^P(\mathscr{T}_-)$ *are*;

$$S_+=\mathscr{T}_+\,,$$

$$S_0=\{E\in\mathscr{T}_n\mid E\cap L=\{0\},\,E\subset(JL)^\perp\}\,,$$

$$S_- = \{E \in \mathscr{T}_n \mid E \cap L = \{0\}, E \not\subset (JL)^\perp\},$$

respectively.

PROOF. It is clear that $S^P(\mathscr{T}_+)$ coincides with S_+. For $A \in G_P$, we have $\langle E, JL \rangle = (E, L)^a = (A(E), A(L))^a = (A(E), L)^a = \langle A(E), JL \rangle$. Thus we get $A(E) \subset (JL)^\perp$ (respectively, $A(E) \not\subset (JL)^\perp$), because $E \perp JL$ (respectively, $E \not\subset (JL)^\perp$). On the other hand, since $E \perp L$, we have $A(E) \cap L = \{0\}$. Then we have $S^P(\mathscr{T}_0) \subset S_0$ (respectively, $S^P(\mathscr{T}_-) \subset S_-$). Since

$$\begin{aligned} \mathscr{T}_n &= S^P(\mathscr{T}_+) \amalg S^P(\mathscr{T}_0) \amalg S^P(\mathscr{T}_-) \\ &= S_+ \amalg S_0 \amalg S_- \end{aligned}$$

are two decompositions of $\mathscr{T}_n$, we obtain $S^P(\mathscr{T}_+) = S_+$, $S^P(\mathscr{T}_0) = S_0$, $S^P(\mathscr{T}_-) = S_-$. □

REMARK. For $S^P(\mathscr{T}_-)$, if $E \not\subset (JL)^\perp$, then $E \cap L = \{0\}$.

(B) Deformations of harmonic maps. Let $\varphi: \Sigma \to HP^n$ be a strongly isotropic harmonic map, and $f: \Sigma \to \mathscr{T}_n$ be a horizontal holomorphic map corresponding to φ. If $f(\Sigma) \subset S^P(\mathscr{T}_-)$, then $\{(\exp\sqrt{-1}tP) \circ f\}_{0 \le t \le \infty}$ provides a continuous deformation to a horizontal holomorphic map into $\mathscr{T}_-$. We shall show that there exists some $L \in CP^{2n+1}$ such that $f(\Sigma) \subset S^P(\mathscr{T}_-)$.

We set $\mathscr{Y}^f = \{L' \in CP^{2n+1} \mid f(z) \not\subset S^P(\mathscr{T}_-)$ for some $z \in \Sigma\}$. Then we have $\mathscr{Y}^f = \{L' \in CP^{2n+1} \mid JL' \perp f(z)$ for some $z \in \Sigma\}$. It suffices to show that $\mathscr{Y}^f$ cannot be equal to CP^{2n+1}.

We define $\mathscr{Y} = \{(L', E) \in CP^{2n+1} \times \mathscr{T}_n \mid JL' \perp E\}$. Let p_1 and p_2 be the projections to CP^{2n+1} and $\mathscr{T}_n$, respectively. Then we get $\mathscr{Y}^f = p_1(p_2^{-1}(f(\Sigma)))$. Since the fibre of p_2 is CP^{n+1}, we hav

$$\begin{aligned} \text{(3.3)} \qquad \dim_C \mathscr{Y}^f &\le \dim_C p_2^{-1} f(\Sigma) \le \dim_C CP^{n+1} + \dim_C f(\Sigma) \\ &\le (n+1) + 1 = n+2 . \end{aligned}$$

From (3.3), it follows that, if $n \ge 2$, the space $\mathscr{Y}^f$ cannot be equal to CP^{2n+1}. It suffices to choose $L \in CP^{2n+1} \setminus \mathscr{Y}^f$. Thus we obtain the following proposition.

PROPOSITION 3.3. *If $n \ge 2$, then any horizontal holomorphic map into $\mathscr{T}_n$ can be deformed continuously through horizontal holomorphic maps to a horizontal holomorphic map into $\mathscr{T}_{n-1}$.*

Hence we obtain the following statement for harmonic maps.

THEOREM 3.4. *Let $\varphi: \Sigma \to HP^n$ be any strongly isotropic harmonic map. Then φ can be deformed continuously to a strongly isotropic harmonic map $\Sigma \to HP^1$.*

PROOF. From Proposition 3.3, if $n \geq 2$, we see that any strongly isotropic harmonic map $\Sigma \to HP^n$ can be deformed continuously to a strongly isotropic harmonic map $\Sigma \to HP^{n-1}$. By induction on dimension n, we obtain the theorem. □

4. Deformations of harmonic maps of quaternionic mixed pairs.

(A) Morse-Bott theory over twistor space CP^{2n+1}. Let G and $\mathfrak{g}$ be as in Section 3. We can regard CP^{2n+1} as an orbit of the adjoint representation of G as follows: If we let L_0 a fixed element of CP^{2n+1} and set $\eta = \sqrt{-1}\pi_{L_0} - \sqrt{-1}\pi_{JL_0}$, then we have $CP^{2n+1} \cong Ad(G)\eta$. Here π_{L_0} denotes the Hermitian projection in $C^{2(n+1)}$ onto L_0.

Fix $E \in \mathscr{T}_n$ and put $Q = \sqrt{-1}\pi_E - \sqrt{-1}\pi_{JE} \in \mathfrak{g}$. For each $X = \sqrt{-1}\pi_L - \sqrt{-1}\pi_{JL} \in Ad(G)\eta$ with $L \in CP^{2n+1}$, we consider the height function $h^Q(X) = (X, Q)$ on CP^{2n+1}. Then h^Q is a Morse-Bott function. Comparing the case of $\mathscr{T}_n$ in Section 3, we want to remark that if we choose $L \in CP^{2n+1}$ instead of E as a fixed element, then h^Q admits isolated critical points and our argument does not work at all.

We shall describe non-degenerate critical manifolds of h^Q. A point $X \in Ad(G)\eta$ is a critical point of h^Q if and only if $[X, Q] = 0$. Then a critical point X of h^Q is characterized by $L = L_1 \oplus L_2 \oplus L_3$ with $L_1 \subseteq E$, $L_2 \subseteq JE$, $L_3 \subseteq (E \oplus JE)^\perp$, where $C^{2(n+1)} = E \oplus JE \oplus (E \oplus JE)^\perp$. By the argument similar to Lemma 3.1, we obtain the following lemma.

LEMMA 4.1. *There are three connected non-degenerate critical manifolds of* h^Q;

$$CP_+ = \{L \in CP^{2n+1} \mid L \subset E\} \cong CP^{n-1},$$
$$CP_0 = \{L \in CP^{2n+1} \mid L \subset (E \oplus JE)^\perp\} \cong CP^1,$$
$$CP_- = \{L \in CP^{2n+1} \mid L \subset JE\} \cong CP^{n-1}.$$

REMARK. (1) The twistor fibration $\pi_2 : CP^{2n+1} \to HP^n$ induces a biholomorphic automorphism $CP_- \to CP^{n-1} \subset HP^{n-1}$.

(2) Any holomorphic map $\Sigma \to CP_-$ is always horizontal.

We set $G_Q = \{A \in G^C \mid A(E) = E\}$. We determine the corresponding stable manifolds by the same way as Lemma 3.2.

LEMMA 4.2. *For three critical manifolds in Lemma* 4.1, *the corresponding stable manifolds are*;

$$S^Q(CP_+) = CP_+,$$
$$S^Q(CP_0) = \{L \in CP^{2n+1} \mid L \cap E = \{0\}, L \subset (JE)^\perp\},$$
$$S^Q(CP_-) = \{L \in CP^{2n+1} \mid L \cap E = \{0\}, L \not\subset (JE)^\perp\}.$$

REMARK. For $S^Q(CP_-)$, if $L \not\subset (JE)^\perp$, then $L \cap E = \{0\}$.

(B) Deformations of harmonic maps. Let $\varphi: \Sigma \to HP^n$ be a quaternionic mixed pair, and $g: \Sigma \to CP^{2n+1}$ be a horizontal holomorphic map corresponding to φ. If $g(\Sigma) \subset S^Q(CP_-)$, then $\{(\exp\sqrt{-1}tQ) \circ g\}_{0 \le t \le \infty}$ provides a continuous deformation to a horizontal holomorphic map into CP_-. We shall show that such E exists.

We set $\mathscr{Y}^g = \{E' \in \mathscr{T}_n \mid g(z) \not\subset S^Q(CP_-)$ for some $z \in \Sigma\}$. Then we have $\mathscr{Y}^g = \{E' \in \mathscr{T}_n \mid g(z) \perp JE'\}$.

We define $\mathscr{Y} = \{(E', L) \in \mathscr{T}_n \times CP^{2n+1} \mid L \perp JE'\}$. Let p_1 and p_2 be the projections to $\mathscr{T}_n$ and CP^{2n+1}, respectively. Then we get $\mathscr{Y}^g = p_1(p_2^{-1}(g(\Sigma)))$.

It is not easy to find the fibre $p_2^{-1}(L) = \{E' \in \mathscr{T}_n \mid L \perp JE'\}$. We shall consider two cases; $E' \subset L^\perp$ or $E' \not\subset L^\perp$. Hence we get $p_2^{-1}(L) = \mathscr{F}_1 \amalg \mathscr{F}_2$, where

$$\mathscr{F}_1 = \{E' \in \mathscr{T}_n \mid E' \perp JL, E' \subset L^\perp\} \quad \text{and} \quad \mathscr{F}_2 = \{E' \in \mathscr{T}_n \mid E' \perp JL, E' \not\subset L^\perp\}.$$

First we deal with $\mathscr{F}_1$. Since $E' \perp L$ and $E' \perp JL$, we have $E' \subset (L \oplus JL)^\perp$. Hence we obtain the following lemma.

LEMMA 4.3. *The space $\mathscr{F}_1$ is diffeomorphic to $Sp(n)/U(n)$. In particular, $\dim_C Sp(n)/U(n)$ is equal to $n(n+1)/2$.*

Next we consider $\mathscr{F}_2$. So we see $E' \subset L \oplus (L \oplus JL)^\perp$. Here, for a subspace E of F, we denote by $F \ominus E = F \cap E^\perp$.

LEMMA 4.4. *The space $\mathscr{F}_2$ is diffeomorphic to a vector bundle*

$$\mathscr{S} = \coprod_{E'' \in \mathscr{T}'_{n-1}} \mathrm{Hom}(L, (L \oplus JL)^\perp \ominus (E'' \oplus JE'')) \tag{4.1}$$

over $\mathscr{T}'_{n-1}$ with the fibre $\mathrm{Hom}(L, (L \oplus JL)^\perp \ominus (E'' \oplus JE''))$, *where $\mathscr{T}'_{n-1} = \{E'' \mid (n-1)$-dimensional J-isotropic subspace of $(L \oplus JL)^\perp\}$.*

REMARK. In particular, $\dim_C \mathscr{S} = (n-1)(n+4)/2 + 2$.

PROOF. Let μ and ν be the Hermitian orthogonal projections from E' to L and $(L \oplus JL)^\perp$, respectively. Then we can characterize E' using μ as $E' = (E' \ominus \operatorname{Ker}\mu) \oplus \operatorname{Ker}\mu$. We note that $\dim_C E' \ominus \operatorname{Ker}\mu = 1$ and $\dim_C \operatorname{Ker}\mu = n-1$.

The space $\operatorname{Ker}\mu$ is J-isotropic. Indeed, since $\operatorname{Ker}\mu \subset \nu(E')$, it suffices to show that $\nu(E')$ is a J-isotropic subspace of $(L \oplus JL)^\perp$. For any $v = \mu(v) + \nu(v)$, $w = \mu(w) + \nu(w) \in E'$, we have $0 = \langle v, Jw \rangle = \langle \nu(v), J\nu(w) \rangle$. It follows that $\operatorname{Ker}\mu$ is an $(n-1)$-dimensional J-isotropic subspace of $(L \oplus JL)^\perp$, that is, an element of $\mathscr{T}'_{n-1}$ which is diffeomorphic to $\mathscr{T}_{n-1}$.

We put $W = (L \oplus JL)^\perp \ominus (\operatorname{Ker}\mu \oplus J\operatorname{Ker}\mu)$, then W is 2-dimensional subspace of $(L \oplus JL)^\perp$. Let $V = E' \ominus \operatorname{Ker}\mu$. Thus it suffices to examine V which is a line of $W \oplus L$ such that $V \not\subset L^\perp$.

We see that for all $x \in L$, there is unique $y \in W$ satisfying $x + y \in V$. Indeed, using a linear isomorphism $\mu: V \to L$, since we can write $z = x + \nu(z) \in V$ for any $x \in L$, then we take $\nu(z) = y$. Then we have a linear map $\delta_V: L \to W$ defined by $\delta_V(x) = y$ for $x \in V$.

Hence we obtain a smooth map $\mathscr{F}_2 \ni E' \to (\operatorname{Ker}\mu, \delta_V) \in \mathscr{S}$.

Now let us examine its inverse map. For any $E'' \in \mathscr{T}'_{n-1}$ and $\delta \in \operatorname{Hom}(L, (L \oplus JL)^{\perp} \ominus (E'' \oplus JE''))$, we put $E' = V \oplus E''$, where $V = \{x + \delta(x) \mid x \in L\}$. Then we see that $E' \in \mathscr{F}_2$. Indeed, it is clear that $E' \perp JL$ and $E' \not\subset L^{\perp}$. We show that E' is J-isotropic. It is enough to show $V \perp JE''$. For any $z = x + \delta(x) \in V$ and $s \in JE''$, we get $\langle z, s \rangle = \langle x, s \rangle + \langle \delta(x), s \rangle = 0$, because $L \perp JE''$ and $\{(L \oplus JL)^{\perp} \ominus (E'' \oplus JE'')\} \perp JE''$.

Hence we obtain the inverse map $\mathscr{S} \ni (E'', \delta) \to E' \in \mathscr{F}_2$.

Thus we obtain a diffeomorphism $\mathscr{F}_2 \ni E' \leftrightarrow (E'', \delta) \in \mathscr{S}$. We note that $\operatorname{Hom}(L, (L \oplus JL)^{\perp} \ominus (E'' \oplus JE'')) \cong \operatorname{Hom}(\boldsymbol{C}, \boldsymbol{C}^2) \cong \boldsymbol{C}^2$. □

It is sufficient to estimate the dimension of the fibre of p_2 from above by the larger dimension of $\mathscr{F}_1$ and $\mathscr{F}_2$. Hence we have

(4.2) $$\dim_{\boldsymbol{C}} \mathscr{Y}^g \leq (\tfrac{1}{2}(n-1)(n+4) + 2\} + 1 = \tfrac{1}{2}(n+1)(n+2)\,.$$

From (4.5), it follows that, if $n \geq 2$, the space $\mathscr{Y}^g$ cannot be equal to $\mathscr{T}_n$. Then we can choose $E \in \mathscr{T}_n \setminus \mathscr{Y}^g$. Thus we obtain the following proposition.

PROPOSITION 4.5. *If $n \geq 2$, then any horizontal holomorphic map into $\boldsymbol{C}P^{2n+1}$ can be deformed continuously through horizontal holomorphic maps to a (horizontal) holomorphic map into $\boldsymbol{C}P^{n-1} \subset \boldsymbol{C}P^{2n-1}$.*

Using Proposition 4.5 recursively, we obtain the following statement for harmonic maps.

THEOREM 4.6. *Any quaternionic mixed pair $\varphi: \Sigma \to \boldsymbol{H}P^n$ can be deformed continuously to a holomoprhic map $\Sigma \to \boldsymbol{C}P^1 \subset \boldsymbol{H}P^1$.*

5. Conclusion.

From Theorems 3.4 and 4.6, we obtain the next theorem.

THEOREM 5.1. *Let $\varphi: \Sigma \to \boldsymbol{H}P^n$ be a harmonic map. If φ is strongly isotropic or a quaternionic mixed pair, then φ can be deformed continuously through harmonic maps to a harmonic map $\Sigma \to \boldsymbol{H}P^1 = S^4$.*

Now we let $\Sigma = S^2$. Loo, Verdier in case $n = 4$, and Kotani in case $n \geq 3$, showed the following.

THEOREM 5.2. *If $n \geq 3$, the space of harmonic 2-spheres in the unit n-spheres with fixed energy is path-connected.*

Combining Theorems 5.1 and 5.2, we complete the proof of Theorem A.

It is very interesting to investigate the deformations of harmonic 2-spheres in $\boldsymbol{H}P^n$ of class (IV) and to determine the connectedness problem of the space of *all* harmonic 2-spheres in $\boldsymbol{H}P^n$.

Note added in proof, May 1994. Recently the author obtained similar results on the connectedness of spaces of harmonic 2-spheres in a real Grassmann manifold of 2-planes $Gr_2(\boldsymbol{R}^{n+2})$, an n-dimensional complex hyperquadric $Q_n(\boldsymbol{C})$ and more generally classical Riemannian symmetric spaces of inner type.

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