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Uniformly Bounded Solutions of Functional Differential Equations

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1. Introduction.

We deal with uniform boundedness and uniform ultimate boundedness of the solutions of functional differential equations with finite delay. Sufficient conditions on Liapunov functionals for these boundedness are obtained.

We consider the functional differential equation with finite delay

$$x'=f(t, x_t)\,, \tag{1}$$

where $x_t(s)=x(t+s)$ for $-h\leqq s\leqq 0$. Suppose that $f: \boldsymbol{R}_+\times C\to\boldsymbol{R}^n$ is completely continuous, where $\boldsymbol{R}_+$ is the set of nonnegative real numbers and C is the space of continuous functions $\varphi: [-h, 0]\to\boldsymbol{R}^n$ with the norm $\|\varphi\|=\sup_{-h\leqq s\leqq 0}|\varphi(s)|$ and any norm $|\cdot|$ in $\boldsymbol{R}^n$. For $t_0\in\boldsymbol{R}_+$ and $\varphi\in C$, $x(t, t_0, \varphi)$ denotes a solution of (1) with initial condition $x_{t_0}(t, t_0, \varphi)=\varphi$. Existence and other fundamental properties of the solutions of (1) under the conditions above are found in [1–3, 5, 6].

DEFINITION. (a) The solutions of (1) are uniformly bounded if, for each $B_1>0$, there is a $B_2=B_2(B_1)>0$ such that, for $t_0\in\boldsymbol{R}_+$ and $\varphi\in C$ with $\|\varphi\|<B_1$, any solution $x(t, t_0, \varphi)$ satisfies $|x(t, t_0, \varphi)|<B_2$ for $t\geqq t_0$.

(b) The solutions of (1) are uniformly ultimately bounded if there are a $B>0$ and, for each $B_3>0$, a $T=T(B_3)>0$ such that, for $t_0\in\boldsymbol{R}_+$ and $\varphi\in C$ with $\|\varphi\|<B_3$, any solution $x(t, t_0, \varphi)$ satisfies $|x(t, t_0, \varphi)|<B$ for $t\geqq t_0+T$.

For a continuous functional $V: \boldsymbol{R}_+\times C\to\boldsymbol{R}_+$, the derivative of $V(t, \varphi)$ along a solution $x(t)$ of (1) is defined by

$$V'_{(1)}(t, x_t)=\limsup_{\Delta t\to+0}\,[V(t+\Delta t, x_{t+\Delta t})-V(t, x_t)]/\Delta t\,.$$

Throughout this paper, W, $W_i: \boldsymbol{R}_+\to\boldsymbol{R}_+$ are continuous and strictly increasing functions with $W(0)=W_i(0)=0$ and $\lambda_i: \boldsymbol{R}_+\to\boldsymbol{R}_+$ are continuous.

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2. Main results.

We show the uniform boundedness and uniform ultimate boundedness under the condition that the right hand side of (1) relates the derivative of the Liapunov functional. We note that the term $\inf\{|x(s)| : t-h \leqq s \leqq t\}$ is weaker than usual $|x(t)|$ or L^p-norm. We used this infimum term to obtain stability properties earlier in [4].

THEOREM. *Suppose that a continuous functional $V\colon \mathbf{R}_+ \times C \to \mathbf{R}_+$ satisfies*

$$0 \leqq V(t, \varphi) \leqq W(\|\varphi\|),$$

and

$$V'_{(1)}(t, x_t) \leqq \begin{cases} -\mu|f(t, x_t)| - \lambda_1(t), & \text{if } \inf\{|x(s)| : t-h \leqq s \leqq t\} \geqq \gamma, \\ -\mu|f(t, x_t)| + \lambda_2(t), & \text{if } |x(t)| \geqq \gamma, \\ \lambda_3(t), & \text{if } |x(t)| < \gamma, \end{cases}$$

for some μ, $\gamma > 0$, with

$$\inf_{t \in \mathbf{R}_+} \int_t^{t+h} \lambda_1(s)ds > \sup_{t \in \mathbf{R}_+} \int_t^{t+h} \lambda_2(s)ds$$

and

$$\sup_{t \in \mathbf{R}_+} \int_t^{t+h} \lambda_3(s)ds < \infty .$$

Then the solutions of (1) *are uniformly bounded and uniformly ultimately bounded.*

PROOF. Let

$$c_1 = \inf_{t \in \mathbf{R}_+} \int_t^{t+h} \lambda_1(s)ds ,$$

and

$$c_i = \sup_{t \in \mathbf{R}_+} \int_t^{t+h} \lambda_i(s)ds \qquad \text{for} \quad i = 2, 3 .$$

To show uniform boundedness, let $B_1 > 0$ be given. For $t_0 \in \mathbf{R}_+$ and $\varphi \in C$ with $\|\varphi\| < B_1$, let $x(t) = x(t, t_0, \varphi)$ be a solution on $[t_0, \beta)$ and $v(t) = V(t, x_t)$. Take $B_4 = B_1 + \gamma + (2c_2 + c_3)/\mu$, $B_5 = W(B_4) + 2c_2 + c_3$ and $B_2 = B_4 + (B_5 + c_2)/\mu$. Suppose, for contradiction, that there is a $t_1 \in (t_0, \beta)$ such that $|x(t_1)| = B_2$. Then, since $|x(t_0)| < B_1 < B_4$ and $B_2 > B_4$, for some $t_2 \in (t_0, t_1)$, $|x(t_2)| = B_4$ and $|x(t)| > B_4$ on $(t_2, t_1]$. Either (i) $t_1 \leqq t_2 + h$ or (ii) $t_1 > t_2 + h$. If (i) holds, then

$$
\begin{aligned}
0 \leqq v(t_1) &\leqq v(t_2) - \mu \int_{t_2}^{t_1} |f(t, x_t)|dt + \int_{t_2}^{t_1} \lambda_2(t)dt \\
&\leqq v(t_2) - \mu(|x(t_1)| - |x(t_2)|) + \int_{t_2}^{t_2+h} \lambda_2(t)dt \\
&\leqq v(t_2) - \mu(B_2 - B_4) + c_2 = v(t_2) - B_5
\end{aligned}
$$

which implies $v(t_2) \geqq B_5$. If (ii) holds, then $\inf\{|x(s)| : t-h \leqq s \leqq t\} \geqq \gamma$ on $[t_2+h, t_1]$ and

$$
\begin{aligned}
0 \leqq v(t_1) &\leqq v(t_2+h) - \mu \int_{t_2+h}^{t_1} |f(t, x_t)|dt \\
&\leqq v(t_2) - \mu \int_{t_2}^{t_2+h} |f(t, x_t)|dt + \int_{t_2}^{t_2+h} \lambda_2(t)dt - \mu \int_{t_2+h}^{t_1} |f(t, x_t)|dt \\
&\leqq v(t_2) - \mu \int_{t_2}^{t_1} |f(t, x_t)|dt + \int_{t_2}^{t_2+h} \lambda_2(t)dt \leqq v(t_2) - B_5 .
\end{aligned}
$$

Thus, we have

$$v(t_2) \geqq B_5$$

in either case. Since

$$v(t) \leqq v(t_0) + \int_{t_0}^{t} (\lambda_2(s) + \lambda_3(s))ds < W(B_1) + c_2 + c_3 < B_5$$

on $[t_0, t_0+h]$, $t_2 > t_0 + h$ and there is a $t_3 \in (t_0+h, t_2)$ such that $v(t_3) = B_5$ and $v(t) < B_5$ on $[t_0, t_3)$. Then, there is a $t_4 \in [t_0+h, t_3]$ with $|x(t_4)| < \gamma$. For, otherwise, $|x(t)| \geqq \gamma$ on $[t_0+h, t_3]$ and this implies

$$B_5 = v(t_3) \leqq v(t_0+h) + c_2 < W(B_1) + c_2 + c_3 + c_2 < B_5 ,$$

a contradiction. From $W(\|x_{t_3}\|) \geqq v(t_3) = B_5 \geqq W(B_4)$, it follows that $|x(t_5)| \geqq B_4$ for some $t_5 \in [t_3 - h, t_3]$. If $t_4 < t_5$, then there is a $t_6 \in (t_4, t_5)$ such that $|x(t_6)| = \gamma$ and $|x(t)| > \gamma$ on $(t_6, t_5]$. We have

$$
\begin{aligned}
B_5 = v(t_3) &\leqq v(t_5) + \int_{t_5}^{t_3} (\lambda_2(t) + \lambda_3(t))dt \\
&\leqq v(t_6) - \mu \int_{t_6}^{t_5} |f(t, x_t)|dt + \int_{t_6}^{t_5} \lambda_2(t)dt + c_2 + c_3 \\
&< B_5 - \mu(B_4 - \gamma) + 2c_2 + c_3 < B_5 ,
\end{aligned}
$$

a contradiction. The same argument also yields a contradiction if $t_4 > t_5$. Hence, $|x(t)| < B_2$ on $[t_0, \beta)$ with $\beta = \infty$, which is uniform boundedness.

Now, we show uniform ultimate boundedness. Let $B=B_2$ of uniform boundedness for $B_1=\gamma+(c_1+c_2+2c_3)/\mu$. For given $B_3>0$, $\varphi\in C$ with $\|\varphi\|<B_3$ and $t_0\in\boldsymbol{R}_+$, let $x(t)=x(t, t_0, \varphi)$ be a solution on $[t_0, \infty)$. Choose an integer N with $N(c_1-c_2)>W(B_3)$. Then there is an $1\leqq i\leqq N$ such that $v(t_0+2(i-1)h)<v(t_0+2ih)+c_1-c_2$. Otherwise,

$$\begin{aligned} 0&\leqq v(t_0+2Nh)\leqq v(t_0+2(N-1)h)-(c_1-c_2)\\ &\leqq v(t_0)-N(c_1-c_2)<W(B_3)-N(c_1-c_2)<0\,, \end{aligned}$$

which is a contradiction. Suppose that $|x(t)|\geqq\gamma$ on $[t_0+2(i-1)h, t_0+2ih]$. Then, $\inf\{|x(s)|: t-h\leqq s\leqq t\}\geqq\gamma$ on $[t_0+(2i-1)h, t_0+2ih]$, which implies

$$\begin{aligned} v(t_0+2ih)&\leqq v(t_0+(2i-1)h)-\int_{t_0+(2i-1)h}^{t_0+2ih}\lambda_1(t)dt\\ &\leqq v(t_0+2(i-1)h)+\int_{t_0+2(i-1)h}^{t_0+(2i-1)h}\lambda_2(t)dt-c_1<v(t_0+2ih)\,, \end{aligned}$$

a contradiction. Thus, there is a $t_7\in[t_0+2(i-1)h, t_0+2ih]$ with $|x(t_7)|<\gamma$. Suppose, for contradiction, that $\|x_{t_0+2ih}\|\geqq B_1$. Then, there is a $t_8\in[t_0+(2i-1)h, t_0+2ih]$ with $|x(t_8)|\geqq B_1$. We may assume $t_7<t_8$. Then there is a $t_9\in(t_7, t_8)$ such that $|x(t_9)|=\gamma$ and $|x(t)|>\gamma$ on $(t_9, t_8]$, and we have

$$\begin{aligned} v(t_0+2ih)&\leqq v(t_8)+\int_{t_8}^{t_0+2ih}(\lambda_2(t)+\lambda_3(t))dt\\ &\leqq v(t_9)-\mu\int_{t_9}^{t_8}|f(t, x_t)|dt+\int_{t_9}^{t_8}\lambda_2(t)dt+\int_{t_8}^{t_0+2ih}(\lambda_2(t)+\lambda_3(t))dt\\ &\leqq v(t_0+2(i-1)h)-\mu(B_1-\gamma)+\int_{t_0+2(i-1)h}^{t_0+2ih}(\lambda_2(t)+\lambda_3(t))dt\\ &<v(t_0+2ih)\,, \end{aligned}$$

a contradiction. Consequently $\|x_{t_0+2ih}\|<B_1$. Let $T=2Nh$, then $|x(t)|<B$ for $t\geqq t_0+T$. This completes the proof of uniform ultimate boundedness.

REMARK. We can easily see that our theorem is a generalization of Burton's theorem [1, Theorem 4.2.11].

COROLLARY. *If there is a continuous functional* $V: \boldsymbol{R}_+\times C\to\boldsymbol{R}_+$ *satisfying*

$$0\leqq V(t, \varphi)\leqq W(\|\varphi\|)\,,$$

and

$$\begin{aligned} V'_{(1)}(t, x_t)\leqq &-\mu|f(t, x_t)|-\lambda_4(t)W_3(\inf\{|x(s)|: t-h\leqq s\leqq t\})\\ &-\lambda_5(t)W_4(|x(t)|)+\lambda_3(t) \end{aligned}$$

where $\mu>0$, $\lambda_1(t):=\lambda_4(t)W_3(\gamma)+\lambda_5(t)W_4(\gamma)-\lambda_3(t)\geqq 0$ *for some constant* $\gamma>0$, *for* $\lambda_2(t):=\max[\lambda_3(t)-\lambda_5(t)W_4(\gamma), 0]$

$$\inf_{t\in \mathbf{R}_+}\int_t^{t+h}\lambda_1(s)ds>\sup_{t\in \mathbf{R}_+}\int_t^{t+h}\lambda_2(s)ds\,,$$

and

$$\sup_{t\in \mathbf{R}_+}\int_t^{t+h}\lambda_3(s)ds<\infty\,,$$

then the solutions of (1) *are uniformly bounded and uniformly ultimately bounded.*

EXAMPLE. *Consider the scalar equation*

$$x'=-a(t)x^m(t)+b(t)x^m(t-h)+p(t)$$

where $m>0$ *is odd,* $a: \mathbf{R}_+\to\mathbf{R}_+$, b, $p: \mathbf{R}_+\to\mathbf{R}$ *are continuous,* $\int_t^{t+h}|b(s)|ds$ *and* $\int_t^{t+h}|p(s)|ds$ *are bounded for* $t\in\mathbf{R}_+$. *Suppose that the following conditions are satisfied:*

(i) $a(t)\geqq\alpha|b(t+h)|$
for some $\alpha>1$;

(ii) $\eta(t):=a(t)-\alpha|b(t+h)|+\beta|b(t)|-|p(t)|\geqq 0$
for some $0\leqq\beta<\alpha-1$;

(iii) $$\inf_{t\in \mathbf{R}_+}\int_t^{t+h}\eta(s)ds>\sup_{t\in \mathbf{R}_+}\int_t^{t+h}(|p(s)|-(a(s)-\alpha|b(s+h)|))ds\,.$$

Then the solutions are uniformly bounded and uniformly ultimately bounded.

PROOF. Let

$$V(t,\varphi)=|\varphi(0)|+\frac{2\alpha}{\alpha+1-\beta}\int_{-h}^{0}|b(t+s+h)||\varphi(s)|^m ds\,.$$

Then $V(t,\varphi)\leqq W(\|\varphi\|)$ for some function W and

$$\begin{aligned}V'(t,x_t)\leqq&-(a(t)-(2\alpha/(\alpha+1-\beta))|b(t+h)|)|x(t)|^m\\&-((\alpha-1+\beta)/(\alpha+1-\beta))|b(t)||x(t-h)|^m+|p(t)|\\\leqq&-((\alpha-1-\beta)/(\alpha+1-\beta))|f(t,x_t)|\\&-(2/(\alpha+1-\beta))(a(t)-\alpha|b(t+h)|)|x(t)|^m\\&-(2\beta/(\alpha+1-\beta))|b(t)||x(t-h)|^m\\&+(2(\alpha-\beta)/(\alpha+1-\beta))|p(t)|\end{aligned}$$

where $f(t,x_t)$ is the right-hand side of the equation. Take $\gamma=\alpha-\beta$ and $\lambda_1(t)=$

$(2\gamma/(\alpha+1-\beta))\eta(t)$, then all the conditions of the corollary are satisfied and the solutions are uniformly bounded and uniformly ultimately bounded.

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