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Examples of Non-Einstein Yamabe Metrics with Positive Scalar Curvature

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Let M be a compact C^{∞}-manifold with $n=\dim M\geq 3$. For any Riemannian metric g on M, we denote its scalar curvature by S_g, and its volume form by dV_g. Yamabe [9] considered the problem of finding a metric which minimizes the functional $I(g):=\int_M S_g dV_g/(\int_M dV_g)^{(n-2)/n}$ in a given conformal class. Such a metric is called a *Yamabe metric* and has constant scalar curvature. This problem was solved completely by Schoen [7], and we know that there is a Yamabe metric in any conformal class. Conversely, a metric g with constant scalar curvature is a Yamabe metric, if $S_g\leq 0$ or g is an Einstein metric ([5]). The Yamabe metrics conformal to $S^1(r)\times S^{n-1}(1)$ are also known in explicit form ([2], [3], [8]).

In this paper, we give a sufficient condition for a metric to be a Yamabe metric, and examples of non-Einstein Yamabe metrics with positive scalar curvature.

THEOREM. *Let g be a Yamabe metric on a compact C^{∞}-manifold M with $S_g>0$, h a metric on M with constant scalar curvature, and φ a diffeomorphism of M such that $dV_{\varphi^*h}=\gamma dV_g$ for some number γ. If $\varphi^*h\leq(S_g/S_h)g$, then h is also a Yamabe metric. Moreover, if $\varphi^*h<(S_g/S_h)g$, then h is a unique Yamabe metric (up to a homothety) in the conformal class $[h]$ of h.*

REMARK. For any two metrics g and h, there is a diffeomorphism φ such that $dV_{\varphi^*h}=\gamma dV_g$ for some γ (see [4]).

PROOF. It suffices to show the case when $\varphi=id$. For any metric $\tilde{h}=u^{4/(n-2)}h\in[h]$, we have

$$I(\tilde{h})=\frac{\int_M(a_n|\nabla_h u|^2+S_h u^2)dV_h}{\left(\int_M u^p dV_h\right)^{2/p}},$$

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where $a_n=4(n-1)/(n-2)$ and $p=2n/(n-2)$. If $h\leq(S_g/S_h)g$, then

$$I(\tilde{h})=\frac{\int_M (a_n|\nabla_h u|^2+S_h u^2)\gamma dV_g}{\left(\int_M u^p\gamma dV_g\right)^{2/p}}$$

$$\geq\gamma^{1-2/p}\frac{S_h}{S_g}\frac{\int_M (a_n|\nabla_g u|^2+S_g u^2)dV_g}{\left(\int_M u^p dV_g\right)^{2/p}}=\gamma^{1-2/p}\frac{S_h}{S_g}I(u^{p-2}g)$$

$$\geq\gamma^{1-2/p}\frac{S_h}{S_g}I(g)=I(h)\,.$$

Therefore h minimizes $I|_{[h]}$ or h is a Yamabe metric. Moreover, if $h<(S_g/S_h)g$, then $I(\tilde{h})=I(h)$ holds only when u is a constant, namely, h is a unique Yamabe metric in $[h]$.

q.e.d.

Our result applies typically in the following

COROLLARY. *Let $\{g_t \mid T\leq t\leq T'\}$ be a variation of Riemannian metrics on M with constant scalar curvature satisfying the conditions*: (1) *g_T is a Yamabe metric*; (2) *$S_{g_t}>0$ for $t<T'$*; *and* (3) *$S_{g_{T'}}\equiv 0$. Then g_t is also a Yamabe metric for any t sufficiently close to T'.*

PROOF. By the proof of Moser [4, Theorem], it is clear that there is a family $\{\varphi_t \mid T\leq t\leq T'\}$ of diffeomorphisms, which is continuous with respect to the parameter t, such that $dV_{\varphi_t^*g_t}=\gamma_t dV_g$ for some γ_t. Therefore the assertion above follows from our theorem.

q.e.d.

Now, let us give such examples with $\varphi=id$.

EXAMPLE 1. Let $\pi:(M,g)\to(B,\check{g})$ be a Riemannian submersion with totally geodesic fibers, g_t the canonical variation of g, and A the O'Neill tensor (see [1], [6], etc.). Suppose g_T is Einstein for some T, $S_{g_T}>0$ and $A\not\equiv 0$. Then g_t is a Yamabe metric on M for any $t\geq S_{\check{g}}/|A|^2-T$.

EXAMPLE 2. Let $\{X_1, X_2, X_3\}$ be a left invariant orthonormal frame of the standard metric on $S^3=SU(2)$. For any $t\geq s\geq 1$, define a metric $g_{s,t}$ on S^3 by

$$g_{s,t}(X_1,X_1)=1\,,\quad g_{s,t}(X_2,X_2)=s\,,\quad g_{s,t}(X_3,X_3)=t\,,$$
$$g_{s,t}(X_i,X_j)=0\qquad\text{for}\quad i\neq j\,.$$

Then $S_{g_{s,t}}=2\{2(s+t+st)-(1+s^2+t^2)\}/st$, and $g_{s,t}$ is a Yamabe metric if $t\geq s+\sqrt{s}+1$. We can also construct Yamabe metrics of this type on other simple compact Lie groups.

EXAMPLE 3. Let g_t be a Yamabe metric on S^{n-1} given in Example 1 with a Hopf fibration $\pi: S^{2m+1} \to CP^m$ ($t \geq 2m+1$), $\pi: S^{4q+3} \to HP^q$ ($t \geq (4q+5)/3$) or $\pi: S^{15} \to S^8$ ($t \geq 3$). Then $r^2 d\theta^2 + g_t$ is a Yamabe metric on $S^1 \times S^{n-1}$ if $r \leq 1/\sqrt{n-2}$. The same assertion holds also for $r^2 d\theta^2 + g_{s,t}$, where $g_{s,t}$ is a Yamabe metric on S^3 given in Example 2.

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