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Determining the Levels of Some Special Complexity Classes of Sets in the Kleene Arithmetical Hierarchy

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Abstract. We shall determine their levels of some special classes of sets of strings such as $\{X \subseteq \Sigma^* : \mathbf{P}[X] \neq \mathbf{NP}[X]\}$ in the Kleene arithmetical hierarchy on subclasses of $\mathscr{P}(\Sigma^*)$. We shall show that such several classes are proper Π_2^0, that is, they are Π_2^0 but not Σ_2^0.

Introduction.

We consider classification of some special classes of sets of strings such as $\{X \subseteq \Sigma^* : \mathbf{P}[X] \neq \mathbf{NP}[X]\}$. That is, we determine their levels in the Kleene arithmetical hierarchy on subclasses of $\mathscr{P}(\Sigma^*)$. At first glance, this class is Σ_3^0, but by using an $\mathbf{NP}[X]$-complete set, it is seen that this class is Π_2^0. For the notions and notations used above, see the following sections.

The classes we shall treat with are the following, where X ranges over subsets of Σ^*:

$\mathbf{E0} = \{X : \mathbf{P}[X] \neq \mathbf{NP}[X]\}$,
$\mathbf{E1} = \{X : \mathbf{coNP}[X] \neq \mathbf{NP}[X]\}$,
$\mathbf{E2} = \{X : \mathbf{DEXT}[X] \neq \mathbf{NEXT}[X]\}$,
$\mathbf{E3} = \{X : \mathbf{coNEXT}[X] \neq \mathbf{NEXT}[X]\}$,
$\mathbf{E4} = \{X : \mathbf{P}[X] \neq \mathbf{PH}[X]\}$,
$\mathbf{E5} = \{X : \mathbf{NP}[X] \neq \mathbf{PH}[X]\}$,
$\mathbf{E6} = \{X : \mathbf{NP}[X] \neq \mathbf{PSPACE}[X]\}$,
$\mathbf{E7} = \{X : \mathbf{NP}[X] \neq \mathbf{EXPTIME}[X]\}$,
$\mathbf{E8} = \{X : \mathbf{PH}[X] \neq \mathbf{PSPACE}[X]\}$, and
$\mathbf{E9} = \{X : \mathbf{PSPACE}[X] \neq \mathbf{EXPTIME}[X]\}$.

Their inclusion relation is as follows: $\mathbf{E1} \subset \mathbf{E0}$ ([BGS 75]), here $\subset$ means the proper inclusion. $\mathbf{E3} \subset \mathbf{E2}$ (it can be shown that there exists a recursive oracle A such that $\mathbf{DEXT}[A] \neq \mathbf{NEXT}[A] = \mathbf{coNEXT}[A]$). And $\mathbf{E2} \subset \mathbf{E0}$ ([BWM 82]). Since $\mathbf{NP}[X] \subseteq \mathbf{PH}[X] \subseteq \mathbf{PSPACE}[X] \subseteq \mathbf{EXPTIME}[X]$, we have $\mathbf{E5}$, $\mathbf{E8} \subseteq \mathbf{E6} \subseteq \mathbf{E7}$, and $\mathbf{E9} \subseteq \mathbf{E7}$.

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Since **P**$[X]$ = **NP**$[X]$ (resp. **coNP**$[X]$ = **NP**$[X]$) implies **P**$[X]$ = **PH**$[X]$ (resp. **NP**$[X]$ = **PH**$[X]$), we have **E0** = **E4** and **E1** = **E5**. Clearly, **E5** $\neq$ **E7**. Also, **E6** $\neq$ **E0** ([BDG 90; p. 156]). All **E***i*'s are not empty. For example, for **E9** $\neq \phi$, see e.g., [Orp 83], though Orponen gives a stronger result. As seen below, they are all co-meager. Further, it is well-known that the complement $\neg$**E0** is not empty ([BGS 75]), and also $\neg$**E7** is not empty ([De 76], [He 84]). Therefore, all $\neg$**E***i*'s are not empty. These facts are needed below in this paper.

SUMMARY:

$$\mathbf{E1} \cup \mathbf{E8} \subseteq \mathbf{E6} \subseteq \mathbf{E7} = \mathbf{E1} \cup \mathbf{E8} \cup \mathbf{E9},$$

$$\mathbf{E1} = \mathbf{E5}, \quad \mathbf{E6} \subset \mathbf{E0} = \mathbf{E4},$$

$$\mathbf{E3} \subset \mathbf{E2} \subset \mathbf{E0}.$$

The aim of this paper is to show that all classes **E***i*'s are Π_2^0 but not Σ_2^0, in fact not even F_σ.

§1. Preliminaries.

We use standard notations for structural theory of complexity and recursion theory (see, e.g., [BDG 88], [BDG 90], and [Ro 67]). Let $\Sigma = \{0, 1\}$ be the alphabet, and Σ^* the set of all finite strings over Σ with empty string λ. The elements of Σ^* can be enumerated as follows:

$$\lambda, 0, 1, 00, 01, 10, 11, 000, 001, \cdots.$$

We denote the $(n+1)$st string in the enumeration by z_n. For $X \subseteq \Sigma^*$, sometimes X is identified with the characteristic function $X(n) = 1$ if $z_n \in X$, and $= 0$ otherwise. w, x, y, and z denote strings. Let N be the set of all natural numbers. i, j, k, m, and n denote members of N. Let $\mathscr{P}(\Sigma^*)$ be the class of all subsets of Σ^*. X and Y denote members of $\mathscr{P}(\Sigma^*)$, and with some exceptions we call *classes* subsets of $\mathscr{P}(\Sigma^*)$. As usual, we regard it as the Cantor space. That is, let w be the string $w(0)w(1)\cdots w(n-1)$, where each $w(i)$ is 0 or 1. Then, the basic open sets are $\{U_w : w \in \Sigma^*\}$, where $U_w = \{X : X(i) = w(i)$ for $i = 0, 1, \cdots, n-1\}$.

Let **E** be a class, that is **E** $\subseteq \mathscr{P}(\Sigma^*)$. **E** is Π_2^0 if it can be expressed in the form

$$X \in \mathbf{E} \Leftrightarrow \forall y \exists z R(X, y, z),$$

where $R(X, y, z)$ is a recursive relation ([Ro 67; §15], though Rogers uses the notation $\Pi_2^{(s)}$ instead of Π_2^0). Similarly for Π_k^0 $(k > 0)$. And **E** is Σ_2^0 when it is of the dual form:

$$X \in \mathbf{E} \Leftrightarrow \exists y \forall z R(X, y, z).$$

Similarly for Σ_k^0 $(k > 0)$. **E** is F_σ if it is a countable union of closed sets, and **E** is G_δ if its complement $\neg$**E** $(= (\Sigma^*) - \mathbf{E})$ is F_σ. Here we temporaily use the word 'sets' for subsets of $\mathscr{P}(\Sigma^*)$ according to the traditional usage. Clearly, each Σ_2^0 set is F_σ and each Π_2^0

set is G_δ, but not vice versa.

E is dense if it intersects every basic open set. **E** is nowhere dense if every basic open set contains a basic open set which is disjoint with **E**. **E** is meager if it is a countable union of nowhere dense sets. **E** is co-meager if $\neg$**E** is meager. By the Baire Category Theorem, in $\mathscr{P}(\Sigma^*)$ every co-meager set is not meager.

The special complexity classes such as **P**[X], **NP**[X], etc. occurred in the definitions of our **E**i's will be explained in §3. For further information about these classes, see, e.g., the textbooks: [BDG 88] and [BDG 90].

Prior to our results, similar results (but different from ours) appeared in [Ha 77] and [Gr 80]. For example, Grant showed that $\{i \in N : \phi_i$ is total and $\mathbf{P}[\phi_i] \neq \mathbf{NP}[\phi_i]\}$ is Π_2^0-complete, where $\{\phi_i : i \in N\}$ is a standard enumeration of the partial recursive functions, and Π_2^0 is one of the second levels in the Kleene arithmetical hierarchy on subsets of N (see [Ro 67; §14]; though Rogers uses Π_2 instead of Π_2^0).

§2. The main theorem.

Let $\mathbf{C}[\sim]$ be a class of oracle-dependent sets. $\mathbf{C}[\sim]$ is *recursively presentable* if there is an enumeration of oracle Turing machines $\{M_0^{\sim}, M_1^{\sim}, \cdots, M_k^{\sim}, \cdots\}$ such that for every oracle X

$$\mathbf{C}[X] = \{L(M_k^X) : k \in N\}, \tag{1}$$

where $L(M)$ denotes the set of all strings accepted by the machine M, and (2) the relation "M_k^X accepts y" is recursive with respect to k, y, and oracle X. (We call (2) the *recursive condition* for the enumeration $\{M_k^{\sim} : k \in N\}$.)

This is the relativized version of recursive presentability in [Sch 82].

An oracle-dependent set $H(X)$ is $\mathbf{C}[X]$-*complete* with respect to p-m-reduction [resp. linear reduction] if $H(X) \in \mathbf{C}[X]$ and for each $L \in \mathbf{C}[X]$ there is a function $f: \Sigma^* \to \Sigma^*$ (independent of X) computable in polynomial time [resp. in linear time] of the length of the input such that for every y

$$y \in L \Leftrightarrow f(y) \in H(X).$$

Since $H(X)$ is in $\mathbf{C}[X]$, the relation "$y \in H(X)$" is recursive with respect to y and X. For $\mathbf{C}[X]$, let $\mathbf{coC}[X] = \{L : \neg L \in \mathbf{C}[X]\}$, where $\neg L = \Sigma^* - L$. $\mathbf{C}[X]$ is *polynomially closed* [resp. *linearly closed*] if $f^{-1}(L) \in \mathbf{C}[X]$ for every $L \in \mathbf{C}[X]$ and for every f computable in polynomial time [resp. in linear time].

Let $X \fallingdotseq Y$ mean that the symmetric difference $X \triangle Y$ is finite. **E** is *closed under finite variation* if $X \in \mathbf{E} \Leftrightarrow Y \in \mathbf{E}$ whenever $X \fallingdotseq Y$. Then, clearly we have

LEMMA 2.1. *If* **E** *is closed under finite variation, then so is* $\neg$**E**. *And further, if* **E** *is not empty, then it is dense.* □

THEOREM 1. *Let* $\mathbf{B}[\sim]$ *and* $\mathbf{C}[\sim]$ *be recursively presentable classes, and let*

$\mathbf{E}=\{X:\mathbf{B}[X]\neq\mathbf{C}[X]\}$. *Suppose that the following conditions are satisfied:*

(a) (a1) $\mathbf{B}[X]\subseteq\mathbf{C}[X]$ *for all* X, *or* (a2) $\mathbf{B}[X]=\mathbf{coC}[X]$ *for all* X,

(b) *there exists a* $\mathbf{C}[X]$*-complete set* $H(X)$ *with respect to either* (b1) *p-m-reduction or* (b2) *linear reduction,*

(c) (c1) $\mathbf{B}[X]$ *is polynomially closed, or* (c2) *it is linearly closed,*

(d) $\mathbf{E}$ *is neither meager nor the whole space* $\mathscr{P}(\Sigma^*)$, *and finally*

(e) $\mathbf{E}$ *is closed under finite variation.*

Then, $\mathbf{E}$ *is proper* Π^0_2; *in fact, it is not* F_σ. *Here we combine* (b1) *with* (c1), *and* (b2) *with* (c2).

LEMMA 2.2. *Let* $\mathbf{E}$ *be* F_σ *and assume that it is not meager. Then,* $\mathbf{E}$ *intersects every dense* $\mathbf{D}$: $\mathbf{E}\cap\mathbf{D}\neq\varnothing$.

PROOF. Since $\mathbf{E}$ is F_σ, it can be expressed as follows:

$$\mathbf{E}=\bigcup_{k=0}^{\infty}\mathbf{A}_k,$$

where each $\mathbf{A}_k$ is closed. Since $\mathbf{E}$ is not meager, there is a k such that $\mathbf{A}_k$ is not nowhere dense. So, the closure of $\mathbf{A}_k$ ($=\mathbf{A}_k$ itself) contains a basic open set. Hence, the $\mathbf{A}_k$ intersects every dense set, a fortiori so does $\mathbf{E}$. □

PROOF OF THEOREM 1. We consider the case (a1), (b1), and (c1). Then we have

$$X\notin\mathbf{E}\Leftrightarrow H(X)\in\mathbf{B}[X]. \tag{3}$$

For, suppose $H(X)\in\mathbf{B}[X]$, and let $L\in\mathbf{C}[X]$ be arbitrary. Then, there is a polynomial time computable function f such that for any y

$$y\in L\Leftrightarrow f(y)\in H(X).$$

Since $\mathbf{B}[X]$ is polynomially closed, we have $L\in\mathbf{B}[X]$. So, $\mathbf{C}[X]\subseteq\mathbf{B}[X]$, and hence $\mathbf{B}[X]=\mathbf{C}[X]$. Therefore, $X\notin\mathbf{E}$. The forward direction of (3) is clear. Now, by (3), we have

$$X\in\mathbf{E}\Leftrightarrow\neg\exists k\forall y[y\in H(X)\leftrightarrow M_k^X \text{ accepts } y],$$

where $M_k^{\sim}$'s are the oracle Turing machines associated with $\mathbf{B}[\sim]$ in the definition of its recursive presentability. This shows $\mathbf{E}$ is Π^0_2. Similarly, if (a2) holds instead of (a1), then again we have (3), since $\mathbf{C}[X]\subseteq\mathbf{coC}[X]$ implies $\mathbf{coC}[X]=\mathbf{C}[X]$. Hence, $\mathbf{E}$ is Π^0_2 also.

Now, suppose that $\mathbf{E}$ is F_σ. Since $\neg\mathbf{E}$ is nonempty and closed under finite variation, it is dense, by Lemma 2.1. Since $\mathbf{E}$ is not meager, by Lemma 2.2, we have $\mathbf{E}\cap\neg\mathbf{E}\neq\phi$. This is a contradiction. Consequently, $\mathbf{E}$ can not be F_σ. Similarly for the case that (b2) and (c2) hold. □

§3. Determining the levels of **E***i*'s.

Now, using Theorem 1, we shall show that all **E***i*'s are proper Π_2^0 classes.

Let $P_k^{\sim}$ [resp. $NP_k^{\sim}$] be the k-th deterministic [resp. nondeterministic] polynomial time bounded oracle Turing machine such that the enumeration $\{P_k^{\sim} : k \in N\}$ [resp. $\{NP_k^{\sim} : k \in N\}$] satisfies the recursive condition. Let $E_k^{\sim}$ [resp. $NE_k^{\sim}$] be the k-th deterministic [resp. nondeterministic] 2^{lin} time bounded oracle Turing machine such that the enumeration satisfies the recursive condition, where 2^{lin} means 2^{cn} for some constant numbers c. Let $EP_k^{\sim}$ be the k-th deterministic 2^{poly} time bounded oracle Turing machine such that the enumeration satisfies the recursive condition, where 2^{poly} means $2^{p(n)}$ for some polynomials $p(n)$. Let $PS_k^{\sim}$ be the k-th polynomial space bounded oracle Turing machine such that the enumeration satisfies the recursive condition. We borrow $H_n^{\sim}$ from Schöning's paper [Sch 82; p. 99] in the relativized form. This enumeration also satisfies the recursive condition. Then we have

$\mathbf{P}[X] = \{L(P_k^X) : k \in N\}$,
$\mathbf{NP}[X] = \{L(NP_k^X) : k \in N\}$,
$\mathbf{DEXT}[X] = \{L(E_k^X) : k \in N\}$,
$\mathbf{NEXT}[X] = \{L(NE_k^X) : k \in N\}$,
$\mathbf{PH}[X] = \{L(H_k^X) : k \in N\}$,
$\mathbf{PSPACE}[X] = \{L(PS_k^X) : k \in N\}$, and
$\mathbf{EXPTIME}[X] = \{L(EP_k^X) : k \in N\}$.

The classes $\mathbf{P}[X]$, $\mathbf{NP}[X]$, etc. (including $\mathbf{coNP}[X]$ and $\mathbf{coNEXT}[X]$) occurred in the definitions of **E***i*'s are all recursively presentable ([Sch 82] for non-relativized forms).

Let $K(X)$, $KE(X)$, $KS(X)$, and $JE(X)$ be as follows:

$K(X) = \{0^k 1x10^n : \text{Some computation of } NP_k^X \text{ accepts } x \text{ in } \leqq n \text{ steps}\}$,
$KE(X) = \{0^k 1x10^n : \text{Some computation of } NE_k^X \text{ accepts } x \text{ in } \leqq 2^n \text{ steps}\}$,
$KS(X) = \{0^k 1x10^n : PS_k^X \text{ accepts } x \text{ in } \leqq n \text{ spaces}\}$, and
$JE(X) = \{0^k 1x10^n : EP_k^X \text{ accepts } x \text{ in } \leqq 2^n \text{ steps}\}$.

Then, $K(X)$, $KS(X)$, and $JE(X)$ are $\mathbf{NP}[X]$-complete, $\mathbf{PSPACE}[X]$-complete, and $\mathbf{EXPTIME}[X]$-complete with respect to p-m-reduction, respectively. $KE(X)$ is $\mathbf{NEXT}[X]$-complete with respect to linear reduction.

All the complexity classes occurred in the definitions of **E***i*'s are either polynomially closed or linearly closed, and they all are closed under finite variation.

Now, we use Poizat's result [Po 86]. So, we state an outline of parts of his paper with some slight modification.

We consider arithmetical predicates (i.e., Σ_k^0 or Π_k^0 predicates for some k) of the form $\phi(X)(u)$, where X ranges over $\mathscr{P}(\Sigma^*)$ and u over Σ^*, as before. $\phi(X)(u)$ is *finitely testable* if there exists a number-theoretic function $\alpha : N \to N$ such that for any string u and any set X

$$\forall n \geqq \alpha(|u|)[\phi(X)(u) \leftrightarrow \phi(X \mid n)(u)],$$

where $X \mid n$ is the initial n-segment of X.

Let $\mathbf{C}(X)$ be a set of arithmetical predicates of the form $\phi(X)(u)$. For $\phi(X)(u)$, let

$$\phi[X] = \{u \in \Sigma^* : \phi(X)(u) \text{ holds}\} ,$$

and let

$$\mathbf{C}[X] = \{\phi[X] : \phi(X)(u) \in \mathbf{C}(X)\} .$$

Poizat considers the following 4 hypotheses:

Hypothesis 1. Each predicate in $\mathbf{C}(X)$ is finitely testable.

Hypothesis 2. If $X \doteqdot Y$, then $\mathbf{C}[X] = \mathbf{C}[Y]$.

Hypothesis 3. For any $A \in \mathbf{C}[X]$, if $B \doteqdot A$ then $B \in \mathbf{C}[X]$.

Hypothesis 4. There is a mapping $\# : \mathscr{P}(\Sigma^*) \to \mathscr{P}(\Sigma^*)$ such that (a) $\mathbf{C}[X] = \mathbf{C}[\#X]$, and (b) for any $A \in \mathbf{C}[X]$ there exists a predicate ψ in $\mathbf{C}(X)$ such that $A = \psi[\#X]$ and it has the following property: if $Y \doteqdot \#Z$, then $\psi[Y] \doteqdot \psi[\#Z]$. (In [Po 86], Poizat imposes a stronger condition: if $Y \doteqdot Z$ then $\psi[Y] \doteqdot \psi[Z]$. However, it may be hard to show that any given concrete class satisfies this condition. This modification does not affect the following Theorem.)

Then

POIZAT'S THEOREM. *Let* $\mathbf{C}(X)$ *and* $\mathbf{D}(X)$ *be two sets of arithmetical predicates of the form* $\phi(X)(u)$ *which satisfy the Hypothese* $1 \sim 4$ *with the same mapping* $\# : X \mapsto \#X$. *Let* $\mathbf{C}[X]$ *and* $\mathbf{D}[X]$ *be the corresponding classes of sets, as before. Suppose that there exists an oracle A such that* $\mathbf{C}[A] \neq \mathbf{D}[A]$. *Then, the set* $\{X : \mathbf{C}[X] \neq \mathbf{D}[X]\}$ *is co-meager.*

In order to apply our Theorem 1 we must show that all **E***i*'s are not meager. For this purpose it suffices to show that all **E***i*'s are co-meager. Bennett-Gill [BG 81] noted that **E0** and **E1** are co-meager, and Babai [Ba 87] noted that **E8** is co-meager by applying the Poizat theorem. However, since the Hypothesis 4 needs a slight correction, here we show, as an example, that **E9** is co-meager. As stated before, the class **E9** is not empty, that is, there is an oracle A such that **PSPACE**$[A] \neq$ **EXPTIME**$[A]$. So, for our purpose it suffices to show that both **PSPACE**(X) and **EXPTIME**(X) satisfy the Hypotheses $1 \sim 4$ with the same mapping #.

We do this for **EXPTIME**(X) only. Proofs for other sets are similar.

Let $\phi_i(X)(u) \Leftrightarrow EP_i^X$ accepts u. Then

EXPTIME$(X) = \{\phi_i(X)(u) : i \in N\}$, and

EXPTIME$[X] = \{\phi_i[X] : i \in N\} = \{L(EP_i^X) : i \in N\}$.

Now,

Hypothesis 1. For $\phi_i(X)(u)$, we can take $\alpha(n) = 2^{\beta(n)+1} - 1$, as the α in the definition of finite testability, where $\beta(n) = 2^{p_i(n)}$ is the time bound function for the machine $EP_i^{\sim}$. Because the maximal number of strings of length n in the enumeration of the members of Σ^* is $2^{n+1} - 2$.

Hypothesis 2. Suppose $X \doteqdot Y$, and let $A \in$ **EXPTIME**$[X]$. So, for some i, $u \in A$

iff $\phi_i(X)(u)$. Since $X \doteqdot Y$, there exists a *linear* time bounded oracle Turing machine $M^{\sim}$ such that $X = L(M^Y)$. Then we can readily find an index k such that $EP_i^{M^Y} = EP_k^Y$. Here this equality means that two machines accept the same language. So, $A \in \mathbf{EXPTIME}[Y]$, and hence $\mathbf{EXPTIME}[X] \subseteq \mathbf{EXPTIME}[Y]$. Similarly for the reverse inclusion.

Hypothesis 3. Let $A \in \mathbf{EXPTIME}[X]$, and suppose $B \doteqdot A$. So, there is an index i and a natural number m such that

$$u \in A \text{ iff } \phi_i(X)(u) \text{ iff } EP_i^X \text{ accepts } u, \text{ and } \forall n \geqq m[B(n) = A(n)].$$

Then we shall find a 2^{poly} time bounded oracle Turing machine $T^{\sim}$ such that

$$u \in B \Leftrightarrow T^X \text{ accepts } u\,. \tag{4}$$

Here we use the notation ‘ ’ defined by ‘z_n’$=n$. First of all, for inputs u such that ‘u’$<m$, we define a segment of the machine $T^{\sim}$ by a finite table so that for every input u with ‘u’$<m$ the segment satisfies the condition (4). On any input u with ‘u’$\geqq m$, T^X simulates EP_i^X so that $T^X(u) = EP_i^X(u)$ holds. Then, (4) holds for each of these u. Thus we have $B \in \mathbf{EXPTIME}[X]$.

Hypothesis 4. Let $\#X = \pi(\Sigma^*, X)$. Here π is a pairing function: $\Sigma^* \times \Sigma^* \to \Sigma^*$ which is one-to-one onto and polynomially computable. Further, for given y we can compute the unique u and x in time $O(|y|)$ that $y = \pi(u, x)$.

(a) $\mathbf{EXPTIME}[X] = \mathbf{EXPTIME}[\#X]$. Proof. Let X be given, and suppose $A \in \mathbf{EXPTIME}[X]$. So, there is an index i such that $u \in A \Leftrightarrow EP_i^X$ accepts u. Then, we must find a 2^{poly} time bounded oracle Turing machine $T^{\sim}$ such that

$$u \in A \Leftrightarrow T^{\#X} \text{ accepts } u\,. \tag{5}$$

For any set Y, let $\rho(Y) = \{x \in \Sigma^* : \exists u, y \in \Sigma^*[y \in Y \wedge y = \pi(u, x)]\}$. Then $\rho(Y) = X$ if $Y = \#X$. Now, given input u, T^Y begins to simulate the computation of $EP_i^{\sim}$ on u. Suppose that $EP_i^{\sim}$ enters the query state. Let w be the queried string. Then T^Y writes $\pi(u, w)$ on its oracle tape (this can be done in time $O(2^{p_i(|u|)})$), and queries whether $\pi(u, w) \in Y$. If the answer is yes, then $w \in \rho(Y)$ and so T^Y simulates the yes-branch of the computation of $EP_i^{\sim}$. Otherwise, it simulates the no-branch. After the whole simulation ends, T^Y gives the same output (=an accepting or rejecting state) as this simulation for $EP_i^{\sim}$. This is a *quasi*-simulation for $EP_i^{\rho(Y)}$ on u (it may not be an exact one, for there can be a case that $\pi(u, w) \notin Y$ but for other v $\pi(v, w) \in Y$ and $w \in \rho(Y)$). If Y is of the form $\#Z$, then certainly the output of T^Y is the same as that of EP_i^Z, since $\pi(u, w) \in Y$ iff $w \in Z$. So we have (5). The $T^{\sim}$ is a 2^{poly} time bounded oracle Turing machine. Hence we have $A \in \mathbf{EXPTIME}[\#X]$. Conversely, let $A \in \mathbf{EXPTIME}[\#X]$. Then for some k, $u \in A$ iff $\phi_k(\#X)(u)$. We define a 2^{poly} time bounded oracle Turing machine $M^{\sim}$ as follows: Given input u M^X simulates the computation of $EP_k^{\sim}$ on u. Suppose $EP_k^{\sim}$ enters the query state. Let y be the queried string. M^X calculates w such that $\pi(u, w) = y$. Recall that w is uniquely determined and can be computed in linear time of $|y| + |u|$. And it queries whether $w \in X$. After it enters yes- or no-state, it resumes simulating. Finally, it outputs the same value as $EP_k^{\sim}$. This $M^{\sim}$ is a 2^{poly} time bounded oracle Turing machine and for any u $u \in A$ iff M^X accepts u. Hence $A \in \mathbf{EXPTIME}[X]$.

(b) For each $A \in \mathbf{EXPTIME}[X]$, there is a predicate $\psi(X)(u)$ in $\mathbf{EXPTIME}[X]$ such that (b1) $A = \psi[\#X]$ and (b2) if $Y \doteqdot \#Z$ then $\psi[Y] \doteqdot \psi[\#Z]$. Proof. (b1) Let $A \in \mathbf{EXPTIME}[X]$. Then, there is an index i such that $u \in A$ iff $\phi_i(X)(u)$. We take the machine $T^{\sim}$ obtained in the proof of (a). As was shown above, $u \in A$ iff $T^{\#X}$ accepts u. Let $\psi(X)(u)$ be the predicate "T^X accepts u". Then $\psi(X)(u)$ is in $\mathbf{EXPTIME}(X)$, and we have $A = \psi[\#X]$. (b2) Suppose $Y \doteqdot \#Z$. Then, there is a number m (depending on Y and Z) such that

$$\forall u, w([|u| \geqq m \text{ or } |w| \geqq m] \Rightarrow [\pi(u, w) \in Y \text{ iff } \pi(u, w) \in \#Z \text{ iff } w \in Z]).$$

So, both computations of T^Y and $T^{\#Z}$ on u are identical with that of EP_i^Z on u for any u with $|u| \geqq m$. Hence, $\psi[Y] \doteqdot \psi[\#Z]$.

Thus, we have shown that $\mathbf{EXPTIME}(X)$ satisfies the four Hypotheses.

Consequently, it is seen that all **Ei**'s are co-meager and hence they are not meager. Hence, all the **Ei**'s satisfy the conditions (a) ~ (e) for **E** in Theorem 1. Therefore we have

THEOREM 2. *All the classes* Ei's *are* Π_2^0 *but not* Σ_2^0, *in fact not even* F_σ.

§4. Conclusion.

We have determined the levels of the classes **Ei**'s in the Kleene Arithmetical Hierarchy on subclasses of $\mathscr{P}(\Sigma^*)$. That is, they are proper Π_2^0 classes. However, there are other similar classes whose exact levels we do not know. For example, we want to know the exact level of the class $\mathbf{SEP} = \{X : \mathbf{P}[X] \neq \mathbf{BPP}[X]\}$. (For the definition of **BPP**[X], see [BDG 88] and [BDG 90].) By directly evaluating **SEP** based on the definition of **BPP**[X], we can see that **SEP** is a Σ_3^0 class. However we do not know whether it is Π_2^0, not even whether it is Π_3^0.

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