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Parallelogram Tilings and Jacobi-Perron Algorithm

Shunji ITO and Makoto OHTSUKI

Tsuda College

0. Introduction.

The most simple example of quasiperiodic tilings is known as follows (See [1]). Consider a line $l(\alpha)=\{(x, y) \mid x+\alpha y=0\}$ for each $\alpha\in[0, 1)$. Let $C(\alpha)$ be the set of squares (translates of the fundamental square) that l intersects, and let $S(\alpha)$ be the path along one side of the boundary of $C(\alpha)$. See Figure 1.

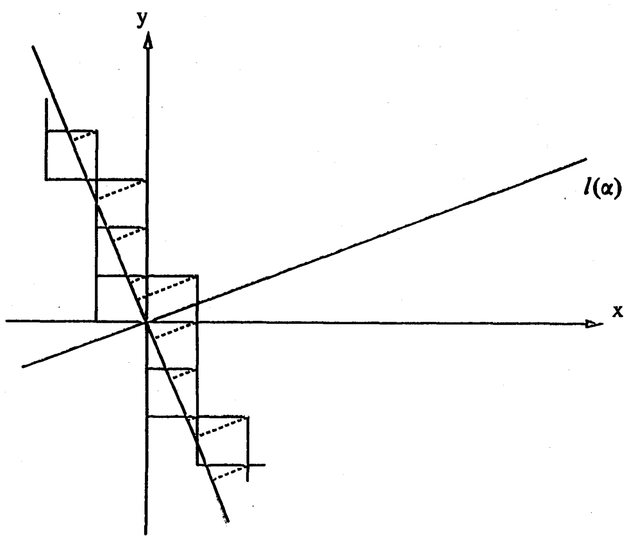

FIGURE 1. Figure of $l(\alpha)$ and projecting images of the vertical and horizontal edges along ${}^t(1, \alpha)$

If we project the stepped curve $S(\alpha)$ onto $l(\alpha)$ along the vector $(1, \alpha)$ then the images of the vertical and horizontal edges in $C(\alpha)$ form a tiling of $l(\alpha)$. We claim that the tiling is quasiperiodic iff α is irrational. We know also the generating method of the stepped curve $S(\alpha)$ by using the continued fraction algorithm and by introducing the substitutions of edges. In this paper we consider the stepped surface of a plane in $\boldsymbol{R}^3$

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constructed from three unit squares as an analogue of stepped curves. If we project the stepped surface onto the plane, then the images form a tiling of the plane constructed from three basis parallelograms. See Figure 2.

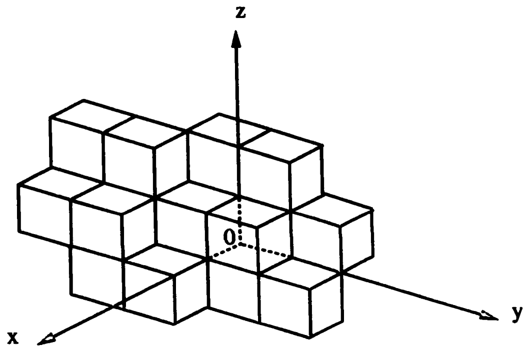


FIGURE 2. Stepped surfaces

The purpose of this paper is to give a generating method of the stepped surface by using Jacobi-Perron algorithm and by introducing substitutions of squares.

As an application of the generating method we have the following theorem:

THEOREM. *Let* $(1, \alpha, \beta)$ *be rationally independent, that is, if* $l+m\alpha+n\beta=0$ *for some* $l, m, n \in \boldsymbol{Z}$ *then* $l=m=n=0$. *Then the tiling of the orthogonal plane of* $(1, \alpha, \beta)$ *induced from the projection of the stepped surface is not periodic but quasi-periodic.*

1. Definition of Jacobi-Perron algorithm.

Let us define an algorithm called Jacobi-Perron algorithm by the following manner (See [2]). Let X be the domain given by $X=[0, 1)\times[0, 1)$ and let us define the transformation T on X by

$$T(\alpha, \beta)=\begin{cases} \left(\dfrac{\beta}{\alpha}-\left[\dfrac{\beta}{\alpha}\right], \dfrac{1}{\alpha}-\left[\dfrac{1}{\alpha}\right]\right) & \text{if } (\alpha, \beta)\in X-I \\ (0, \beta) & \text{if } (\alpha, \beta)\in I, \end{cases} \tag{1-1}$$

where I is given by $I=\{(0, \beta) \mid \beta\in[0, 1)\}$. By using the following integer valued functions

$$a(\alpha, \beta):=\left[\frac{\beta}{\alpha}\right] \quad \text{and} \quad b(\alpha, \beta):=\left[\frac{1}{\alpha}\right]$$

on $X-I$, we define for each $(\alpha, \beta)\in X-I$ a sequence of digits ${}^t(a_n, b_n)$ by

$${}^t(a_n, b_n):=(a(T^{n-1}(\alpha, \beta)), b(T^{n-1}(\alpha, \beta))) \qquad \text{if} \quad T^{n-1}(\alpha, \beta)\in X-I\,.$$

For each $(\alpha, \beta)\in X-I$ which has an infinite sequence of digits $\{{}^t(a_n, b_n) \mid n\geq 0\}$, the sequence satisfies the following properties:

(1) $a_n \in N \cup \{0\}$ and $b_n \in N$,

(2) $0 \leq a_n \leq b_n$,

(3) if $a_n = b_n$ then $a_{n+1} \neq 0$.

The triple $(X, T, (a(\alpha, \beta), b(\alpha, \beta)))$ is called *Jacobi-Perron algorithm*, and the sequence of digits of (α, β) is called the *name* of (α, β) with respect to Jacobi-Perron algorithm and we denote $(\alpha_n, \beta_n) := T^{n-1}(\alpha, \beta)$.

REMARK 1.1. We define a simple continued fraction algorithm S on I as follows: for each $(0, \beta) \in I$

$$S(0, \beta) = \begin{cases} \left(0, \dfrac{1}{\beta} - \left[\dfrac{1}{\beta}\right]\right) & \text{if } \beta \neq 0 \\ (0, 0) & \text{if } \beta = 0 . \end{cases}$$

For each $(0, \beta) \in I$ the sequence of digits, called partial quotient, is given by

$$b_n := \left[\frac{1}{\beta_{n-1}}\right]$$

where β_{n-1} is the second coordinate of $S^{n-1}(0, \beta)$. Therefore even if there exists an n such that $T^n(\alpha, \beta) \in I$ we are able to get the infinite squence of digits $\{b_{n+m} \mid m = 1, 2, \cdots\}$ by using S, except the case that $\beta \in Q$. In other words, we consider the simple continued fraction algorithm as a subalgorithm of Jacobi-Perron algorithm. In this paper, we will discuss this exceptional case in Remarks.

Let us introduce the family of matrices as follows: for each pair of integers (a, b)

$$A_{\binom{a}{b}} := \begin{pmatrix} b & 0 & 1 \\ 1 & 0 & 0 \\ a & 1 & 0 \end{pmatrix}. \tag{1-2}$$

Then we see

$${}^tA^{-1}_{\binom{a}{b}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -a & -b \\ 0 & 1 & 0 \end{pmatrix}. \tag{1-3}$$

Using these matrices, we have the following proposition.

PROPOSITION 1.1. *Assume that $T^k(\alpha, \beta) \notin I$, $0 \leq k \leq n-1$. Then we have*

$$\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \alpha\alpha_1 \cdots \alpha_{n-1} A_{\binom{a_1}{b_1}} A_{\binom{a_2}{b_2}} \cdots A_{\binom{a_n}{b_n}} \begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix},$$

where $\begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix}$ *is the name of* (α, β) *with respect to Jacobi-Perron algorithm.*

PROOF. From the definition of the algorithm (1-1) and (1-2), we have

$$\begin{pmatrix} 1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} = \frac{1}{\alpha} A^{-1}_{\binom{a_1}{b_1}} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} \qquad \text{if} \quad (\alpha, \beta) \notin I,$$

that is,

$$\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \alpha A_{\binom{a_1}{b_1}} \begin{pmatrix} 1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} \qquad \text{if} \quad (\alpha, \beta) \notin I.$$

Therefore, we have the conclusion.

REMARK 1.2. For the case of $\alpha = 0$, we introduce the family of matrices as follows:

$$A_b := \begin{pmatrix} b & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1-4}$$

Then from the definition of the continued fraction algorithm, we have

$$\begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix} = \frac{1}{\beta} A^{-1}_{b_1} \begin{pmatrix} 1 \\ 0 \\ \beta \end{pmatrix} \qquad \text{if} \quad \beta \neq 0.$$

Therefore, putting $(b_1, b_2, \cdots, b_n)$ the sequence of the partial quotients of β, we have

$$\begin{pmatrix} 1 \\ 0 \\ \beta \end{pmatrix} = \beta A_{b_1} \begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix} = \beta \beta_1 \cdots \beta_{m-1} A_{b_1} A_{b_2} \cdots A_{b_m} \begin{pmatrix} 1 \\ 0 \\ \beta_m \end{pmatrix},$$

where β_k is the second coordinate of $S^k(0, \beta)$.

For each n, let us introduce a transformation $\varphi_{\binom{a_n}{b_n}}$ from $\boldsymbol{R}^3$ whose coordinate is denoted by ${}^t(x_n, y_n, z_n)$ to $\boldsymbol{R}^3$ whose coordinate is denoted by ${}^t(x_{n-1}, y_{n-1}, z_{n-1})$ as follows:

$$\begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix} = \varphi_{\binom{a_n}{b_n}} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} := {}^t A^{-1}_{\binom{a_n}{b_n}} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}.$$

LEMMA 1.2. *Let* $(\alpha, \beta) \in X - I$ *and* (a_1, b_1) *be the first digits of* (α, β). *Then we have*

$$\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ \alpha_1 \\ \beta_1 \end{pmatrix}\right) = \alpha\left(\varphi_{\binom{a_1}{b_1}}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}\right). \tag{1-5}$$

The proof is easy from the definition of $\varphi_{\binom{a_n}{b_n}}$ and Proposition 1.1. Hereafter we denote also $\boldsymbol{\alpha}_n = {}^t(1, \alpha_n, \beta_n)$, $\boldsymbol{x}_n = {}^t(x_n, y_n, z_n)$. For each $(\alpha, \beta) \in X$, let us denote the plane which is orthogonal to $\boldsymbol{\alpha} = {}^t(1, \alpha, \beta)$ by

$$P(\alpha, \beta) = \{\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{\alpha}) = 0\}$$

and let us define the domain $\mathbb{P}(\alpha, \beta)$ by

$$\mathbb{P}(\alpha, \beta) = \{\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{\alpha}) > 0\}.$$

Then from Lemma 1.2 we have

COROLLARY 1.3. *On the same assumption as in Lemma* 1.2 *we have*

$$\varphi_{\binom{a_1}{b_1}}(P(\alpha_1, \beta_1)) = P(\alpha, \beta)$$

and

$$\varphi_{\binom{a_1}{b_1}}(\mathbb{P}(\alpha_1, \beta_1)) = \mathbb{P}(\alpha, \beta).$$

REMARK 1.3. In the case of $\alpha = 0$, we also define $\varphi_b : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ by

$$\varphi_b\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = {}^t A_b^{-1}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

Then Lemma 1.2 and Corollary 1.3 are also valid.

2. Substitutions on stepped surfaces.

Let E_1, E_2 and E_3 be unit squares spanned by $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, $\{\boldsymbol{e}_3, \boldsymbol{e}_1\}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, that is,

$$E_1 := \{\lambda \boldsymbol{e}_2 + \mu \boldsymbol{e}_3 \mid 0 \le \lambda, \mu \le 1\},$$
$$E_2 := \{\lambda \boldsymbol{e}_3 + \mu \boldsymbol{e}_1 \mid 0 \le \lambda, \mu \le 1\},$$
$$E_3 := \{\lambda \boldsymbol{e}_1 + \mu \boldsymbol{e}_2 \mid 0 \le \lambda, \mu \le 1\},$$

where

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For each $(\alpha, \beta) \in X$, let us define $\mathscr{S}(\alpha, \beta)$, which is a subset of $\boldsymbol{Z}^3 \times \{E_1, E_2, E_3\}$, as follows:

$$\mathscr{S}(\alpha,\beta):=\left\{(x,S)\;\middle|\;\begin{array}{c}S\in\{E_1,E_2,E_3\},\,x\in Z^3,\,x+S\subset \mathbb{P}(\alpha,\beta)\\ \text{and } x-e_i\notin \mathbb{P}(\alpha,\beta) \text{ if } S=E_i\end{array}\right\},$$

and let us define $\mathscr{G}(\alpha,\beta)$ to be the family of all finite subsets of $\mathscr{S}(\alpha,\beta)$, that is,

$$\mathscr{G}(\alpha,\beta):=\left\{\sum_{\lambda\in\Lambda}(x_\lambda,S_\lambda)\;\middle|\;\begin{array}{c}\sharp\Lambda<\infty,\,(x_\lambda,S_\lambda)\in\mathscr{S}(\alpha,\beta),\\ (x_\lambda,S_\lambda)\neq(x_{\lambda'},S_{\lambda'}) \text{ if } \lambda\neq\lambda'\end{array}\right\},$$

where an element of $\mathscr{G}(\alpha,\beta)$ is denoted as a formal sum. For each $(\alpha,\beta)\in X-I$, let us define a map $\Sigma_{\binom{a_1}{b_1}}$ from $\mathscr{G}(\alpha_1,\beta_1)$ to $\mathscr{G}(\alpha,\beta)$ as follows:

$$\Sigma_{\binom{a_1}{b_1}}:\begin{array}{ll}(0,E_1)\to & \sum_{1\le k\le b_1}(e_1-ke_2,E_1)+\sum_{1\le j\le a_1}(e_3-je_2,E_3)+(0,E_2)\\ (0,E_2)\to & (0,E_3)\\ (0,E_3)\to & (0,E_1)\end{array}$$

and for $(x_1,S)\in\mathscr{S}(\alpha_1,\beta_1)$, we define

$$\Sigma_{\binom{a_1}{b_1}}(x_1,S):=\varphi_{\binom{a_1}{b_1}}(x_1)+\Sigma_{\binom{a_1}{b_1}}(0,S)$$

and for $\sum_{\lambda\in\Lambda}(x_\lambda,S_\lambda)\in\mathscr{G}(\alpha_1,\beta_1)$, we define

$$\Sigma_{\binom{a_1}{b_1}}\left(\sum_{\lambda\in\Lambda}(x_\lambda,S_\lambda)\right):=\sum_{\lambda\in\Lambda}\left(\Sigma_{\binom{a_1}{b_1}}(x_\lambda,S_\lambda)\right),$$

where $y+(x,S)$ means $(y+x,S)$.

Then we see by the following two lemmas that $\Sigma_{\binom{a_1}{b_1}}$ is a map from $\mathscr{G}(\alpha_1,\beta_1)$ to $\mathscr{G}(\alpha,\beta)$. See Figure 3.

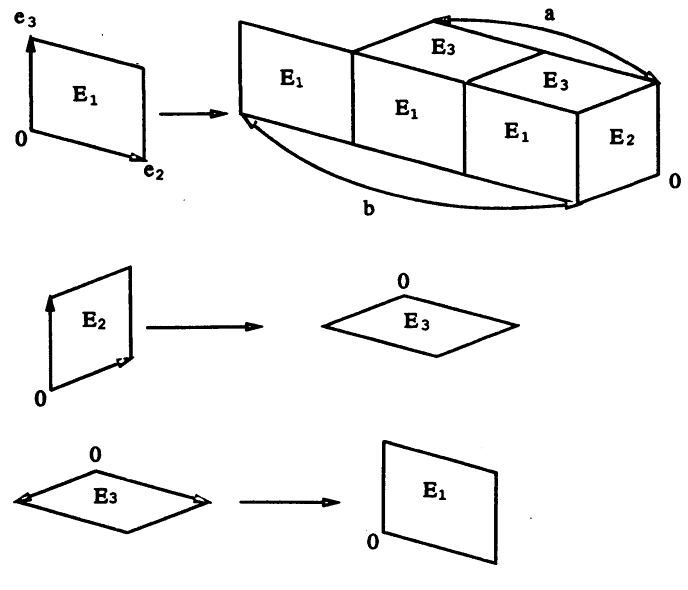


FIGURE 3. Figure of substitutions $\Sigma_{\binom{a_1}{b_1}}(0,E_i)$, $i=1,2,3$

LEMMA 2.1. *For each* $(\boldsymbol{x}_1, S_1) \in \mathscr{S}(\alpha_1, \beta_1)$, $\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, S_1)$ *is an element of* $\mathscr{G}(\alpha, \beta)$.

PROOF. (i) The case that $\boldsymbol{x}_1 + E_1 \subset \mathbb{P}(\alpha_1, \beta_1)$ and $\boldsymbol{x}_1 - \boldsymbol{e}_1 \notin \mathbb{P}(\alpha_1, \beta_1)$. Remarking $\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1 + E_1) \subset \mathbb{P}(\alpha, \beta)$ by Lemma 1.2, we see that

$$\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) \in \mathbb{P}(\alpha, \beta),$$
$$\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1 + \boldsymbol{e}_2) = \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) + \boldsymbol{e}_3 - a_1\boldsymbol{e}_2 \in \mathbb{P}(\alpha, \beta),$$
$$\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1 + \boldsymbol{e}_3) = \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) + \boldsymbol{e}_1 - b_1\boldsymbol{e}_2 \in \mathbb{P}(\alpha, \beta).$$

Therefore

$$\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) + \boldsymbol{e}_3 - j\boldsymbol{e}_2 \in \mathbb{P}(\alpha, \beta) \qquad (0 \le j \le a_1),$$
$$\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) + \boldsymbol{e}_1 - k\boldsymbol{e}_2 \in \mathbb{P}(\alpha, \beta) \qquad (0 \le k \le b_1).$$

Hence we have

$$\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, E_1) \subset \mathbb{P}(\alpha, \beta).$$

On the other hand, we see that

$$\begin{aligned}(\boldsymbol{\alpha}, \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) - \boldsymbol{e}_2) &= (\boldsymbol{\alpha}, \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1 - \boldsymbol{e}_1)) && \text{(by (1-3))}\\ &= \frac{1}{\alpha}(\boldsymbol{\alpha}_1, \boldsymbol{x}_1 - \boldsymbol{e}_1) && \text{(by Lemma 1.2)}\\ &< 0.\end{aligned}$$

Therefore, we see $\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) - \boldsymbol{e}_2 \notin \mathbb{P}(\alpha, \beta)$ and so $\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) - k\boldsymbol{e}_2 \notin \mathbb{P}(\alpha, \beta)$ for $1 \le k$. This means

$$\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, E_1) \subset \mathscr{S}(\alpha, \beta).$$

(ii) The case of $\boldsymbol{x}_1 + E_2 \subset \mathbb{P}(\alpha_1, \beta_1)$ and $\boldsymbol{x}_1 - \boldsymbol{e}_2 \notin \mathbb{P}(\alpha_1, \beta_1)$. From $\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, E_2) = \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) + E_3$, it is sufficient to see that

$$(\boldsymbol{\alpha}, \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) - \boldsymbol{e}_3) < 0$$

holds. We know from $(\boldsymbol{\alpha}_1, \boldsymbol{x}_1 - \boldsymbol{e}_2) < 0$, Lemma 1.2 and (1-3) that

$$\begin{aligned}0 &> (\boldsymbol{\alpha}, \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) - \varphi_{\binom{a_1}{b_1}}(\boldsymbol{e}_2))\\ &= (\boldsymbol{\alpha}, \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) + a_1\boldsymbol{e}_2 - \boldsymbol{e}_3).\end{aligned}$$

Therefore from $(\boldsymbol{\alpha}, \boldsymbol{e}_2) > 0$ we have

$$(\boldsymbol{\alpha}, \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) - \boldsymbol{e}_3) = (\boldsymbol{\alpha}, \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) - \varphi_{\binom{a_1}{b_1}}(\boldsymbol{e}_2)) - (\alpha, a_1\boldsymbol{e}_2) < 0.$$

(iii) The case of $x_1+E_3\subset \mathbb{P}(\alpha_1, \beta_1)$ and $x_1-e_3\notin \mathbb{P}(\alpha_1, \beta_1)$.
It is sufficient to see that $(\alpha, \varphi_{\binom{a_1}{b_1}}(x)-e_1)<0$. We know from $(\alpha_1, x_1-e_3)<0$ and Lemma 1.2 and (1.6) that

$$\begin{aligned} 0&>(\alpha, \varphi_{\binom{a_1}{b_1}}(x_1)-\varphi_{\binom{a_1}{b_1}}(e_3)) \\ &=(\alpha, \varphi_{\binom{a_1}{b_1}}(x_1)-e_1+b_1e_2) . \end{aligned}$$

Therefore, we have from $(\alpha, e_2)>0$,

$$(\alpha, \varphi_{\binom{a_1}{b_1}}(x_1)-e_1)=(\alpha, \varphi_{\binom{a_1}{b_1}}(x_1)-\varphi_{\binom{a_1}{b_1}}(e_3))-(\alpha, b_1e_2)<0 .$$

(q.e.d.)

LEMMA 2.2. *If* $(x_1, S)\neq(x'_1, S')$, *then*

$$\Sigma_{\binom{a_1}{b_1}}(x_1, S)\cap\Sigma_{\binom{a_1}{b_1}}(x'_1, S')=\emptyset .$$

PROOF. (i) Suppose that

$$\Sigma_{\binom{a_1}{b_1}}(x_1, E_1)\cap\Sigma_{\binom{a_1}{b_1}}(x'_1, E_1)\neq\emptyset \qquad (x_1\neq x'_1) ,$$

that is, suppose that
(1-i) there exist k, j, $1\le k, j\le b_1$, such that

$$\varphi_{\binom{a_1}{b_1}}(x_1)+e_1-ke_2+E_1=\varphi_{\binom{a_1}{b_1}}(x'_1)+e_1-je_2+E_1 ,$$

(1-ii) there exist k, j, $0\le k, j\le a_1$, such that

$$\varphi_{\binom{a_1}{b_1}}(x_1)+e_3-ke_2+E_2=\varphi_{\binom{a_1}{b_1}}(x'_1)+e_3-je_2+E_2 ,$$

(1-iii) or

$$\varphi_{\binom{a_1}{b_1}}(x_1)+E_3=\varphi_{\binom{a_1}{b_1}}(x'_1)+E_3 .$$

In case of (1-i), the assumption yields

$$\varphi_{\binom{a_1}{b_1}}(x_1-x'_1)=(k-j)e_2 .$$

Therefore, we see that if $k>j$ then

$$\begin{aligned} &x_1-x'_1=(k-j)e_1 , \quad \text{that is} , \\ &x_1-e_1=x'_1+(k-j-1)e_1\in \mathbb{P}(\alpha, \beta) . \end{aligned}$$

This contradicts

$$x_1-e_1\notin \mathbb{P}(\alpha_1, \beta_1) .$$

The assumption (1-ii) leads to a contradiction similar to (1-i) and the assumption (1-iii) contradicts $x_1\neq x'_1$.

(2) Suppose that

$$\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, E_1) \cap \Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}'_1, E_2) \neq \varnothing \qquad (\boldsymbol{x}_1 \neq \boldsymbol{x}'_1),$$

that is, suppose that there exists j, $0 \leq j \leq a_1$, such that

$$\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}_1) + \boldsymbol{e}_3 - j\boldsymbol{e}_2 + E_3 = \varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}'_1) + E_3 .$$

Therefore, we have

$$\boldsymbol{x}'_1 - \boldsymbol{x}_1 = (a_1 - j)\boldsymbol{e}_1 + \boldsymbol{e}_2 .$$

This contradicts $\boldsymbol{x}'_1 - \boldsymbol{e}_2 \notin \mathbb{P}(\alpha_1, \beta_1)$.

(3) The assumption that

$$\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, E_1) \cap \Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}'_1, E_3) \neq \varnothing$$

leads to a contradiction similar to (2).

(4) We see from the definition of $\Sigma_{\binom{a_1}{b_1}}$ that the cases

$$\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, E_i) \cap \Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}'_1, E_i) \neq \varnothing \qquad (i, j = 2, 3)$$

do not occur. (q.e.d.)

By Lemma 2.1 and Lemma 2.2 we see that the map $\Sigma_{\binom{a_1}{b_1}}$ is well defined as a map from $\mathscr{G}(\alpha_1, \beta_1)$ to $\mathscr{G}(\alpha, \beta)$. From now on, the map $\Sigma_{\binom{a_1}{b_1}}$ is called the *substitution* associated with Jacobi-Perron algorithm.

LEMMA 2.3. *For any* $(\boldsymbol{x}, S) \in \mathscr{S}(\alpha, \beta)$, *there exists* $(\boldsymbol{x}_1, S_1) \in \mathscr{S}(\alpha_1, \beta_1)$ *such that*

$$(\boldsymbol{x}, S) \in \Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, S_1) .$$

PROOF. (1) Assume that $(\boldsymbol{x}, E_2) \in \mathscr{S}(\alpha, \beta)$, that is,

$$\boldsymbol{x} + E_2 \subset \mathbb{P}(\alpha, \beta) \quad \text{and} \quad \boldsymbol{x} - \boldsymbol{e}_2 \notin \mathbb{P}(\alpha, \beta) .$$

Put $\boldsymbol{x}_1 = \varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x})$, then

$$\begin{aligned}(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}) - \boldsymbol{e}_1, \boldsymbol{\alpha}_1) &= (\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x} - \boldsymbol{e}_2), \boldsymbol{\alpha}_1) \qquad \text{(by Lemma 1.2)} \\ &= \alpha(\boldsymbol{x} - \boldsymbol{e}_2, \boldsymbol{\alpha}) < 0 .\end{aligned}$$

This means

$$(\boldsymbol{x}, E_2) \in \Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1, E_1) .$$

(2) Assume that $(\boldsymbol{x}, E_1) \in \mathscr{S}(\alpha, \beta)$, that is,

$$\boldsymbol{x} + E_1 \subset \mathbb{P}(\alpha, \beta) \quad \text{and} \quad \boldsymbol{x} - \boldsymbol{e}_1 \notin \mathbb{P}(\alpha, \beta) .$$

From $(\boldsymbol{x} - \boldsymbol{e}_1, \boldsymbol{\alpha}) < 0$, there exists k such that

$$(\boldsymbol{x}-\boldsymbol{e}_1+(k-1)\boldsymbol{e}_2, \boldsymbol{\alpha})<0 \quad \text{and} \quad (\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2, \boldsymbol{\alpha})>0\,,$$

and k satisfies $1 \le k \le b_1+1$. Because, from $-1=-(\boldsymbol{e}_1, \boldsymbol{\alpha})$ and $(\boldsymbol{x}, \boldsymbol{\alpha})>0$ we know that $0>(\boldsymbol{x}-\boldsymbol{e}_1, \boldsymbol{\alpha})>-1$. Therefore we see $1 \le k=[(\boldsymbol{x}-\boldsymbol{e}_1, \boldsymbol{\alpha})/\alpha]+1 \le [1/\alpha]+1$. In the case of $1 \le k \le b_1$, we take $(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2), E_1)$, then we see that $(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2), \boldsymbol{\alpha})=\alpha(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2, \boldsymbol{\alpha})>0$ and

$$(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2)-\boldsymbol{e}_1, \boldsymbol{\alpha}_1)=(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2)-\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{e}_2), \boldsymbol{\alpha}_1)<0\,.$$

That is, $(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2), E_1)\in\mathscr{S}(\alpha_1, \beta_1)$. Therefore, we see from the definition of $\Sigma_{\binom{a_1}{b_1}}$ that

$$\begin{aligned}
&\Sigma_{\binom{a_1}{b_1}}(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2), E_1)\\
&\quad=\sum_{1\le j\le b_1}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2+\boldsymbol{e}_1-j\boldsymbol{e}_2, E_1)+\sum_{1\le j'\le a_1}(\boldsymbol{x}+\boldsymbol{e}_1+k\boldsymbol{e}_2+\boldsymbol{e}_3-j'\boldsymbol{e}_2, E_3)\\
&\qquad+(\boldsymbol{x}+\boldsymbol{e}_1+k\boldsymbol{e}_2, E_2)\\
&\quad\ni(\boldsymbol{x}, E_1).
\end{aligned}$$

In case of $k=b_1+1$, we take

$$(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+k\boldsymbol{e}_2)-\boldsymbol{e}_1+\boldsymbol{e}_3, E_3)\,.$$

Then we see that

$$\begin{aligned}
&(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+(b_1+1)\boldsymbol{e}_2)-\boldsymbol{e}_1+\boldsymbol{e}_3, \boldsymbol{\alpha}_1)\\
&\quad=(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+(b_1+1)\boldsymbol{e}_2)-\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{e}_2)+\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{e}_1)-b_1\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{e}_2), \boldsymbol{\alpha}_1)\\
&\quad=\alpha(\boldsymbol{x}, \boldsymbol{\alpha})>0\,.
\end{aligned}$$

That is,

$$(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+b_1\boldsymbol{e}_2)+\boldsymbol{e}_3, E_3)\in\mathscr{S}(\alpha_1, \beta_1)\,.$$

Therefore, we see that

$$\Sigma_{\binom{a_1}{b_1}}(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_1+b_1\boldsymbol{e}_2)+\boldsymbol{e}_3, E_3)=(\boldsymbol{x}, E_1)\,.$$

(3) Assume that $(\boldsymbol{x}, E_3)\in\mathscr{S}(\alpha, \beta)$, that is,

$$\boldsymbol{x}+E_3\in\mathbb{P}(\alpha, \beta) \quad \text{and} \quad \boldsymbol{x}-\boldsymbol{e}_3\notin\mathbb{P}(\alpha, \beta)\,.$$

From $(\boldsymbol{x}-\boldsymbol{e}_3, \boldsymbol{\alpha})<0$, there exists k such that $(\boldsymbol{x}-\boldsymbol{e}_3+(k-1)\boldsymbol{e}_2, \boldsymbol{\alpha})<0$, $(\boldsymbol{x}-\boldsymbol{e}_3+k\boldsymbol{e}_2, \boldsymbol{\alpha})>0$ and k satisfies $1\le k\le a_1+1$. Because from $-\beta=(-\boldsymbol{e}_3, \boldsymbol{\alpha})$ and $(\boldsymbol{x}, \boldsymbol{\alpha})>0$, we know $0>(\boldsymbol{x}-\boldsymbol{e}_3, \boldsymbol{\alpha})>-\beta$. Therefore, we see $1\le k=[(\boldsymbol{x}-\boldsymbol{e}_3, \boldsymbol{\alpha})/\alpha]+1\le[\beta/\alpha]+1$. In the case of $1\le k\le a_1$, we take

$$(\varphi_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_3+k\boldsymbol{e}_2), E_1)$$

and in the case of $k=a_1+1$, we take

$$(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_3+k\boldsymbol{e}_2),\ E_2)\,.$$

Then we see

$$\Sigma_{\binom{a_1}{b_1}}(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_3+k\boldsymbol{e}_2),\ E_1)\ni(\boldsymbol{x},\ E_3)$$

and

$$\Sigma_{\binom{a_1}{b_1}}(\varphi^{-1}_{\binom{a_1}{b_1}}(\boldsymbol{x}-\boldsymbol{e}_3+k\boldsymbol{e}_2),\ E_2)\ni(\boldsymbol{x},\ E_3)\,,$$

respectively. (q.e.d.)

Let us define a geometrical realization map $\mathscr{K}$ from $\mathscr{G}(\alpha, \beta)$ to the set of compact sets of $\boldsymbol{R}^3$ as follows:

$$\mathscr{K}((\boldsymbol{x},\ S)):=\boldsymbol{x}+S\,,$$

$$\mathscr{K}\left(\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda,\ S_\lambda)\right):=\bigcup_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda+S_\lambda)\,.$$

Let us denote

$$\mathscr{S}(\alpha,\ \beta):=\bigcup_{(\boldsymbol{x},S)\in\mathscr{S}(\alpha,\beta)}\mathscr{K}((\boldsymbol{x},\ S))$$

and call it the stepped surface of the plane orthogonal to $\boldsymbol{\alpha}={}^t(1,\ \alpha,\ \beta)$. Then from Lemma 2.2 and Lemma 2.3 we have

PROPOSITION 2.4. *For any $(\alpha, \beta)\in X-I$ the stepped surface $\mathscr{S}(\alpha, \beta)$ is invariant under the substitution $\Sigma_{\binom{a_1}{b_1}}$ in the following sense*:

$$\mathscr{S}(\alpha,\ \beta)=\bigcup_{(\boldsymbol{x}_1,S)\in\mathscr{S}(\alpha_1,\beta_1)}\mathscr{K}(\Sigma_{\binom{a_1}{b_1}}(\boldsymbol{x}_1,\ S))\,.$$

REMARK 2.1. If $(\alpha, \beta)\in I-\{(0, 0)\}$, let us define the substitution Σ_{b_1} from $\mathscr{G}(\alpha_1, \beta_1)$ to $\mathscr{G}(\alpha, \beta)$ by

$$\Sigma_{b_1}:\begin{array}{lcl}(0,\ E_1)&\to&(0,\ E_3)+\sum_{i=1}^{b_1}(\boldsymbol{e}_1-i\boldsymbol{e}_3,\ E_1)\\(0,\ E_2)&\to&(0,\ E_2)\\(0,\ E_3)&\to&(0,\ E_1)\end{array}$$

and for $(\boldsymbol{x}_1, S)\in\mathscr{S}(\alpha_1, \beta_1)$ we define

$$\Sigma_{b_1}(\boldsymbol{x}_1,\ S)=\varphi_{b_1}(\boldsymbol{x}_1)+\Sigma_{b_1}(0,\ S)\,,$$

where φ_{b_1} is given in Remark 1.3. Note that the stepped surface $\mathscr{S}(0, \beta)$ is constructed only by the unit square E_1 and E_3. By an analogous proof, it is easy to see that Lemma

2.1, 2.2, 2.3 and Proposition 2.1 also hold for the substitution Σ_{b_1}.

3. A generating method of stepped surfaces.

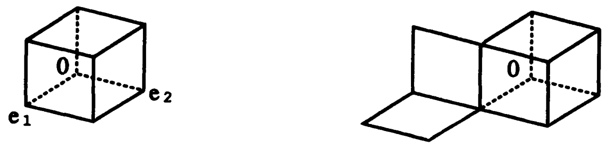
In this section, we discuss a generating method of the stepped surface $\mathscr{S}(\alpha, \beta)$ by using substitutions $\Sigma_{\binom{a_n}{b_n}}$, $n \geq 1$. We mention firstly the statement of the theorem. Let us introduce a subset $\mathscr{L}$ of X. We say $(\alpha, \beta) \in \mathscr{L}$ if the name of (α, β) satisfies the following condition:

There exists an n_0 such that for all $k \geq 0$,

(1) $b_{n_0+3k} = a_{n_0+3k}$,

(2) $b_{n_0+3k+1} - a_{n_0+3k+1} \geq 1$ and $a_{n_0+3k+1} \neq 0$,

(3) $a_{n_0+3k+2} = 0$.



$\mathscr{U}$ $\mathscr{V}$

FIGURE 4. Figure of $\mathscr{U}$ and $\mathscr{V}$

THEOREM 3.1. *Let us denote* $\mathscr{U} = \sum_{i=1}^{3} (e_i, E_i)$ *and* $\mathscr{V} = \mathscr{U} + (e_1 - e_2, E_1) + (e_1 - e_2, E_3)$ (See Figure 4). *The stepped surface* $\mathscr{S}(\alpha, \beta)$ *is generated in the following way*:

(1) $\mathscr{S}(\alpha, \beta) = \lim_{n\to\infty} \mathscr{K}(\Sigma_{\binom{a_1}{b_1}} \Sigma_{\binom{a_2}{b_2}} \cdots \Sigma_{\binom{a_n}{b_n}}(\mathscr{U}))$ *if* $(\alpha, \beta) \in X - \mathscr{L}$ *and* $(\alpha_n, \beta_n) \in X - I$ *for all* n.

(2) $\mathscr{S}(\alpha, \beta) = \lim_{k\to\infty} \mathscr{K}(\Sigma_{\binom{a_1}{b_1}} \Sigma_{\binom{a_2}{b_2}} \cdots \Sigma_{\binom{a_{n_0+3k+2}}{b_{n_0+3k+2}}}(\mathscr{V}))$ *if* $(\alpha, \beta) \in \mathscr{L}$.

(3) $\mathscr{S}(\alpha, \beta) = \lim_{n\to\infty} \mathscr{K}(\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m-1}}{b_{m-1}}} \Sigma_{b_m} \cdots \Sigma_{b_{m+n}} (\sum_{k\in \mathbf{Z}} (e_1 + ke_2, E_1) + (e_3 + ke_2, E_3)))$, *if there exists an* m *such that* $(\alpha_m, \beta_m) \in I$ *and* $\beta_m \notin \mathbf{Q}$.

To prove the theorem, we will prepare several lemmas. Firstly let us introduce a set $\mathscr{C}_0$ as follows:

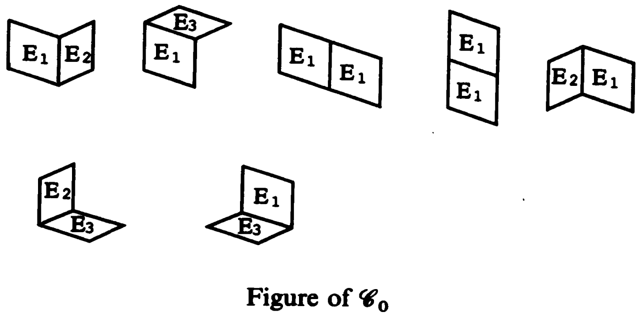


Figure of $\mathscr{C}_0$

and introduce a subset $\mathscr{C}$ of $\mathscr{G}(\alpha, \beta)$:

$$\mathscr{C}=\{T_z\gamma \mid \gamma\in\mathscr{C}_0, z\in \boldsymbol{Z}^3, T_z\gamma\in\mathscr{G}(\alpha, \beta)\},$$

where T_z is a translation map given by

$$T_z\Big(\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda, S_\lambda)\Big)=\sum_{\lambda\in\Lambda}(\boldsymbol{x}_\lambda+z, S_\lambda).$$

We say that a set δ of $\mathscr{G}(\alpha, \beta)$ is *$\mathscr{C}$-covered* if there exists a finite subset $\{\gamma_\lambda \mid \lambda\in\Lambda\}$ of $\mathscr{C}$ satisfying the following properties:

(1) For any $\gamma_\lambda, \gamma_\mu\in\delta$, there exist $\gamma_i\in\mathscr{C}$, $i=1, 2, \cdots, n$, such that $\gamma_\lambda=\gamma_1$, $\gamma_i\cap\gamma_{i+1}\neq\varnothing$ $(i=1, 2, \cdots, n-1)$ and $\gamma_\mu=\gamma_n$,

(2) $\bigcup_{\lambda\in\Lambda}\mathscr{K}(\gamma_\lambda)=\mathscr{K}(\delta)$.

LEMMA 3.1. *Assume that $\delta\in\mathscr{G}(\alpha_1, \beta_1)$ is $\mathscr{C}$-covered. Then $\Sigma_{\binom{a_1}{b_1}}(\delta)$ is also $\mathscr{C}$-covered.*

PROOF. For each $\gamma\in\delta$, it is not difficult to see from the definition of $\Sigma_{\binom{a_1}{b_1}}$ and

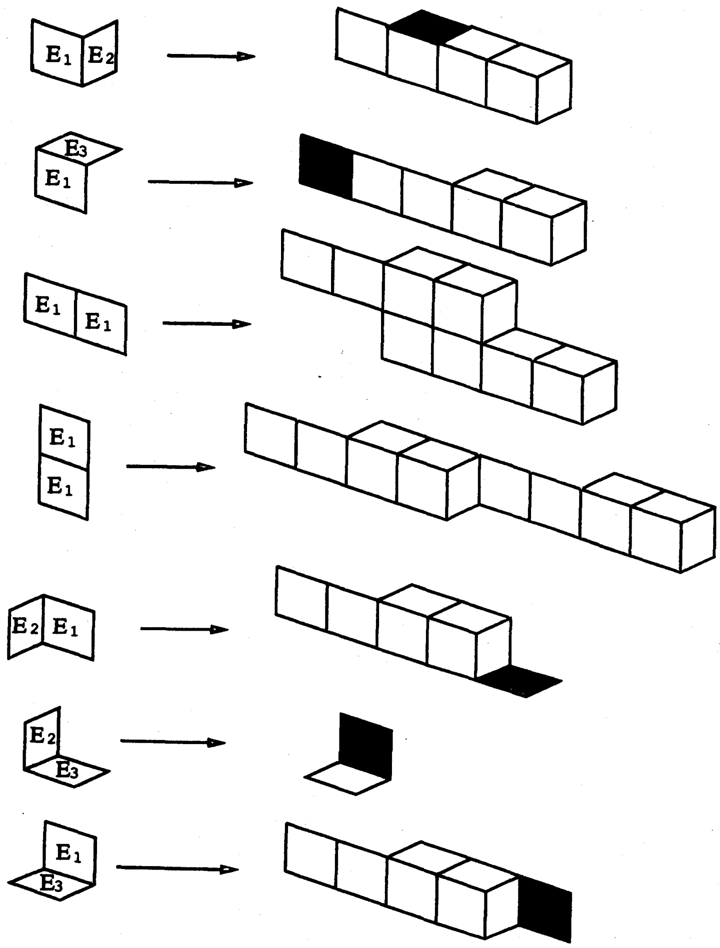


FIGURE 5. Figure of $\Sigma_{\binom{a}{b}}(\gamma)$, $\gamma\in\mathscr{C}_0$

Lemma 2.2 that $\Sigma_{\binom{a_1}{b_1}}(\gamma)$ is $\mathscr{C}$-covered (See Figure 5). Therefore, we see that for any $\mathscr{C}$-covered set δ, $\Sigma_{\binom{a_1}{b_1}}(\delta)$ is $\mathscr{C}$-covered.

DEFINITION 3.1. A $\mathscr{C}$-covered set δ is called a *$\mathscr{C}$-covered cell* (resp. *annulus*) if its geometric realization $\mathscr{K}(\delta)$ is a topological cell (resp. annulus).

PROPOSITION 3.1. (1) *If $\delta\in\mathscr{G}(\alpha_1, \beta_1)$ is a $\mathscr{C}$-covered cell, then $\Sigma_{\binom{a_1}{b_1}}(\delta)$ is a $\mathscr{C}$-covered cell.*

(2) *If $\delta\in\mathscr{G}(\alpha_1, \beta_1)$ is a $\mathscr{C}$-covered annulus, then $\Sigma_{\binom{a_1}{b_1}}(\delta)$ is a $\mathscr{C}$-covered annulus.*

PROOF. Suppose that $\mathscr{K}(\Sigma_{\binom{a_1}{b_1}}(\delta))$ is not a $\mathscr{C}$-covered cell, that is, the set $\mathscr{K}(\mathscr{S}(\alpha, \beta))-\mathscr{K}(\Sigma_{\binom{a_1}{b_1}}(\delta))$ has a bounded component D_1 and one unbounded component D_2. Then we are able to choose $(x, S)\in\mathscr{G}(\alpha, \beta)$ and $(x', S')\in\mathscr{G}(\alpha, \beta)$ such that $\mathscr{K}(x, S)\subset D_1$ and $\mathscr{K}(x', S')\subset D_2$. By Lemma 2.3, there exist $(x_1, S_1)\in\mathscr{G}(\alpha_1, \beta_1)$ and $(x'_1, S'_1)\in\mathscr{G}(\alpha_1, \beta_1)$ such that

$$\Sigma_{\binom{a_1}{b_1}}(x_1, S_1)\ni(x, S)\,, \qquad \Sigma_{\binom{a_1}{b_1}}(x'_1, S'_1)\ni(x', S')\,.$$

On the other hand, from the assumption that δ is a topological cell, and from the fact that (x_1, S_1) and (x'_1, S'_1) do not belong to δ, there exists a chain $\{\gamma_i \mid \gamma_i\in\mathscr{G}(\alpha_1, \beta_1)$ and $\gamma_i\in\mathscr{C},\ i=1, 2, \cdots, m\}$ such that $\gamma_1\cap(x_1, S_1)\neq\varnothing$, $\gamma_i\cap\gamma_{i+1}\neq\varnothing$, $\gamma_m\cap(x'_1, S'_1)\neq\varnothing$, $\gamma_i\cap\delta=\varnothing$. Therefore the sequence $\Sigma_{\binom{a_1}{b_1}}(\gamma_i)$, $i=1, 2, \cdots, m$ satisfy the following property:

(1) $\Sigma_{\binom{a_1}{b_1}}(\gamma_1)\ni(x, S)$,

(2) $\Sigma_{\binom{a_1}{b_1}}(\gamma_i)\cap\Sigma_{\binom{a_1}{b_1}}(\gamma_{i+1})\neq\varnothing$,

(3) $\Sigma_{\binom{a_1}{b_1}}(\gamma_m)\ni(x', S')$,

(4) $\Sigma_{\binom{a_1}{b_1}}(\gamma_i)\cap\Sigma_{\binom{a_1}{b_1}}(\delta)=\varnothing$.

This means that D_1 and D_2 are connected by using pieces of $\mathscr{C}_0$ which are in the outside of $\mathscr{K}(\Sigma_{\binom{a_1}{b_1}}(\delta))$. This is a contradiction. The proof of (2) in Proposition 3.1 is easy from (1).

LEMMA 3.2. *Let $(\alpha, \beta)\in X$ such that there exist infinitely many indices $n_1<m_1<n_2<m_2<\cdots$ such that for all j, $\Sigma_{\binom{a_{n_j}}{b_{n_j}}}\Sigma_{\binom{a_{n_j+1}}{b_{n_j+1}}}\cdots\Sigma_{\binom{a_{m_j}}{b_{m_j}}}(\mathscr{U})-\mathscr{U}$ contains a $\mathscr{C}$-covered annulus, then $\lim_{n\to\infty}\mathscr{K}(\Sigma_{\binom{a_1}{b_1}}\Sigma_{\binom{a_2}{b_2}}\cdots\Sigma_{\binom{a_n}{b_n}}(\mathscr{U}))=\mathscr{S}(\alpha, \beta)$.*

PROOF. From the assumption and that $\mathscr{U}$ is $\mathscr{C}$-covered, we see

$$\Sigma_{\binom{a_{n_1}}{b_{n_1}}}\cdots\Sigma_{\binom{a_{m_1}}{b_{m_1}}}(\mathscr{U})-\mathscr{U}$$

contains a $\mathscr{C}$-covered annulus. Noticing that

$$\Sigma_{\binom{a_1}{b_1}}\cdots\Sigma_{\binom{a_n}{b_n}}(\mathscr{U})\supset\Sigma_{\binom{a_1}{b_1}}\cdots\Sigma_{\binom{a_{n-1}}{b_{n-1}}}(\mathscr{U})\,,$$

we see

(i) $\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_1}}{b_{m_1}}}(\mathscr{U}) \supset \mathscr{U}$,

(ii) $\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_1}}{b_{m_1}}}(\mathscr{U}) - \mathscr{U}$ contains a $\mathscr{C}$-covered annulus.

In general, we have from the assumption that

$$\Sigma_{\binom{a_{n_j}}{b_{n_j}}} \Sigma_{\binom{a_{n_j+1}}{b_{n_j+1}}} \cdots \Sigma_{\binom{a_{m_j}}{b_{m_j}}}(\mathscr{U}) - \mathscr{U}$$

contains a $\mathscr{C}$-covered annulus and by Proposition 3.1,

(i)′ $\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_j}}{b_{m_j}}}(\mathscr{U}) \supset \Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{n_j-1}}{b_{n_j-1}}}(\mathscr{U})$,

(ii)′ $\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_j}}{b_{m_j}}}(\mathscr{U}) - \Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_j-1}}{b_{m_j-1}}}(\mathscr{U})$ contains a $\mathscr{C}$-covered annulus.

Thus the sequence $\{\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_j}}{b_{m_j}}} \mathscr{U}\}$ is increasing so that

$$\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_j}}{b_{m_j}}}(\mathscr{U}) - \Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_{m_j-1}}{b_{m_j-1}}}(\mathscr{U})$$

contains the annulus in (ii)′. (q.e.d.)

To obtain Theorem 3.1(1), we will show that for $(\alpha, \beta) \in X - \mathscr{L}$ and $(\alpha_n, \beta_n) \in X - \mathscr{L}$ for all n, we are able to choose the indices $n_i < m_i$ in Lemma 3.2. For this purpose we prepare several notations and lemmas. We decompose X into four parts:

$$X_A := \{(\alpha, \beta) \mid \alpha - \beta \geq 0,\ 1 - \alpha - \beta \geq 0\},$$

$$X_B := \{(\alpha, \beta) \mid \alpha - \beta < 0,\ 1 - \alpha - \beta \geq 0\},$$

$$X_C := \{(\alpha, \beta) \mid \alpha - \beta \geq 0,\ 1 - \alpha - \beta < 0\},$$

$$X_D := \{(\alpha, \beta) \mid \alpha - \beta < 0,\ 1 - \alpha - \beta < 0\}.$$

Let us denote the configurations around $\mathscr{U}$ as follows:

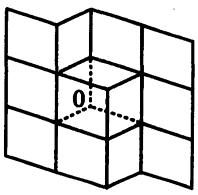
$$\delta_A := \mathscr{U}' + (e_1 - e_2 + e_3, E_1) + (e_1 - e_2 - e_3, E_1) + (e_3, E_2) + (e_2 - e_3, E_1) + (e_2 - e_3, E_2) + (e_2 + e_3, E_1),$$

$$\delta_B := \mathscr{U}' + (e_1 - e_2 - e_3, E_1) + (-e_2 + e_3, E_1) + (-e_2 + e_3, E_3) + (e_2, E_3) + (e_1 + e_2 - e_3, E_1) + (e_2 + e_3, E_1),$$

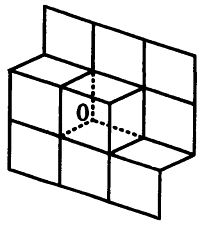
$$\delta_C := \mathscr{U}' + (e_1 - e_2, E_3) + (e_1 - e_3, E_2) + (e_1 - e_2 + e_3, E_1) + (e_3, E_2) + (e_2 - e_3, E_1) + (e_2 - e_3, E_2) + (e_2 + e_3 - e_1, E_2) + (e_2 + e_3 - e_1, E_3),$$

$$\delta_D := \mathscr{U}' + (e_1 - e_2, E_3) + (e_1 - e_3, E_2) + (e_1 + e_2 - e_3, E_1) + (e_2, E_3) + (e_3 - e_2, E_1) + (e_3 - e_2, E_3) + (e_2 + e_3 - e_1, E_2) + (e_2 + e_3 - e_1, E_3),$$

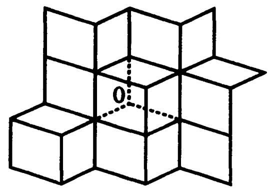
where $\mathscr{U}' = \mathscr{U} + (e_1 - e_2, E_1) + (e_1 - e_3, E_1) + (e_2, E_1) + (e_3, E_1)$. (See Figure 6.)



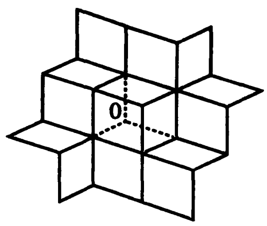
δ_A



δ_B



δ_C



δ_D

FIGURE 6. Figure of δ_A, δ_B, δ_C and δ_D

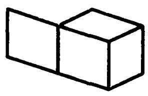
Then we have the following lemma.

LEMMA 3.3. *For each* $U \in \{A, B, C, D\}$, *if* $(\alpha, \beta) \in X_U$ *then*

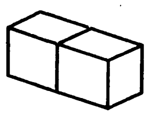
$$\delta_U \subset \mathscr{S}(\alpha, \beta).$$

The proof is easy. For example, to see that $(-e_2+e_1+e_3, E_1) \in \mathscr{S}(\alpha, \beta)$ when $(\alpha, \beta) \in X_A$, it is enough to check the following inequality $(-e_2+e_1+e_3, \alpha)>0$ and $(-e_2+e_1+e_3-e_1, \alpha)<0$. These inequalities follow from the fact that the point $(\alpha, \beta) \in X_A$. The other case can be checked in the same way.

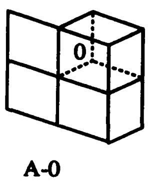
From now on, we will observe patterns of growth of configurations around $\mathscr{U}$ under $\Sigma_{\binom{a_1}{b_1}}, \cdots, \Sigma_{\binom{a_n}{b_n}}$. For this purpose, let us introduce a class of subsets of δ_A, δ_B, δ_C and δ_D as follows:



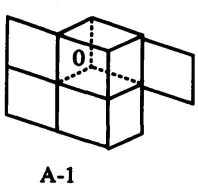
$\mathscr{U}_1$



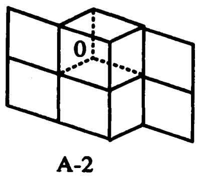
$\mathscr{U}_2$



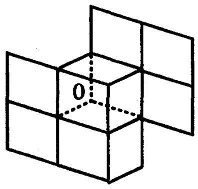
A-0



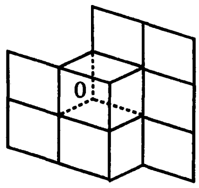
A-1



A-2



A-3



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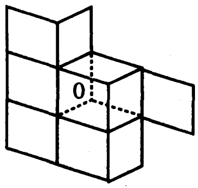
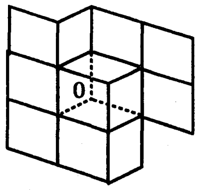
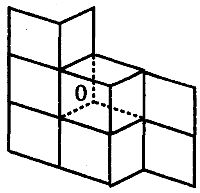


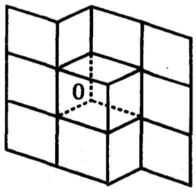
A-5



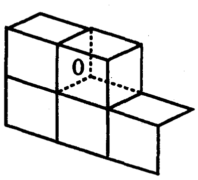
A-6



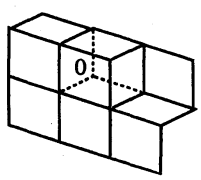
A-7



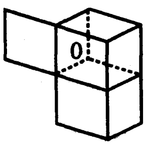
A-8 $= \delta_A$



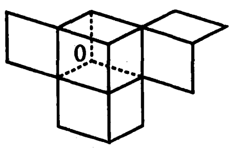
B-0



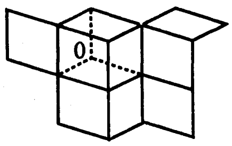
B-1



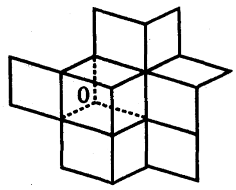
C-0



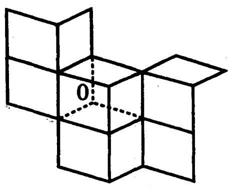
C-1



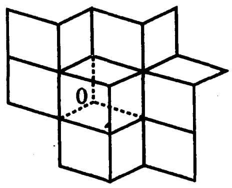
C-2



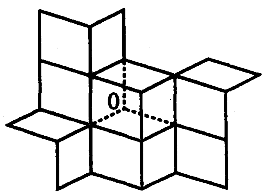
C-3



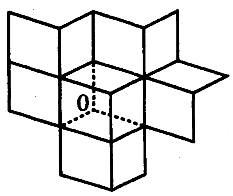
C-4



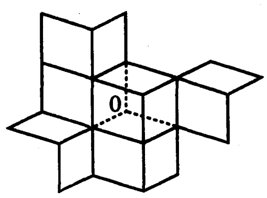
C-5



C-6



C-7



C-8

Figure of $\mathscr{U}_1$, $\mathscr{U}_2$, $(A\text{-}0), \cdots, (D\text{-}4)$ (To be continued)

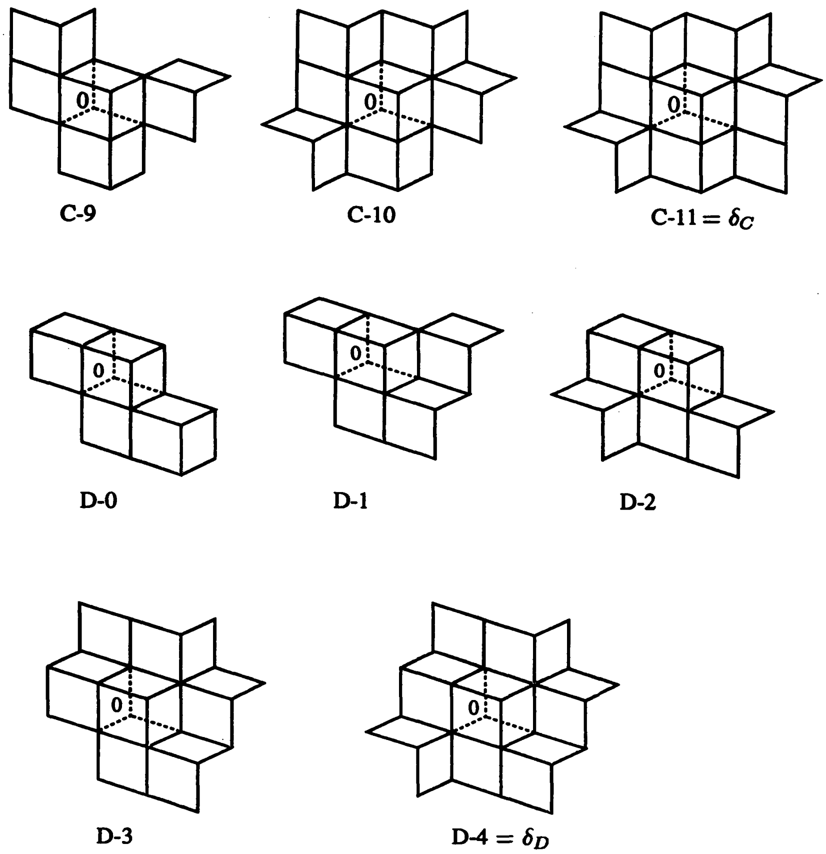
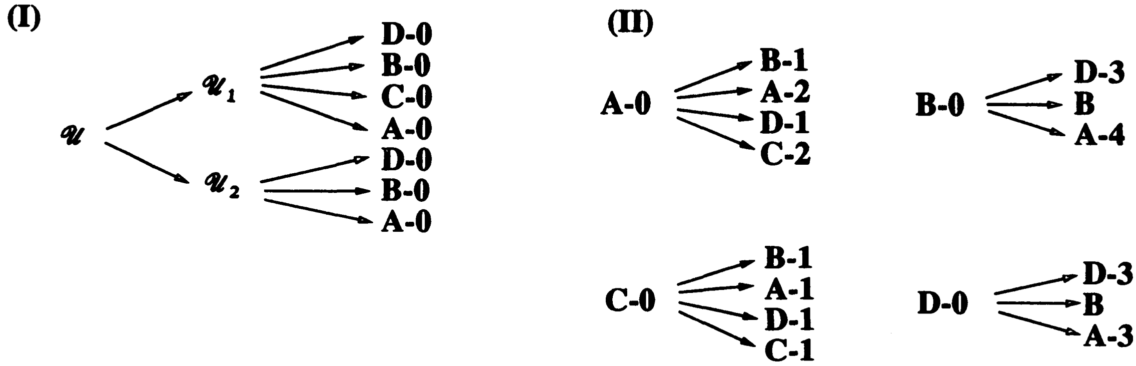
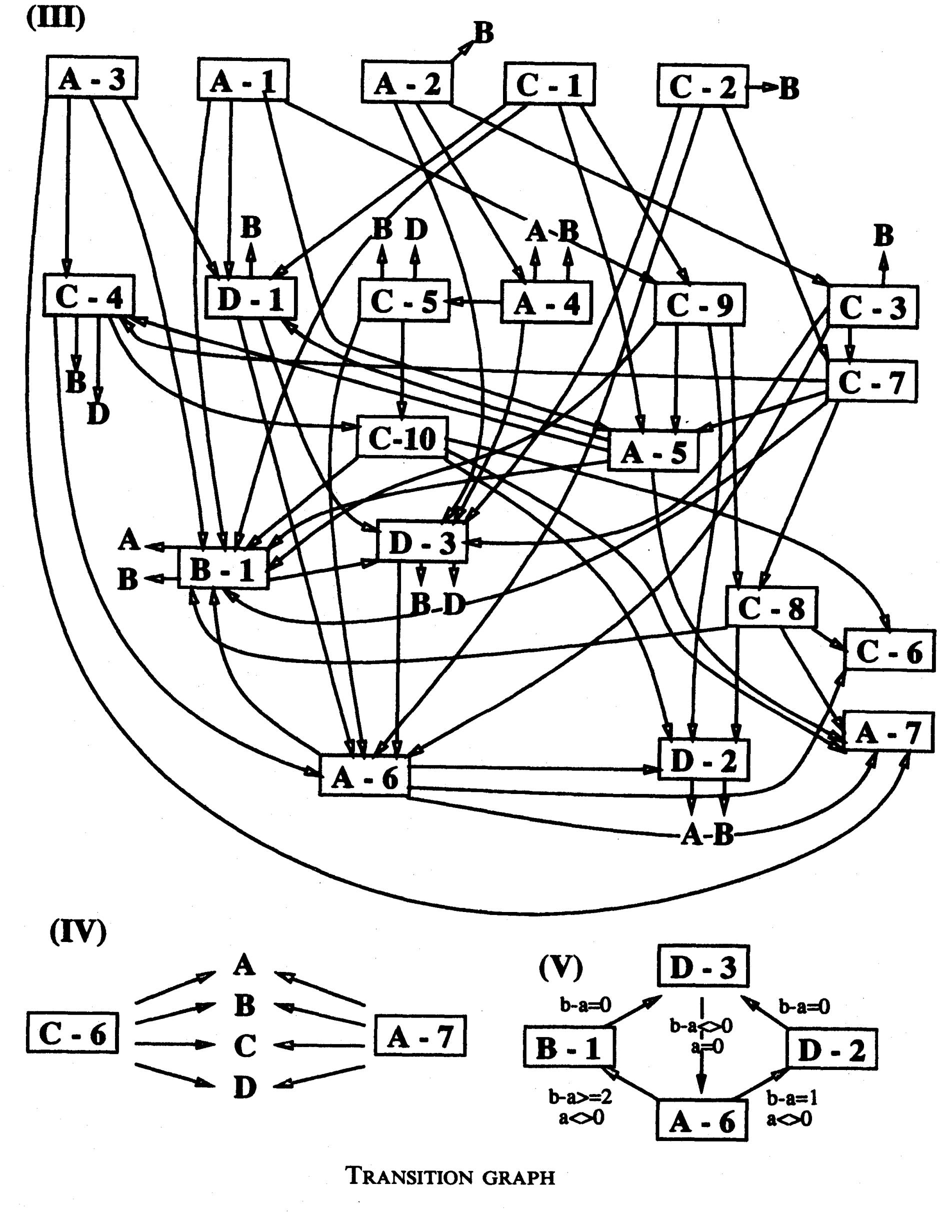


Figure of $\mathscr{U}_1$, $\mathscr{U}_2$, $(A\text{-}0), \cdots, (D\text{-}4)$ (Continued)

Let us consider $\mathscr{U}$, $\mathscr{U}_1$, $\mathscr{U}_2$, $(A\text{-}0), \cdots, (D\text{-}4)$ as states. Then we have the following lemma.

LEMMA 3.4. *We get the following transition graph, of which the transition of states from* $(U\text{-}k)$ *to* $(V\text{-}j)$ *with condition* C: $(U\text{-}k) \xrightarrow{C} (V\text{-}j)$ *means that* $\Sigma_{\binom{a}{b}}(U\text{-}k) \supset (V\text{-}j)$ *for* (a, b) *satisfying the condition* C.





TRANSITION GRAPH

The proof is obtained by the observation of tedious combinations of growth of configuration patterns by the substitution $\Sigma_{\binom{a}{b}}$. Therefore, the proof is given as an appendix in the final section.

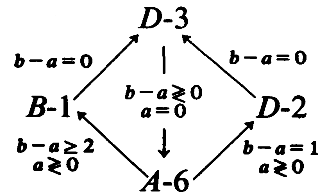
From Lemma 3.4 we have the following result.

COROLLARY 3.5. *Assume that* $(\alpha, \beta) \notin \mathscr{L}$, *then for any n there exists an n′ such that*

(1) $\Sigma_{\binom{a_n}{b_n}} \Sigma_{\binom{a_{n+1}}{b_{n+1}}} \cdots \Sigma_{\binom{a_{n+n'}}{b_{n+n'}}} \mathscr{U} \supset \mathscr{U}$,

(2) $\Sigma_{\binom{a_n}{b_n}} \Sigma_{\binom{a_{n+1}}{b_{n+1}}} \cdots \Sigma_{\binom{a_{n+n'}}{b_{n+n'}}} \mathscr{U} - \mathscr{U}$ *is a* $\mathscr{C}$*-covered annulus.*

PROOF. The transition graph in Lemma 3.4 contains a cycle:



Therefore, if $(\alpha, \beta)\notin\mathscr{L}$, we know that for any n there exists n' and $\delta_E\in\{\delta_A, \delta_B, \delta_C, \delta_D\}$ such that $\Sigma_{\binom{a_n}{b_n}}\Sigma_{\binom{a_{n+1}}{b_{n+1}}}\cdots\Sigma_{\binom{a_{n+n'}}{b_{n+n'}}}\mathscr{U}\supset\delta_E$. Notice the fact that if $E=A$ or B then $\delta_E-\mathscr{U}$ is a $\mathscr{C}$-covered annulus. Therefore, we have the conclusion. If $E=C$ or D then $\delta_E-\mathscr{U}$ is an annulus but not $\mathscr{C}$-covered, because $(e_1-e_3, E_2)+(e_1-e_2, E_3)$ does not belong to $\mathscr{C}$. But it is easy to see that $\Sigma_{\binom{a}{b}}(\delta_E-\mathscr{U})$ is $\mathscr{C}$-covered.

LEMMA 3.6. *Assume that* $(\alpha, \beta)\in\mathscr{L}$, *that is, there exists* n *such that for all* $k\in N\cup\{0\}$

(1) $b_{n+3k}=a_{n+3k}$,

(2) $b_{n+3k+1}-a_{n+3k+1}\geq 1$,

(3) $a_{n+3k+2}=0$.

Then

(1) $\Sigma_{\binom{a_{n+3k}}{b_{n+3k}}}\Sigma_{\binom{a_{n+3k+1}}{b_{n+3k+1}}}\cdots\Sigma_{\binom{a_{n+3(k+2)-1}}{b_{n+3(k+2)-1}}}\mathscr{V}\supset\mathscr{V}$,

(2) $\Sigma_{\binom{a_{n+3k}}{b_{n+3k}}}\Sigma_{\binom{a_{n+3k+1}}{b_{n+3k+1}}}\cdots\Sigma_{\binom{a_{n+3(k+2)-1}}{b_{n+3(k+2)-1}}}\mathscr{V}-\mathscr{V}$ *is a* $\mathscr{C}$*-covered annulus.*

PROOF. We know that if $(\alpha, \beta)\in X_C$ or X_D then $\mathscr{V}\in\mathscr{G}(\alpha, \beta)$, and if $a_1=b_1$ then $(\alpha, \beta)\in X_D$. Therefore we know that $\mathscr{V}\in\mathscr{G}(\alpha_{n+3k+2}, \beta_{n+3k+2})$ and $\Sigma_{\binom{a_1}{b_1}}\cdots\Sigma_{\binom{a_{n+3k+2}}{b_{n+3k+2}}}(\mathscr{V})\in\mathscr{G}(\alpha, \beta)$ (by Lemma 2.4). By the analogous method as in Lemma 3.4, we see that for any $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \end{pmatrix}$ such that $c_1=d_1$, $c_4=d_4$, $d_2-c_2\geq 1$, $c_2\neq 0$, $d_5-c_5\geq 1$, $c_5\neq 0$ and $c_3=c_6=0$,

(1) $\Sigma_{\binom{c_1}{d_1}}\Sigma_{\binom{c_2}{d_2}}\cdots\Sigma_{\binom{c_6}{d_6}}\mathscr{V}\supset\mathscr{V}$,

(2) $\Sigma_{\binom{c_1}{d_1}}\Sigma_{\binom{c_2}{d_2}}\cdots\Sigma_{\binom{c_6}{d_6}}\mathscr{V}-\mathscr{V}$ is a $\mathscr{C}$-covered annulus.

Therefore we have the conclusion.

PROOF OF THE THEOREM. In case of $(\alpha, \beta)\in X-\mathscr{L}$ and $(\alpha_m, \beta_m)\notin I$ for all m, we see from Corollary 3.5 that (α, β) satisfies the assumption of Lemma 3.2. Therefore we see that

$$\mathscr{S}(\alpha, \beta)=\lim_{n\to\infty}\mathscr{K}(\Sigma_{\binom{a_1}{b_1}}\Sigma_{\binom{a_2}{b_2}}\cdots\Sigma_{\binom{a_n}{b_n}}(\mathscr{U})).$$

In case of $(\alpha, \beta)\in\mathscr{L}$, we see from Lemma 3.6 and Lemma 3.2 that

$$\mathscr{S}(\alpha, \beta)=\lim_{k\to\infty}\mathscr{K}(\Sigma_{\binom{a_1}{b_1}}\Sigma_{\binom{a_2}{b_2}}\cdots\Sigma_{\binom{a_{n+3k+2}}{b_{n+3k+2}}}(\mathscr{V}))\,.$$

REMARK 3.1. Assume that $(\alpha, \beta)\in I$ and $\beta\notin \boldsymbol{Q}$ then from Remark 2.1 we see that the set

$$\lim_{n\to\infty} \mathscr{K}(\Sigma_{b_1}\Sigma_{b_2}\cdots\Sigma_{b_n}((\boldsymbol{e}_1, E_1)+(\boldsymbol{e}_3, E_3)))$$

is the stepped surface of the belt

$$\{(x, y, z) \mid ({}^t(x, y, z), {}^t(1, 0, \beta))=0 \text{ and } 0\le y\le 1\}\,.$$

The stepped surface $\mathscr{S}(0, \beta)$ is given by

$$\lim_{n\to\infty} \mathscr{K}(\Sigma_{b_1}\Sigma_{b_2}\cdots\Sigma_{b_n}\left(\sum_{k\in \boldsymbol{Z}}((\boldsymbol{e}_1+k\boldsymbol{e}_2, E_1)+(\boldsymbol{e}_3+k\boldsymbol{e}_2, E_3))\right).$$

In case of $(\alpha, \beta)\in I$ and $\beta\in \boldsymbol{Q}$, that is, there exists an m such that $(\alpha_m, \beta_m)=(0, 0)$, $\mathscr{S}(\alpha, \beta)$ is given by

$$\mathscr{K}(\Sigma_{b_1}\Sigma_{b_2}\cdots\Sigma_{b_m}\left(\sum_{(k, j)\in \boldsymbol{Z}^2}(\boldsymbol{e}_1+k\boldsymbol{e}_2+j\boldsymbol{e}_3, E_1)\right).$$

Now we discuss about the quasi-periodicity of the tiling

$$\bigcup_{(\boldsymbol{x}, S)\in \mathscr{S}(\alpha, \beta)} \pi\mathscr{K}(\boldsymbol{x}, S)\,.$$

DEFINITION 3.2. A tiling by $\mathscr{S}(\alpha, \beta)$ is said to be *quasi-periodic* if for any $r>0$, there exists $R>0$ such that any configuration $\mathscr{K}(\gamma)\in\mathscr{G}(\alpha, \beta)$ whose diameter is smaller than r occurs somewhere in a neighborhood of any point of radius R.

THEOREM 3.8. *Let $(1, \alpha, \beta)$ be rationally independent, that is, assume that if $l+m\alpha+n\beta=0$ for some $l, m, n\in \boldsymbol{Z}$ then $l=m=n=0$. Then the tiling by $\mathscr{S}(\alpha, \beta)$ is not periodic but quasi-periodic.*

Before the proof of this theorem, we introduce a notation. For two configurations γ and δ of $\mathscr{G}(\alpha, \beta)$, we write $\gamma\succ\delta$ if γ and δ have the following relation: there exists $\boldsymbol{z}\in \boldsymbol{Z}^3$ such that

$$\mathscr{K}(T_{\boldsymbol{z}}\gamma)\supset\mathscr{K}(\delta).$$

PROOF OF THE THEOREM. From the assumption of the rational independence we know that $(\alpha_n, \beta_n)\notin I$ for all n. Firstly, let us assume that $(\alpha, \beta)\in X-\mathscr{L}$. By Theorem 3.1(1), we see that for any $r>0$ there exists N such that

$$\Sigma_{\binom{a_1}{b_1}}\cdots\Sigma_{\binom{a_N}{b_N}}\mathscr{U}\succ\gamma$$

for any $\gamma\in\mathscr{G}$ whose diameter is smaller than r. From the fact that

$$\Sigma_{\binom{a}{b}}(0, E_1)\succ\mathscr{U}\,,$$

$$\Sigma_{\binom{a}{b}}(0, E_2) \succ (0, E_1),$$
$$\Sigma_{\binom{a}{b}}(0, E_3) \succ (0, E_2),$$

we see that for any $r>0$ there exists M such that for each $i=1, 2, 3$

$$\Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_M}{b_M}}(0, E_i) \succ \gamma$$

for any $\gamma \in \mathcal{G}$ whose diameter is smaller than r. Let us take R as

$$R = \max_{i=1,2,3} diam(\pi \Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_M}{b_M}} \mathcal{K}(0, E_i)).$$

Then from Lemma 2.3 we see that for any $(z, S) \in \mathcal{S}(\alpha, \beta)$ there exists $(x, E_i) \in \mathcal{S}(\alpha, \beta)$ such that

$$(z, S) \in \Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_M}{b_M}}(x, E_i).$$

Therefore the neighborhood U_R of z with radius R includes the set

$$\pi \mathcal{K} \Sigma_{\binom{a_1}{b_1}} \cdots \Sigma_{\binom{a_M}{b_M}}(x, E_i).$$

This is the conclusion of the theorem.

In case of $(\alpha, \beta) \in \mathcal{L}$, it is easy to see that for any sequence satisfying the admissible conditions $a_1 = b_1$, $b_2 - a_2 \geq 1$, $a_2 \neq 0$, $a_3 = 0$ in Lemma 3.6,

$$\Sigma_{\binom{a_1}{b_1}} \Sigma_{\binom{a_2}{b_2}} \Sigma_{\binom{a_3}{b_3}}(0, E_1) \succ \mathcal{V}.$$

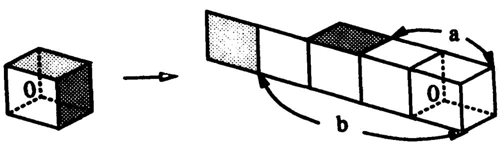
Therefore we obtain the conclusion by an analogous discussion. (q.e.d.)

4. Appendix.

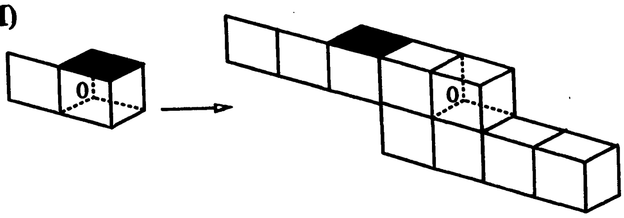
In this section, we give a sketch of proof of Lemma 3.4.

LEMMA 4.1. *By the substitution* $\Sigma_{\binom{a}{b}}$, *each configuration on the left hand side is translated to the one on the right hand side*:

(I)



(II)



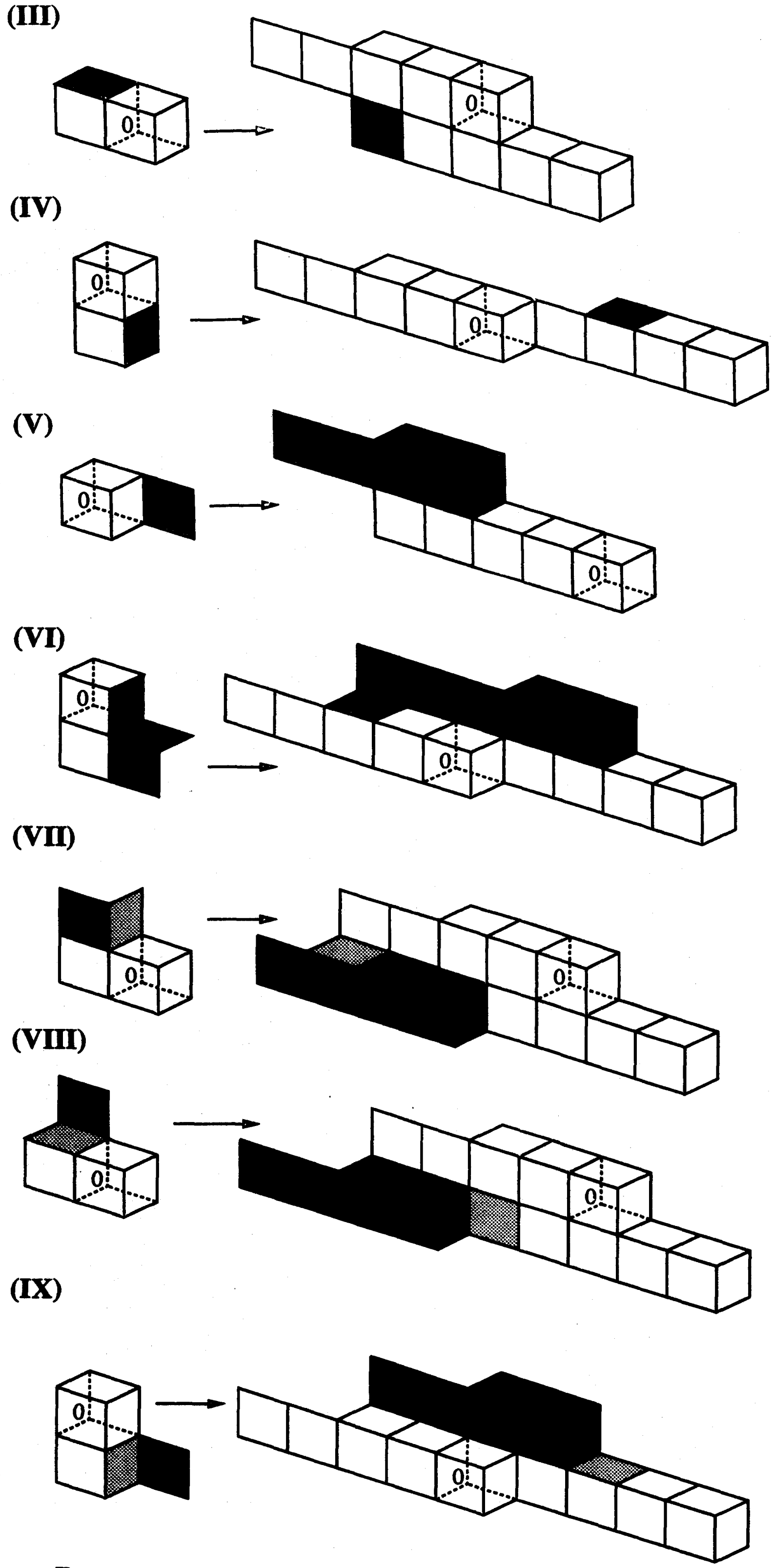


FIGURE 7. Patterns of configurations (To be continued)

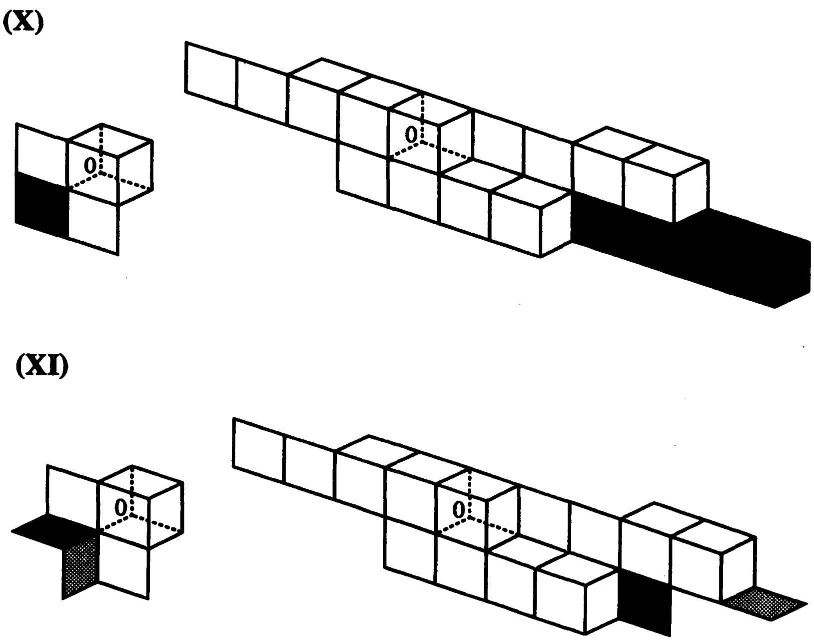
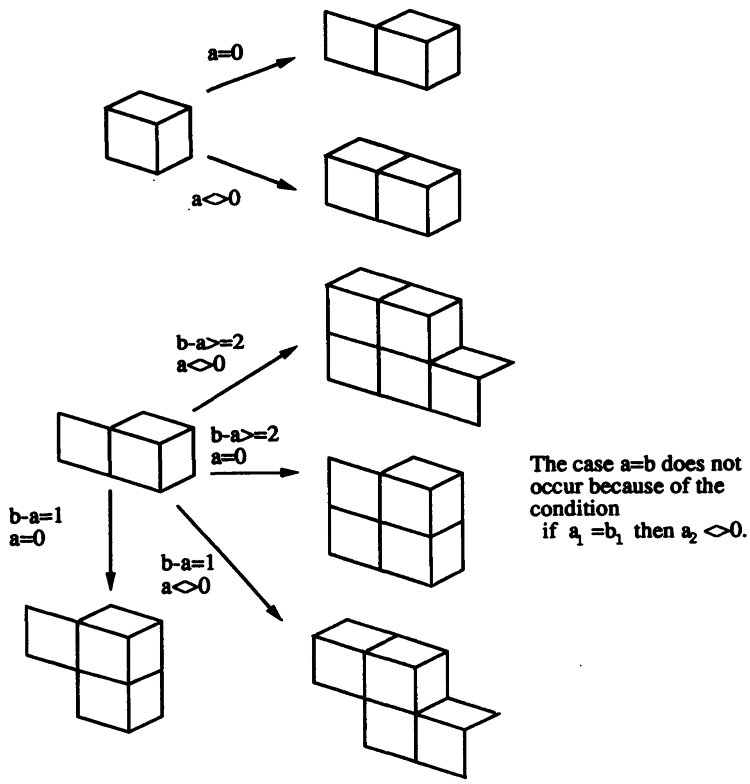


FIGURE 7. Patterns of configurations (Continued)

Using (I) (II) (III) in Lemma 4.1, we have the following transition:



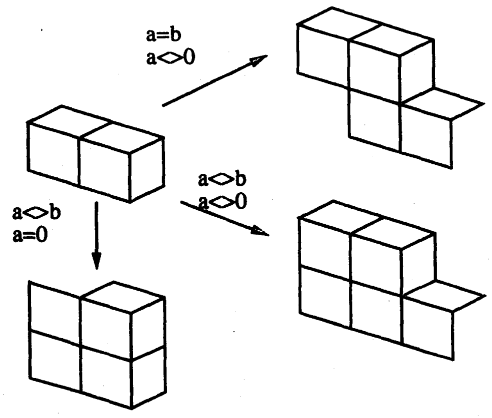


FIGURE 8

This is nothing but the statement (I) in Lemma 3.4.

Using Lemma 4.1, we obtain the statement (II) (III) (IV) analogously. For example, by using (X) in Lemma 4.1, we obtain the following transition:

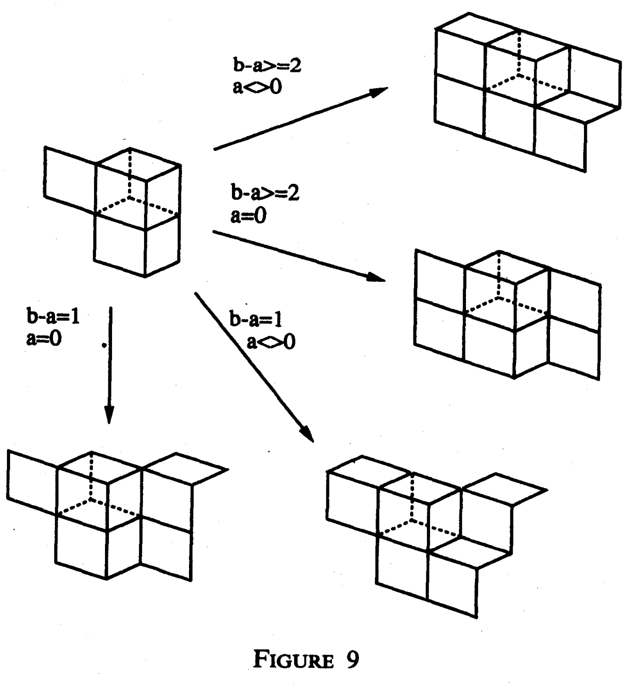
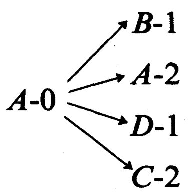


FIGURE 9

This is nothing but



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Present Address:
DEPARTMENT OF MATHEMATICS, TSUDA COLLEGE,
TSUDA-MACHI, KODAIRA-SHI, TOKYO, 187 JAPAN.