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Rauzy's Conjecture on Billiards in the Cube

Pierre ARNOUX, Christian MAUDUIT, Iekata SHIOKAWA
and Jun-ichi TAMURA

*Université Paris 7, Université de Lyon 1,
Keio University and International Junior College*

§1. Introduction.

We consider the billiards in the cube I^3 with $I=[0, 1]$. Let a particle start at a point $Q\in\bigcup_{i=1}^{3}F_i$ with constant velocity along a vector $v=(\alpha_1, \alpha_2, \alpha_3)$ and reflect at each face specularly, where $F_i:=\{(x_1, x_2, x_3)\mid x_i=0,\ 0\le x_j<1\ \ (j\neq i)\}$ $(i=1, 2, 3)$. Throughout this paper, we assume that

i) $\alpha_1, \alpha_2, \alpha_3>0$ are linearly independent over the field of rationals and

ii) the (forward) path of the particle never touches the edges of the cube.

If we label the two faces perpendicular to the x_i-ax as i and write down the label of the faces which the particle hits in order of collision, we have an infinite sequence $w(v, Q)$ of 1, 2, and 3. The complexity of an infinite sequence $w\in\{1, 2, 3\}^{N}$ is the function $p(n; w)$ defined as the number of distinct blocks $\in\{1, 2, 3\}^n$ appearing in w. In particular, we put $p(n; v, Q)=p(n; w(v, Q))$. Then the authors proved in [1] the following theorem conjectured by G. Rauzy [2–3] in 1981.

THEOREM. *Let v and Q satisfy the conditions* i) *and* ii). *Then the complexity of the sequence $w(v, Q)$ is given by*

$$p(n; v, Q)=n^2+n+1 \qquad (n\ge 1)\,.$$

The proof in [1] is based on a dynamical system associated with billiards in the cube. In this paper, we give an alternative proof, which is more elementary and independent of the ergodic arguments.

§2. The sequence $\{p_n\}_{n\ge 1}$ and $\{q_n\}_{n\ge 1}$.

By symmetry with respect to the faces, the word $w(v, Q)$ remains unchanged, if we replace the cube by the torus $\boldsymbol{R}^3/\boldsymbol{Z}^3$ and imagine that the particle does not reflect at the faces but passes through them. If we attach $i\in\{1, 2, 3\}$ to the intersection points of

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the half line $l := \{tv+Q \mid t>0\}$ to the planes $X_i := \{(x_1, x_2, x_3) \mid x_i \in \boldsymbol{N}\}$ and trace them along l, we obtain the sequence $w(v, Q)$ defined above. More precisely, if we define $\{t_n\}_{n\geq 0}$ by

$$\{t_n v+Q\}_{n\geq 1} = \{tv+Q \mid t>0\} \cap \bigcup_{i=1}^{3} X_i$$

with $t_0 = 0 < t_1 < t_2 < \cdots$ and write

$$w(v, Q) = w_1 w_2 \cdots w_n \cdots, \qquad w_n \in \{1, 2, 3\},$$

we have $t_n v+Q \in X_{w_n}$ $(n \geq 1)$. We remark that the condition ii) implies

$$\{t_n v+Q\}_{n\geq 1} \cap X_i \cap X_j = \emptyset \qquad (i, j \in \{1, 2, 3\}, i \neq j). \tag{1}$$

For each $n \geq 1$, let $P_n \in \boldsymbol{N}^3$ and $Q_n \in \bigcup_{i=1}^{3} F_i$ be defined by $P_1 = (1, 1, 1)$, $Q_1 = Q$,

$$\{tv+Q \mid t_{n-1} \leq t < t_n\} \subset P_n - P_1 + I^3, \quad \text{and} \quad Q_n = t_{n-1}v + Q - (P_n - P_1).$$

Then by definition

$$w(v, Q_n) = w_n w_{n+1} w_{n+2} \cdots. \tag{2}$$

Let π denote the projection of $\boldsymbol{R}^3$ onto the plane $\Pi = \{(x_1, x_2, x_3) \mid \sum_{i=1}^{3} \alpha_i x_i = 0\}$ along v and let $H = \pi(I^3)$. If two points x and y in $\boldsymbol{R}^3$ satisfy the relation $x - y = \sum_{i=1}^{3} k_i e_i$ for some $k_i \in \boldsymbol{Z}$ with $\sum_{i=1}^{3} k_i = 0$, we write $x \equiv y \pmod{H}$. This defines an equivalence relation in $\boldsymbol{R}^3$. We put

$$H^* = H \setminus ([e_1+e_3, e_3] \cup [e_3, e_2+e_3] \cup [e_2+e_3, e_2]),$$

where $e_1 = \pi(1, 0, 0)$, $e_2 = \pi(0, 1, 0)$, $e_3 = \pi(0, 0, 1)$, and $[a, b]$ is the closed segment joining a to b. Then the family of hexagons $\{\sum_{i=1}^{3} k_i e_i + H^* \mid k_i \in \boldsymbol{Z}, \sum_{i=1}^{3} k_i = 0\}$ forms a tiling of Π, and hence for any $x \in \Pi$ there corresponds a unique $x^* \in H^*$ such that $x \equiv x^* \pmod{H}$. H^* can be considered as the two-dimensional torus.

We put $q_n = \pi(Q_n)$. Then

$$q_{n+1} = q_n - e_{w_n}, \qquad q_n \in \pi(G_{w_n}), \tag{3}$$

where $G_i = \{(x_1, x_2, x_3) \mid x_i = 1, 0 \leq x_j < 1\ (j \neq i)\}$ $(i = 1, 2, 3)$. Noting that

$$x \in F_i \quad \text{if and only if} \quad x + e_i \in G_i \qquad (i = 1, 2, 3),$$

we have $q_n = q_n^* \in H^*$ $(n \geq 1)$. However, $\pi(P_n)$ $(n \geq 1)$ are not always in H^*, and so we define $p_n = \pi(P_n)^*$. Then, since $\pi(P_{n+1}) = \pi(P_n) + e_{w_n}$, we have

$$p_{n+1} \equiv p_n + e_i \qquad (\text{mod}\, H) \tag{4}$$

for any $i = 1, 2, 3$, so that

$$p_n + q_n \equiv p_1 + q_1 \qquad (n \geq 1). \tag{5}$$

We remark that the sequence $\{q_n\}_{n\geq 1}$ depends on v and Q, however $\{p_n\}_{n\geq 1}$ depends only on v.

LEMMA 1. *Both of the sequences* $\{p_n\}_{n\geq 1}$ *and* $\{q_n\}_{n\geq 1}$ *are dense in* H.

PROOF. We put $Q=(\beta_1, \beta_2, \beta_3)$. Then $tv+Q\in X_1$ if and only if $t\alpha_1+\beta_1\in N$, so that $\{Q_n\}_{n\geq 1}\cap F_1=\{(0, \langle\alpha_1^{-1}\alpha_2 k+\gamma_2\rangle, \langle\alpha_1^{-1}\alpha_3 k+\gamma_3\rangle\}_{k\geq 1}$ for some fixed γ_2 and γ_3, where $\langle x\rangle$ is the fractional part of x. Hence $\{Q_n\}_{n\geq 1}\cap F_1$ is dense in F_1 by Kronecker's theorem. Similarly, $\{Q_n\}_{n\geq 1}\cap F_i$ is dense in F_i $(i=2, 3)$. Therefore $\{q_n\}_{n\geq 1}$ is dense in H. So is $\{p_n\}_{n\geq 1}$, by (5).

§3. Decomposition of the hexagon H.

We put

$$m_n=\bigcup_{i=1}^{3}[p_n-e_i, p_n]^* \qquad (n\geq 1),$$

and define

$$M_n=\bigcup_{k=1}^{n} m_k\left(=\bigcup_{i=1}^{3}[p_1-e_i, p_1+(n-1)e_i]^*\right) \qquad (n\geq 1).$$

m_n is the union of three segments in H^* starting at p_{n-1}, not intersecting each other, and ending at p_n. M_n consists of three segments starting at three points p_1-e_i $(i=1, 2, 3)$ which coincide mod H, winding around H^*, intersecting only at $p_1, p_2, \cdots$ and p_{n-1}, and ending at p_n. So M_n forms a mesh which decomposes the hexagon H into subpolygons. The set of all these subpolygons will be denoted by Δ_n, namely, Δ_n is the set of all connected components of $H\backslash(\partial H\cup M_n)$, where ∂A denotes the boundary of a set A in Π. We note that the condition i) implies $p_n\notin M_{n-1}$ $(n\geq 2)$, $p_1\notin\partial H$, and ii) implies

$$\{q_n\}_{n\geq 1}\cap\left(\partial H\cup\bigcup_{n=1}^{\infty}M_n\right)=\varnothing .$$

REMARK 1. Every element in Δ_n $(n\geq 1)$ is a convex polygon, since it is an intersection of a finite number of half-planes in Π. Moreover, it can be proved that any element in Δ_n is triangle, quadrangle, pentagon, or hexagon whose sides are parallel with e_1, e_2, or e_3. However, the latter fact will not be used to prove the theorem.

LEMMA 2. $p(n; v, Q)=\#\Delta_n$ $(n\geq 1)$.

PROOF. For any $h\geq 1$ and $k\geq 0$, we have by (3) and (5)

$$q_h=\sum_{j=0}^{k-1}e_{w_{h+j}}+q_{h+k}\equiv q_1-q_{k+1}+q_{h+k}$$

$$\equiv p_{k+1}-p_1+q_{h+k} \pmod{H}, \qquad k\geq 0$$

noting that $e_i\equiv e_j \pmod{H}$ for any $i,j=1,2,3$, and so

$$q_h\in(p_k-p_1+\pi(G_{w_{h+k-1}}))^*, \qquad k\geq 1. \tag{6}$$

Thus it follows from (2) that $w_hw_{h+1}\cdots w_{h+n-1}=\sigma_1\sigma_2\cdots\sigma_n$ for some $\sigma_1\sigma_2\cdots\sigma_n\in\{1,2,3\}^n$ if and only if

$$q_h\in\bigcap_{k=1}^{n}(p_k-p_1+\pi(G_{\sigma_k}))^*.$$

Here we have for $k\geq 1$

$$\bigcup_{i=1}^{3}\partial(p_k-p_1+\pi(G_i))^*=(p_k-p_1+\partial H)^*\cup(p_k-p_1+m_1)^*$$

$$=m_{k-1}\cup m_k, \tag{7}$$

where $m_0=\partial H$, and so

$$\bigcup_{k=1}^{n}\bigcup_{i=1}^{3}\partial(p_k-p_1+\pi(G_i))^*=\bigcup_{k=1}^{n}(m_{k-1}\cup m_k)=\partial H\cup M_n.$$

Hence we get

$$p(n)=\#\left\{\sigma_1\sigma_2\cdots\sigma_n\in\{1,2,3\}^n \;\middle|\; \bigcap_{k=1}^{n}(p_k-p_1+\pi(G_{\sigma_k}))^*\neq\emptyset\right\},$$

which implies $p(n;v,Q)\leq\#\Delta_n$.

To prove $p(n;v,Q)\geq\#\Delta_n$, it is enough to show by Lemma 1 that, if q_i and q_j belong to distinct elements in Δ_n $(n\geq 1)$, then

$$w_iw_{i+1}\cdots w_{i+n-1}\neq w_jw_{j+1}\cdots w_{j+n-1}.$$

This is true for $n=1$. Assume that the statement holds for some $n\geq 1$. Let $q_i\in\delta$ and $q_j\in\delta'$ for some $\delta,\delta'\in\Delta_{n+1}$ with $\delta\neq\delta'$. Then $\delta\subset\delta_n$ and $\delta'\subset\delta'_n$ for some $\delta_n,\delta'_n\in\Delta_n$. If $\delta_n\neq\delta'_n$, the statement holds for $n+1$ by induction hypothesis. Suppose that $\delta_n=\delta'_n$. Then $\delta\subset\gamma$ and $\delta'\subset\gamma'$ for some connected components γ and γ' of $H\backslash(m_n\cup m_{n+1})$ adjacent each other. Taking (7) into account, we have $\gamma\subset(p_{n+1}-p_1+\pi(G_\sigma))^*$ and $\gamma'\subset(p_{n+1}-p_1+\pi(G_{\sigma'}))^*$ for some $\sigma,\sigma'\in\{1,2,3\}$. Here we note that

$$T=\bigcup_{\tau=1}^{3}\left\{m+\pi(G_\tau)\;\middle|\; m=\sum_{i=1}^{3}k_ie_i,\ k_i\in\mathbf{Z},\ \sum_{i=1}^{3}k_i=0\right\}$$

forms a tiling of Π, where $m+\pi(G_\tau)$ and $m'+\pi(G_\tau)\in T$ $(m\neq m')$ are not adjacent each other; so that, for any τ and $m+\pi(G_\tau)\in T$, any distinct connected components γ_1 and γ_2 in $(m+\pi(G_\tau))^*$ are not adjacent each other. Threfore, we get $\sigma\neq\sigma'$; which together with (6) implies $w_{i+n} \neq w_{j+n}$. This completes the proof of Lemma 2.

REMARK 2. The above proof shows that every set $\bigcap_{k=1}^{n}(p_k - p_1 + \pi(G_{\sigma_k}))^*$ is connected unless it is empty, and so

$$\Delta_n = \left\{ \bigcap_{k=1}^{n} (p_k - p_1 + \pi(G_{\sigma_k}))^* \,\middle|\, \sigma_1 \sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n \right\} \setminus \{\varnothing\} \qquad (n \geq 1).$$

Hence, noting that the diameter of each $\delta_n \in \Delta_n$ tends to zero as $n \to \infty$ by Lemma 1, we have $w(v, Q) \neq w(v, Q')$ for $Q \neq Q'$, and

$$q = q_1 = \bigcap_{n=1}^{\infty} (p_n - p_1 + \pi(G_{w_n}))^* = \bigcap_{\substack{n=1 \\ q \in \delta_n \in \Delta_n}}^{\infty} \delta_n .$$

These facts will not be used to prove the theorem.

§4. Proof of the theorem.

We have to show by Lemma 2 that

$$\# \Delta_n = n^2 + n + 1 \qquad (n \geq 1).$$

This is true for $n = 1, 2$. Let $n \geq 3$. The mesh M_n decomposes H into $\# \Delta_n$ polygons and in the next step m_{n+1} divides some of these polygons into subpolygons in $\Delta_{n+1} \setminus \Delta_n$. We put $d(n+1) = \# \Delta_{n+1} - \# \Delta_n$. Since $d(2) = 4$, it is enough to show that

$$d(n+1) - d(n) = 2 \qquad (n \geq 2).$$

We shall count $d(n)$ by means of the intersection points $\bigcup_{i=1}^{3} [p_n - e_i, p_n]^* \cap (\partial H \cup M_{n-1})$. We write $m_n = \bigcup_{i=1}^{3} l_{n,i}$ where $l_{n,i} = [p_n - e_i, p_n]^*$. Since $p_n \notin M_{n-1}$, there is a $\delta_{n-1} \in \Delta_{n-1}$ with $p_n \in \delta_{n-1} \setminus \partial \delta_{n-1}$. Then δ_{n-1} is divided into three polygons in Δ_n by $l_{n,i}$ $(i = 1, 2, 3)$. Let $s_{n,i}$ be defined by $\{s_{n,i}\} = l_{n,i} \cap \partial \delta_{n-1}$ $(i = 1, 2, 3)$, so that $\bigcup_{i=1}^{3} [p_n - e_i, s_{n,i}]^*$ is the part of m_n outside of δ_{n-1}. Since the elements in Δ_{n-1} are convex and $[p_n - e_i, s_{n,i}]^*$ $(i = 1, 2, 3)$ never intersects each other except at p_{n-1}, the number of elements in $\Delta_n \setminus \Delta_{n-1}$ produced by these segments coincides with $\# \bigcup_{i=1}^{3} [p_n - e_i, s_{n,i}]^* \cap (\partial H \cup M_{n-1})$, counting the points on ∂H appropriately. $d(n+1)$ is counted similarly. Noting that the contribution of these intersection points on ∂H as well as p_{k-1} and p_k to $d(k)$ is the same for $k = n$ and $n+1$, we get

$$d(n+1) - d(n) = \# (M_n \cap m_{n+1})' - \# (M_{n-1} \cap m_n)',$$

where $A' = A \setminus \{p_k\}_{k \geq 1}$. Here $\# (M_n \cap m_{n+1})' = \# (M_{n-1} \cap m_{n+1})'$, since $m_n \cap m_{n+1} = \{p_n\}$. Therefore it is enough to show that

$$\# (L_{n-1, w_n} \cap m_{n+1})' = \# (L_{n-1, w_n} \cap m_n)', \tag{8}$$

$$\# (L_{n-1, i} \cap m_{n+1})' = \# (L_{n-1, i} \cap m_n)' + 1 \qquad (i \neq w_n) \tag{9}$$

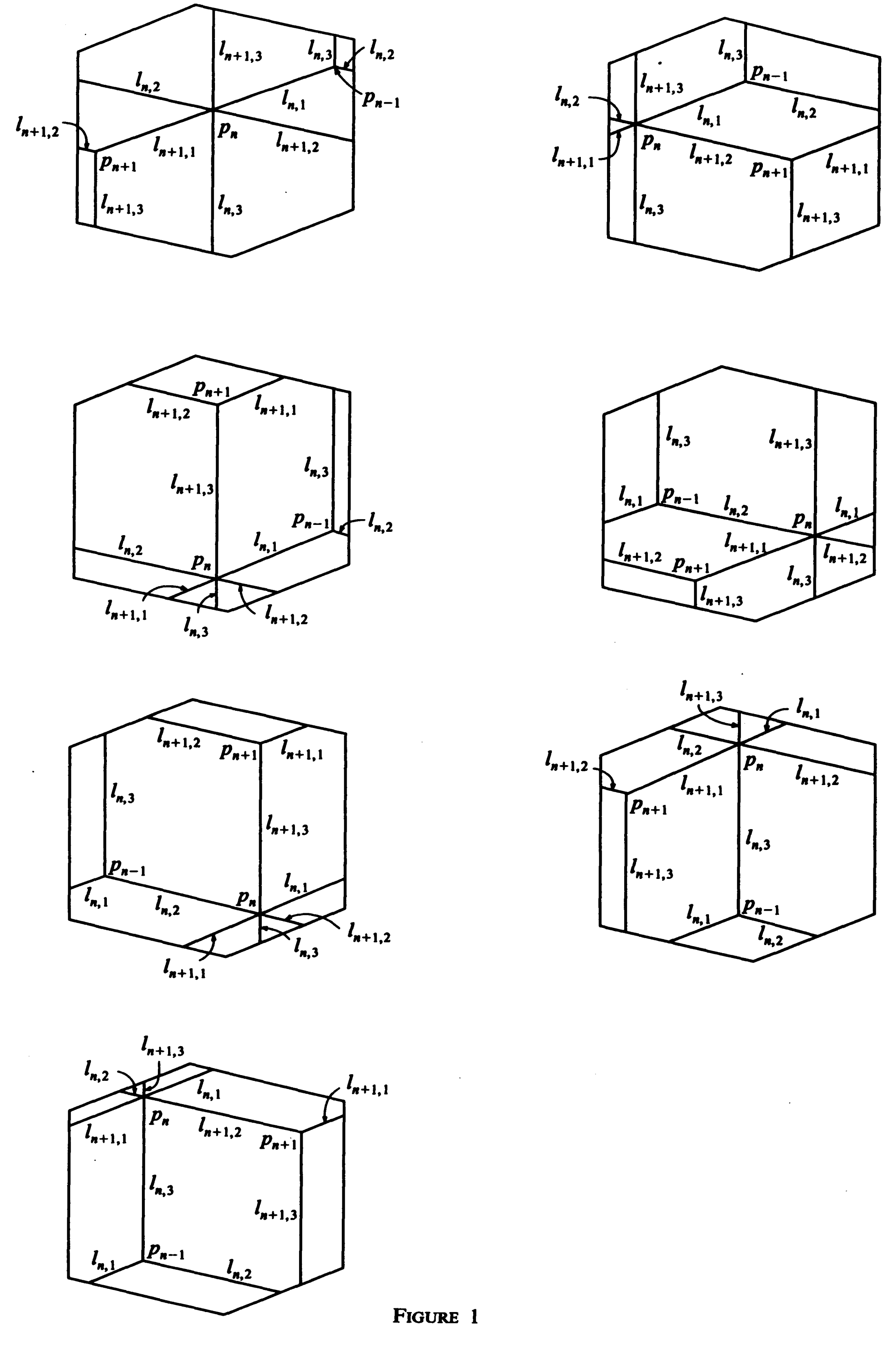
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FIGURE 1

where

$$M_{n-1}=\bigcup_{i=1}^{3} L_{n-1,i}, \qquad L_{n,i}=[p_1-e_i, p_1+(n-1)e_i]^* .$$

We assume as we may that $\alpha_1>\alpha_2>\alpha_3>0$. Then $(w_{n-1}, w_n)\in\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3.2)\}$ (cf. Fig. 1).

We shall prove (8) and (9) only in the first case $(w_{n-1}, w_n)=(1, 1)$, since the remaining cases can be treated in just the same way. Let $w_{n-1}=w_n=1$. To prove (8), we may exclude $l_{k,1}$ from m_k $(k=n, n+1)$, since $l_{k,1}$ is parallel with $L_{n-1,1}$ (cf. Fig. 2).

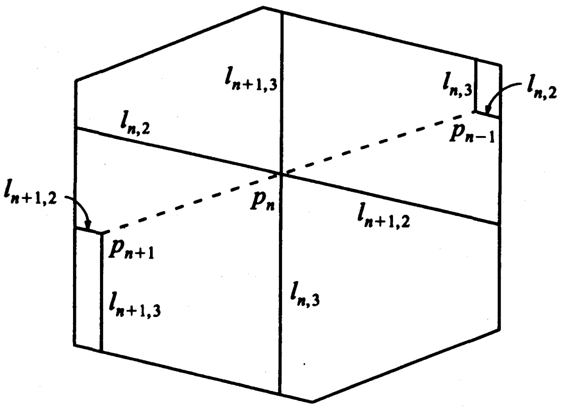

FIGURE 2

Let ϕ be the projection of $[p_n-e_2, p_n]\cup[p_n-e_3, p_n]$ onto $[p_{n+1}-e_2, p_{n+1}]\cup[p_{n+1}-e_3, p_{n+1}]$ along e_1. We regard ϕ as a bijection of $l_{n,2}\cup l_{n,3}$ onto $l_{n+1,2}\cup l_{n+1,3}$ by identifying points on π by mod$H$. Then we see that $\phi(p_{k-1})=p_k$ $(k=n, n+1)$ and that $x\in(L_{n-1,1}\cap(l_{n,2}\cup l_{n,3}))'$ if and only if $\phi(x)\in(L_{n-1,1}\cap(l_{n+1,2}\cup l_{n+1,3}))'$; and hence (8) follows. Next we prove (9) with $i=2$. For this we may exclude $l_{k,2}$ from m_k $(k=n, n+1)$, by the same reason as above (cf. Fig. 3).

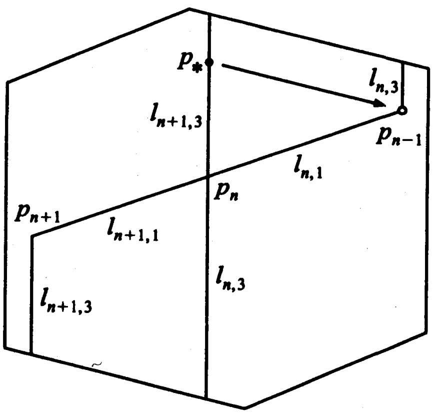

FIGURE 3

Let ψ denote the projection of $l_{n+1,1} \cup l_{n+1,3}$ onto $l_{n,1} \cup l_{n,3}$ along e_2. Clearly, $p_{n+1}, \psi(p_{n+1}) \notin L_{n-1,2}$, and $p_n = \psi(p_n) \notin L_{n-1,2}$. Furthermore, for $x \neq p_*$, $x \in (L_{n-1,2} \cap (l_{n+1,1} \cup l_{n+1,3}))'$ if and only if $\psi(x) \in (L_{n-1,1} \cap (l_{n,1} \cup l_{n,3}))'$. However, a point on $L_{n-1,2}$ starting at $p_1 - e_2$ and going along $L_{n-1,2}$ must cross $l_{n+1,3}$ at p_* to get p_{n-1} (cf. Fig. 3). This implies that $\psi^{-1}(p_{n-1}) \in l_{n+1,3}$, and therefore (9) with $i=2$ follows. The proof is similar for $i=3$, cf. Fig. 4.

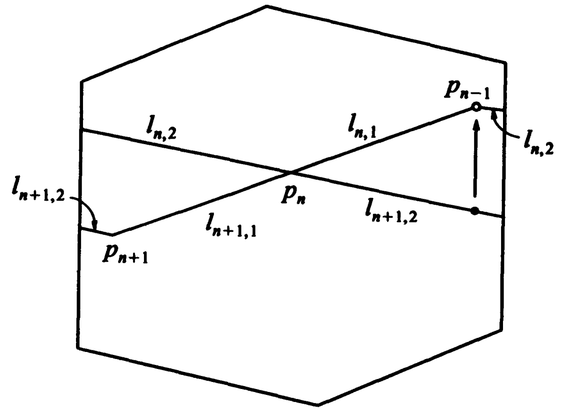

FIGURE 4

The proof of the theorem is now completed.

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Present Addresses:
Pierre ARNOUX
UFR DE MATHÉMATIQUES, UNIVERSITÉ PARIS 7,
2 PLACE JUSSIEU, 75251 PARIS CEDEX 05, FRANCE.

Christian MAUDUIT
INSTITUT DE MATHÉMATIQUES, UNIVERSITÉ DE LYON 1,
43 BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE Cedex.

Iekata SHIOKAWA
DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY,
HIYOSHI, YOKOHAMA, 223 JAPAN.

Jun-ichi TAMURA
FACULTY OF GENERAL EDUCATION, INTERNATIONAL JUNIOR COLLEGE,
EKODA 4–15–1, NAKANO-KU, TOKYO, 165 JAPAN.