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Mean Value Results for the Non-Symmetric Form of the Approximate Functional Equation of the Riemann Zeta-Function

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1. Statement of results.

Let $s=\sigma+it$ $(0\leqq\sigma\leqq 1,\ t\geqq 1)$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, $d(n)$ the number of positive divisors of the integer n, γ the Euler constant and $\exp(2\pi i\alpha)=e(\alpha)$. We first define

$$R(s;\, t/2\pi)=\zeta^2(s)-\sum_{n\leqq t/2\pi} d(n)n^{-s}-\chi^2(s)\sum_{n\leqq t/2\pi} d(n)n^{s-1}\,,$$

where

$$\chi(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\,.$$

As for this function $R(s;\, t/2\pi)$, Motohashi (see (1) of [6]) proved the following "weak form" of the Riemann-Siegel formula for $\zeta^2(s)$:

$$\chi(1-s)R(s;\, t/2\pi)=(t/2\pi)^{-1/4}\sum_{n=1}^{\infty} d(n)h(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4) + O(t^{-1/2}\log t)\,, \tag{1.1}$$

where

$$h(n)=(2/\pi)^{1/2}\int_0^{\infty}(y+n\pi)^{-1/2}\cos(y+\pi/4)dy\,.$$

Kiuchi and Matsumoto (see Theorem 1 of [3]) started from this formula, and proved an asymptotic formula for the mean square of $|R(1/2+it;\, t/2\pi)|$:

$$\int_1^T |R(1/2+it;\, t/2\pi)|^2 dt=\sqrt{2\pi}\left\{\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-1/2}\right\}T^{1/2}+O(T^{1/4}\log T)\,. \tag{1.2}$$

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Moreover, Motohashi sketched the proof for a stronger approximation formula of (1.1) in [6], and gave a complete proof in [7]. By using this "full form" of Motohashi's formula (see Theorem 6 of [7]), the error term of the formula (1.2) was improved to $O(\log^5 T)$ by Kiuchi [4].

Now we define the function $R^*(s; lt/2\pi k)$ which is a generalization of $R(s; t/2\pi)$. Let k and l be integers with $1 \leqq l \leqq k$ and $(k, l)=1$, and we define

$$R^*(s; lt/2\pi k)=\zeta^2(s)-\sum_{n\leqq lt/2\pi k} d(n)n^{-s}-\chi^2(s)\sum_{n\leqq kt/2\pi l} d(n)n^{s-1}.$$

The aim of this paper is to calculate mean value results of the error term $R^*(s; lt/2\pi k)$. Our starting point is the following "non-symmetric form" of the Riemann-Siegel formula for $\zeta^2(s)$, which was proved in Motohashi (see Theorem 7 of [7]). In order to mention his result, we need some other notation. Let a and b be integers with $a\geqq 1$ and $(a, b)=1$. For $x\geqq 1$, we put

$$\Delta(x; b/a)=\sum_{n\leqq x}{}' d(n)e(bn/a)-a^{-1}x(\log(x/a^2)+2\gamma-1)-E(0; b/a), \tag{1.3}$$

where $\sum'$ indicates that the last term is to be halved if x is an integer. $E(0; b/a)$ is the value at $s=0$ of the analytic continuation of

$$E(s; b/a)=\sum_{n=1}^{\infty} d(n)e(bn/a)n^{-s}.$$

It is well-known that

$$E(0; b/a)\ll a\log(2a) \qquad \text{(see (2.6.3) of [7])}. \tag{1.4}$$

Then Motohashi's "non-symmetric form" of the Riemann-Siegel formula for $\zeta^2(s)$ states as follows: For $t\geqq 2$, we have, uniformly for $kl\leqq t(\log t)^{-20}$,

$$\begin{aligned}\chi(1-s)R^*(s; lt/2\pi k)&=M(s; l/k)+\overline{M(1-\bar{s}; k/l)}\\ &\quad+O((l/k)^{1/2-\sigma}(kl/t)^{1/2}\log^3 t),\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}M(s; l/k)=&-e^{-\pi i/4}(t/2\pi)^{-1/2}(l/k)^{-s}\Delta(lt/2\pi k; -k/l)\\ &+\frac{1}{2}e^{-\pi i/4}(kl/2\pi t)^{1/4}(l/k)^{1/2-s}\sum_{n=1}^{\infty} d(n)e(\bar{k}n/l)n^{-1/4}\sin(2(2\pi tn/kl)^{1/2}+\pi/4)\\ &\cdot\int_0^{\infty} e^{iy/kl}(y+n\pi)^{-3/2}dy;\end{aligned}$$

and $\bar{k}$ is defined by $k\bar{k}\equiv 1 \pmod{l}$. In the section 3 we will give an analogue of the formula (1.1) for $R^*(s; lt/2\pi k)$ (see (3.3)). Then we can calculate the mean square of

$|R^*(1/2+it;\, lt/2\pi k)|$ in the following way, which is an analogue of (1.2).

THEOREM 1. *For* $1 \leqq l \leqq k$, $(k, l)=1$, $kl \leqq T(\log T)^{-20}$ *and* $T \geqq 2$, *we have*

$$\int_2^T |R^*(1/2+it;\, lt/2\pi k)|^2 dt$$
$$= \sqrt{2\pi}\left\{\sum_{n=1}^{\infty} d^2(n) H_{k,l}^2(n) n^{-1/2}\right\} T^{1/2} + O((kl)^{3/4} T^{1/4} \log^3 T)\,,$$

where

$$H_{k,l}(n) = (kl)^{-1/4}(2/\pi)^{1/2} \int_0^{\infty} \{y + (n\pi/kl)\}^{-1/2} \cos(y + 2\pi(\bar{k}n/l) + \pi/4)\,dy\,. \tag{1.6}$$

COROLLARY. *For* $1 \leqq l \leqq k$, $(k, l)=1$, $kl \leqq t(\log t)^{-20}$ *and* $t \geqq 2$, *we have*

$$|R^*(1/2+it;\, lt/2\pi k)| = \Omega((kl)^{1/4} t^{-1/2})\,.$$

As has been observed by Jutila (see p. 105 of [1]) when l/k is very close to 1 (e.g. $l/k = 1 + O(t^{-1/2})$),

$$|R^*(s;\, lt/2\pi k)| = \Omega(\log t)\,.$$

The content of this corollary includes the Ω-result which is deduced from the formula (1.2).

Next we consider the mean square of $R^*(1/2+it;\, lt/2\pi k)$ itself. Let $w = t/2\pi$ and $f(w) = 2w - 2w \log w + 1/4$. From (1.5), we get

$$R^*(1/2+it;\, lt/2\pi k)^2$$
$$= \chi^2(1/2+it)\{M(1/2+it;\, l/k) + \overline{M(1/2+it;\, k/l)} + O((kl/t)^{1/2} \log^3 t)\}^2\,.$$

It follows from Stirling's formula that

$$\chi^2(1/2+it) = e(f(w)) + O(1/t)\,, \tag{1.7}$$

so the χ^2-factor of the above formula can be considered as an "oscillatory factor". Because of this factor, it is natural to expect that the integral of $R^*(1/2+it;\, lt/2\pi k)^2$ is smaller than that of $|R^*(1/2+it;\, lt/2\pi k)|^2$. In fact, we obtain the following estimate:

THEOREM 2. *For* $1 \leqq l \leqq k \ll T^{1/3}$ *and* $T \geqq 2$, *we have*

$$\int_2^T R^*(1/2+it;\, lt/2\pi k)^2 dt \ll (kl)^{3/4} T^{1/4+\varepsilon} + (kl)^{5/2} T^{-1/2}\,.$$

REMARK. Kiuchi and Matsumoto proved the following mean value result for $R(1/2+it;\, t/2\pi)$ (see Theorem 2 of [3]):

$$\int_1^T R(1/2+it;\, t/2\pi)^2 dt \ll T^{1/4+\varepsilon}. \tag{1.8}$$

Comparing (1.2) and (1.8), we find that the leading term of (1.2) disappears in (1.8). We find the same phenomenon between Theorems 1 and 2.

In what follows, ε denotes an arbitrarily small positive number, not necessarily the same at each occurrence.

2. Application of Meurman's method.

Jutila (see [2], or (2.6.7) of [7]) proved the following formula, which is an analogue of Voronoi formula for (1.3):

$$\Delta(x;\, b/a) = (\pi\sqrt{2})^{-1} a^{1/2} x^{1/4} \sum_{n=1}^{\infty} d(n) e(-\bar{b}n/a) n^{-3/4} \cdot \cos(4\pi\sqrt{nx}/a - \pi/4) + O(a^{3/2} x^{-1/4}), \tag{2.1}$$

where $x \geqq a^2(\log 2a)^3$, and $\bar{b}$ is defined by $b\bar{b} \equiv 1 \pmod{a}$. According to Meurman's paper [5], we transform (2.1) into

$$\Delta(x;\, b/a) = (\pi\sqrt{2})^{-1} a^{1/2} x^{1/4} \sum_{n \leqq M} d(n) e(-\bar{b}n/a) n^{-3/4} \cos(4\pi\sqrt{nx}/a - \pi/4) + O(a^{1/2} x^{1/4}(|S_1| + |S_2|)) + O(a^{3/2} x^{-1/4}), \tag{2.2}$$

where

$$S_1 = \sum_{n > M} \{d(n) e(-\bar{b}n/a) - a^{-1}(\log(n/a^2) + 2\gamma)\} n^{-3/4} e(\pm 2\sqrt{nx}/a),$$

$$S_2 = a^{-1} \sum_{n > M} (\log(n/a^2) + 2\gamma) n^{-3/4} e(\pm 2\sqrt{nx}/a).$$

Now, let us assume that $M \geqq 2x$, and let us put

$$G(\xi) = \sum_{M < n \leqq \xi} e(\pm 2\sqrt{nx}/a).$$

By using [8, Lemma 4.8], we have

$$G(\xi) = \int_M^{\xi} e(\pm 2\sqrt{xy}/a) dy + O(1).$$

So, we get

$$G(\xi) \ll a x^{-1/2} \xi^{1/2}. \tag{2.3}$$

On the other hand we have, by partial summation,

$$S_2=-\int_M^\infty G(\xi)\{a^{-1}(\log(\xi/a^2)+2\gamma)\xi^{-3/4}\}'d\xi+a^{-1}\lim_{A\to\infty}G(A)(\log(A/a^2)+2\gamma)A^{-3/4}.$$

Hence, by (2.3), we obtain, for $M\geqq 2x$,

$$S_2\ll ax^{-1/2}. \tag{2.4}$$

We note that, from (1.3) and (1.4),

$$\sum_{n\leqq\xi}(d(n)e(-\bar{b}n/a)-a^{-1}(\log(n/a^2)+2\gamma))=\Delta(\xi;\,-\bar{b}/a)+O(a\log\xi)\qquad(\xi>1).$$

Then, by partial summation, and with

$$|\Delta(x;\,-\bar{b}/a)|\ll a^{2/3}x^{1/3+\varepsilon} \tag{2.5}$$

for any $a\leqq x$, we have

$$S_1\ll a^{-1}x^{1/2}\left|\int_M^\infty\Delta(\xi;\,-\bar{b}/a)\xi^{-5/4}e(\pm 2\sqrt{x\xi}/a)d\xi\right|+x^{1/2}M^{-1/4}\log M.$$

Now, we estimate the integral,

$$\int_L^{2L}\Delta(\xi;\,-\bar{b}/a)\xi^{-5/4}e(\pm 2\sqrt{x\xi}/a)d\xi,\qquad(L\geqq M).$$

To do this, we make use of the following formula

$$\Delta(x;b/a)=(\pi\sqrt{2})^{-1}a^{1/2}x^{1/4}\sum_{n\leqq N}d(n)e(-\bar{b}n/a)n^{-3/4}\cos(4\pi\sqrt{nx}/a-\pi/4)$$
$$+O(ax^{1/2+\varepsilon}N^{-1/2})+O(ax^{\varepsilon})$$

(for any $a\leqq x$ and $1\leqq N\ll x^A$ $(A>0)$), which is a truncated version of (2.1) (The proof of this formula proceeds in the same way as in Ivić's book [2], pp, 86–88). Then we see that the above integral is

$$\ll a^{3/2}L^{-1/2}x^{-1/4+\varepsilon}\|x\|^{-1}+aL^{-1/4+\varepsilon},$$

where $\|x\|$ is the distance between x and its nearest integer. Hence we have

$$S_1\ll a^{1/2}M^{-1/2}x^{1/4+\varepsilon}\|x\|^{-1}+x^{1/2}M^{-1/4+\varepsilon}.$$

And this is $O(ax^{-1/2})$, provided that $M\gg a^{-1}x^5\|x\|^{-2}$. From this, (2.2) and (2.4), we obtain the following Lemma:

LEMMA 1. *Let $a\leqq x$, and we put*

$$E_a(M;x)=\Delta(x;b/a)-(\pi\sqrt{2})^{-1}a^{1/2}x^{1/4}\sum_{n\leqq M}d(n)e(-\bar{b}n/a)n^{-3/4}\cos(4\pi\sqrt{nx}/a-\pi/4).$$

Then we have

$$E_a(M;x)\ll\begin{cases} a^{3/2}x^{-1/4} & \text{if } M\gg a^{-1}x^5\|x\|^{-2},\\ ax^{\varepsilon}+ax^{1/2+\varepsilon}M^{-1/2} & \text{otherwise}.\end{cases}$$

3. An analogue to the formula (1.1).

Applying the formula (2.1) with $a=l$, $b=-k$ and $x=lt/2\pi k$, we see that

$$\Delta(lt/2\pi k;\ -k/l)=(\pi\sqrt{2})^{-1}l^{1/2}(lt/2\pi k)^{1/4}\sum_{n=1}^{\infty}d(n)e(\bar{k}n/l)n^{-3/4}\cos(2(2\pi tn/kl)^{1/2}-\pi/4)$$
$$+O(k^{1/4}l^{5/4}t^{-1/4}).$$

From this, it follows that

$$M(s;\ l/k)=i(2\pi k)^{-1/2}(t/2\pi)^{-1/4}(l/k)^{1/4-s}\sum_{n=1}^{\infty}d(n)e(\bar{k}n/l)n^{-1/4}\sin(2(2\pi tn/kl)^{1/2}+\pi/4)$$
$$\cdot\int_0^{\infty}e^{i(y-\pi/4)}(y+(n\pi/kl))^{-1/2}dy+O(k^{1/4+\sigma}l^{5/4-\sigma}t^{-3/4}).$$

Similarly as above, we have

$$\overline{M(1-\bar{s};\ k/l)}=-i(2\pi l)^{-1/2}(t/2\pi)^{-1/4}(k/l)^{s-3/4}\sum_{n=1}^{\infty}d(n)e(-\bar{k}n/l)n^{-1/4}$$
$$\cdot\sin(2(2\pi tn/kl)^{1/2}+\pi/4)\int_0^{\infty}e^{-i(y-\pi/4)}(y+(n\pi/kl))^{-1/2}dy+O(l^{1/4+\sigma}k^{5/4-\sigma}t^{-3/4}).$$

Hence, for $t\geqq 2$ and $kl\leqq t(\log t)^{-20}$, in case $\sigma=1/2$, the formula (1.5) can be written as

$$\chi(1/2-it)R^*(1/2+it;\ lt/2\pi k)=(t/2\pi)^{-1/4}(k/l)^{it}$$
$$\cdot\sum_{n=1}^{\infty}d(n)H_{k,l}(n)n^{-1/4}\sin(2(2\pi tn/kl)^{1/2}+\pi/4)+O((kl/t)^{1/2}\log^3 t), \tag{3.1}$$

where $H_{k,l}(n)$ is defined by (1.6). Integrating by parts, we have

$$H_{k,l}(n)=-\sqrt{2}(\pi\sqrt{n})^{-1}(kl)^{1/4}\cos(2\pi\bar{k}n/l-\pi/4)+O((kl)^{5/4}n^{-3/2}). \tag{3.2}$$

Combining (2.1) and Lemma 1, we have

$$\sum_{n>M}d(n)e(-\bar{b}n/a)n^{-3/4}\cos(4\pi\sqrt{nx}/a-\pi/4)\ll ax^{-1/2}+a^{-1/2}x^{-1/4}|E_a(M;x)|.$$

Substituting $a=l$, $b=\pm\bar{k}$ and $x=lt/2\pi k$, we see that the right-hand side turns into

$$\ll(kl/t)^{1/2}+k^{1/4}l^{-3/4}t^{-1/4}|E_l(M;\ lt/2\pi k)|.$$

Therefore, from this, (3.1) and (3.2), it follows that for $kl \leqq t(\log t)^{-20}$,

$$\chi(1/2-it)R^*(1/2+it;\, lt/2\pi k)=(t/2\pi)^{-1/4}(k/l)^{it} \cdot \sum_{n\leqq M} d(n)H_{k,l}(n)n^{-1/4}\sin(2(2\pi tn/kl)^{1/2}+\pi/4)+D_{k,l}(M;\, lt/2\pi k)\,, \tag{3.3}$$

where

$$D_{k,l}(M;\, lt/2\pi k) \ll (kl/t)^{1/2}\log^3 t+(kl)^{5/4}t^{-1/4}M^{-3/4+\varepsilon} + (k/lt)^{1/2}\,|\,E_l(M;\, lt/2\pi k)\,|\,. \tag{3.4}$$

4. Proof of Theorem 1.

In this section we assume that $M \gg l^{-1}X^5\|X\|^{-2}$ $(X=lT/2\pi k)$. From (3.3), we have

$$\int_T^{2T} |\,R^*(1/2+it;\, lt/2\pi k)\,|^2 dt = I_1+O(|\,I_1\,|^{1/2}|\,I_2\,|^{1/2}+|\,I_2\,|)\,, \tag{4.1}$$

where

$$I_1=\sum\sum_{m,n\leqq M} d(m)d(n)H_{k,l}(m)H_{k,l}(n)(mn)^{-1/4} \cdot \int_T^{2T} (t/2\pi)^{-1/4}\sin(2(2\pi tm/kl)^{1/2}+\pi/4)\sin(2(2\pi tn/kl)^{1/2}+\pi/4)dt\,,$$

$$I_2=\int_T^{2T} |\,D_{k,l}(M;\, lt/2\pi k)\,|^2 dt\,.$$

By using (3.4) and Lemma 1, we have

$$I_2 \ll kl\log^6 T+(kl)^{5/2}T^{1/2}M^{-3/2+\varepsilon}\,, \tag{4.2}$$

provided that $1\leqq k\leqq l$ and $kl\leqq T(\log T)^{-20}$. Next, we have

$$I_1=\frac{1}{2}\sum_{n\leqq M} d^2(n)H_{k,l}^2(n)n^{-1/2}\int_T^{2T}(t/2\pi)^{-1/2}dt$$

$$+\frac{1}{2}\sum_{n\leqq M} d^2(n)H_{k,l}^2(n)n^{-1/2}\int_T^{2T}(t/2\pi)^{-1/2}\sin(4(2\pi tn/kl)^{1/2})dt$$

$$+\frac{1}{2}\sum\sum_{\substack{m,n\leqq M\\ m\neq n}} d(m)d(n)H_{k,l}(m)H_{k,l}(n)(mn)^{-1/4}\int_T^{2T}(t/2\pi)^{-1/2}\sin(2(\sqrt{m}+\sqrt{n})(2\pi t/kl)^{1/2})dt$$

$$+\frac{1}{2}\sum\sum_{\substack{m,n\leqq M\\ m\neq n}} d(m)d(n)H_{k,l}(m)H_{k,l}(n)(mn)^{-1/4}\int_T^{2T}(t/2\pi)^{-1/2}\cos(2(\sqrt{m}-\sqrt{n})(2\pi t/kl)^{1/2})dt$$

$$=I_{1,1}+I_{1,2}+I_{1,3}+I_{1,4}\,, \qquad \text{say}\,.$$

From (3.2), we have

$$I_{1,1}=\sqrt{2\pi}\left\{\sum_{n=1}^{\infty} d^2(n)H_{k,l}^2(n)n^{-1/2}\right\}(\sqrt{2T}-\sqrt{T})+O((klT)^{1/2}M^{-1/2+\varepsilon})\,. \tag{4.3}$$

Since

$$\int_T^{2T}(t/2\pi)^{-1/2}e(2(\sqrt{m}\pm\sqrt{n})(t/2\pi kl)^{1/2})dt \ll \sqrt{kl}\,|\sqrt{m}\pm\sqrt{n}\,|^{-1} \qquad (m\neq n)\,,$$

we see that

$$I_{1,2}\ll kl\,. \tag{4.4}$$

By using (3.2), and the argument of the proof of Theorem 3 in [7] (see also [3], or [4]), we have

$$\begin{aligned} I_{1,3}+I_{1,4} &\ll kl\sum_{m,n\leq M}\sum d(m)d(n)(mn)^{-3/4}\{(\sqrt{m}+\sqrt{n})^{-1}+|\sqrt{m}-\sqrt{n}\,|^{-1}\} \\ &\ll kl\log^5 M\,. \end{aligned} \tag{4.5}$$

Hence, from (4.3)–(4.5), we obtain

$$I_1=\sqrt{2\pi}\left\{\sum_{n=1}^{\infty} d^2(n)H_{k,l}^2(n)n^{-1/2}\right\}(\sqrt{2T}-\sqrt{T})+O(kl\log^5 M+(klT)^{1/2}M^{-1/2+\varepsilon})\,. \tag{4.6}$$

Now we put $M=(lT/2\pi k)^8$. Then (4.6) implies $I_1=O((klT)^{1/2})$. From this and (4.2), the second term on the right-hand side of (4.1) is estimated by

$$\ll (kl)^{3/4}T^{1/4}\log^3 T\,.$$

Substituting this estimate and (4.6) into (4.1), we have

$$\begin{aligned} &\int_T^{2T}|R^*(1/2+it;\,lt/2\pi k)|^2dt \\ &=\sqrt{2\pi}\left\{\sum_{n=1}^{\infty} d^2(n)H_{k,l}^2(n)n^{-1/2}\right\}(\sqrt{2T}-\sqrt{T})+O((kl)^{3/4}T^{1/4}\log^3 T)\,. \end{aligned}$$

We complete the proof of Theorem 1.

5. Proof of Theorem 2.

From (3.3) and Schwarz's inequality, it follows that

$$\int_T^{2T} R^*(1/2+it;\, lt/2\pi k)^2 dt = J + O(|I_1|^{1/2}|I_2|^{1/2} + |I_2|), \tag{5.1}$$

where

$$J = \int_T^{2T} (t/2\pi)^{-1/2} e((t/\pi)\log(k/l))\chi^2(1/2+it) \cdot \left\{ \sum_{n\leq M} d(n)H_{k,l}(n)n^{-1/4}\sin(2(2\pi tn/kl)^{1/2}+\pi/4)\right\}^2 dt .$$

In this section we put $M=(lT/2\pi k)^{1-\varepsilon}$. Then from (4.6) we have $I_1 = O((klT)^{1/2})$. By using Lemma 1, (3.4) and this estimate, we have

$$|I_1|^{1/2}|I_2|^{1/2} + |I_2| \ll (kl)^{3/4}T^{1/4+\varepsilon} \tag{5.2}$$

for $k \ll T^{1/3}$. From (1.7), it follows that

$$J = \int_T^{2T} (t/2\pi)^{-1/2} e(f(w)+2w\log(k/l))\left\{ \sum_{n\leq M} d(n)H_{k,l}(n)n^{-1/4}\sin(2(2\pi tn/kl)^{1/2}+\pi/4)\right\}^2 dt$$
$$+ O\left(\int_T^{2T} t^{-3/2}\left|\sum_{n\leq M} d(n)H_{k,l}(n)n^{-1/4}\sin(2(2\pi tn/kl)^{1/2}+\pi/4)\right|^2 dt\right)$$
$$= J_1 + J_2, \qquad \text{say}.$$

By using (2.5) and (3.2), we see that, for $k \ll T^{1/3}$,

$$J_2 \ll k^{1/3}l^{5/6}T^{-1/3+\varepsilon} + (kl)^{5/2}T^{-1/2}. \tag{5.3}$$

Similarly as in the case of I_1, we have

$$J_1 = \frac{1}{2}\sum_{n\leq M} d^2(n)H_{k,l}^2(n)n^{-1/2}\int_T^{2T}(t/2\pi)^{-1/2}e(f(w)+2w\log(k/l))dt$$
$$+\frac{1}{2}\sum_{n\leq M} d^2(n)H_{k,l}^2(n)n^{-1/2}\int_T^{2T}(t/2\pi)^{-1/2}e(f(w)+2w\log(k/l))\sin(4(2\pi tn/kl)^{1/2})dt$$
$$+\frac{1}{2}\sum\sum_{\substack{m,n\leq M\\ m\neq n}} d(m)d(n)H_{k,l}(m)H_{k,l}(n)(mn)^{-1/4}\int_T^{2T}(t/2\pi)^{-1/2}e(f(w)+2w\log(k/l))$$
$$\cdot\sin(2(\sqrt{m}+\sqrt{n})(2\pi t/kl)^{1/2})dt$$
$$+\frac{1}{2}\sum\sum_{\substack{m,n\leq M\\ m\neq n}} d(m)d(n)H_{k,l}(m)H_{k,l}(n)(mn)^{-1/4}\int_T^{2T}(t/2\pi)^{-1/2}e(f(w)+2w\log(k/l))$$
$$\cdot\cos(2(\sqrt{m}-\sqrt{n})(2\pi t/kl)^{1/2})dt$$
$$= J_{1,1}+J_{1,2}+J_{1,3}+J_{1,4}, \qquad \text{say}. \tag{5.4}$$

The right-hand side of (5.4) can be estimated by using the following Lemma:

LEMMA 2 (see (2.3) of [1]). *Let $F(t)$ be real differentiable, $F'(t)$ monotonic, $F'(t) \geqq m > 0$ or $\leqq -m < 0$ in $[a, b]$. Let $G(t)$ be positive monotonic, $|G(t)| \leqq M$ in $[a, b]$. Then*

$$\left| \int_a^b G(t) \exp(iF(t)) dt \right| \ll M/m .$$

Let $F(w) = 2\pi(f(w) + 2w \log(k/l) + 2u\sqrt{w})$ with $u \leqq 2(M/kl)^{1/2}$. Then, by Lemma 2, we have

$$\int_{T/2\pi}^{T/\pi} w^{-1/2} \exp(iF(w)) dw \ll T^{-1/2} .$$

From the cases $u=0$ and $u = \pm 2(n/kl)^{1/2}$, it follows that

$$J_{1,1} + J_{1,2} \ll (kl/T)^{1/2} ,$$

and from the cases $u = \pm(\sqrt{m} \pm \sqrt{n})/(kl)^{1/2}$, it follows that

$$J_{1,3} + J_{1,4} \ll lT^{\varepsilon} .$$

Hence we have $J_1 = O(lT^{\varepsilon} + (kl/T)^{1/2})$. Substituting this estimate, (5.2) and (5.3) into (5.1), we obtain Theorem 2.

References

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