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On the Convergence of the Spectrum of Perron-Frobenius Operators

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1. Introduction.

We will consider $\{F_t\}_{t=1,2,\cdots,\infty}$ a family of piecewise C^2 mappings from an interval I into itself. We denote by P_t the Perron-Frobenius operator corresponding to F_t:

$$\int_I P_t f(x)g(x)dx = \int_I f(x)g(F_t(x))dx \qquad \text{for } f \in L^1 \text{ and } g \in L^\infty ,$$

where we denote by L^1 (resp. L^∞) the set of integrable functions (resp. the set of bounded measurable functions). We denote by $Spec(F_t)$ the spectrum of P_t restricted to BV, the set of bounded functions. Here, as usual, we consider BV as a subset of L^1 by taking L^1-version and the norm

$$V(f) = \inf\{\text{the total variation of } \tilde{f} : \tilde{f} \text{ is a } L^1\text{-version of } f\} + \int_I |f(x)|dx .$$

We assume that F_t converges to F_∞ in piecewise C^1 (the definition will be stated in §2). In this situation, though P_t converges to P_∞ in L^1, P_t does not necessarily converge to P_∞ in BV. This means that general perturbation theories cannot be applied. Nevertheless, using Fredholm matrix which is defined in [10], our main theorem (Theorem A) states that $Spec(F_\infty)$ can be approximated by $Spec(F_t)$.

THEOREM A. *Assume that*
(1) *each F_t is a piecewise C^2 mapping with positive lower Lyapunov number ξ_t $(t=1, 2, \cdots, \infty)$,*
(2) *F_t converges to F_∞ in piecewise C^1.*
Then for z_∞ which satisfies $|z_\infty| < e^{\xi_\infty}$, $z_\infty^{-1} \in Spec(F_\infty)$ if and only if there exists a sequence $\{z_t\}$ such that z_t converges to z_∞ and $z_t^{-1} \in Spec(F_t)$.

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This theorem can be applied to calculate $Spec(F_\infty)$. Take piecewise linear Markov transformations F_t which converges to F_∞ in piecewise C^1. Although the Perron-Frobenius operator P_t does not converge to P_∞ in BV, this theorem states that $Spec(F_\infty)$ can be approximated by $Spec(F_t)$. Therefore we can approximate $Spec(F_\infty)$ easily according to the facts:
(1) the Fredholm matrices of Markov mappings are much simpler than those of non-Markov mappings because we need not to trace the orbits of the division points of the partition,
(2) the Fredholm matrix of a piecewise linear mapping is a finite dimensional matrix.

Applying Theorem A also to the perturbation theory (cf. [7]), we can conclude:
(1) When the dynamical system becomes non-ergodic, then in the neighborhood of the dynamical system there exist eigenvalues which are close to 1. Therefore, as n tends to ∞, $\int f(x)g(F^n(x))d\mu-\int f d\mu\cdot\int g d\mu$ oscillates with long period which decays slowly to zero for $f\in BV$ and $g\in L^\infty$.
(2) When the dynamical system becomes non-mixing but still ergodic, then in the neighborhood of the dynamical system there exist eigenvalues near the some rational root of 1. Therefore, $\int f(x)g(F^n(x))d\mu-\int fd\mu\cdot\int g d\mu$ oscillates with period nearly corresponding to the root and decays slowly to zero.

We considered in [7] the perturbation theory for simple cases such as
(a) β-transformations for which the slopes decrease to 1, and
(b) unimodal linear transformations for which the slopes decrease to $\sqrt{2}$.
The case (a) corresponds to the case (1) and the case (b) to the case (2).

Now we will summarize the results concerning piecewise C^2 mappings. Let F be a piecewise C^2 mapping from an interval I into itself. The spectrum problem of the Perron-Frobenius operator P corresponding to the mapping F as an operator on L^1 is rather trivial: for instance, the unit disk is contained in the spectrum of P (cf. [13]). Hence, we restrict P to BV. This is quite natural, since on the unit circle the spectrum of P as an operator on L^1 coincides with $Spec(F)$ and most of the ergodic properties of the dynamical system can be stated in terms of the spectrum on the unit circle of the Perron-Frobenius operator P (cf. [6], [11]). Moreover, we can study the decay rate of correlation of the dynamical system by $Spec(F)$ (cf. for example [10]).

As proved in [3], all λ such that $|\lambda|<e^{-h(F)}$ are contained in $Spec(F)$, where $h(F)$ is the topological entropy of F. Thus we only need to consider the spectra which satisfies $|\lambda|>e^{-\xi}$, where ξ is the lower Lyapunov number. (Note that ξ equals $h(F)$ when the dynamical system is ergodic.)

In [10], we characterize them in terms of the Fredholm matrix $\Phi(z)$ and its truncation $\Phi_N(z)$:

THEOREM B. *Let z_0 be a complex number such that $|z_0|<e^{\xi}$. Then z_0^{-1} belongs to $Spec(F)$ if and only if there exists a sequence $\{z_N\}_{N=1}^{\infty}$ such that* $\lim_{N\to\infty}z_N=z_0$ *and* $\det(I-\Phi_N(z_N))=0$.

By Theorem B, we can also characterize $Spec(F)$ by the zeta function $\zeta(z)$ (Ruelle-Artin-Mazur zeta function) as follows:

THEOREM C. *The reciprocal* $1/\zeta(z)$ *of the zeta function has analytic extension to the domain* $\{z : |z| < e^{\xi}\}$, *and*

$$Spec(F) \cap \{\lambda : |\lambda| > e^{-\xi}\} = \{z^{-1} : 1/\zeta(z) = 0, |z| < e^{\xi}\} .$$

The key points of these theorems are as follows (cf. [10]):

(1) Signed symbolic dynamics: The structure of the dynamical system can be gotten by tracing the orbits of division points. To trace them, usual symbolic dynamics is insufficient.

(2) Formal piecewise linear transformation: To define a Fredholm matrix, we need piecewise linear transformations which approximates the transformation F, we construct formal piecewise linear transformations on the symbolic dynamics where F is realized.

(3) Renewal equation: This is a well-known notion in Markov processes. We define the Fredholm matrix by constructing a renewal equation.

Roughly saying, the proofs of Theorem B and Theorem C can be proved as follows. We consider a generating function of the form:

$$(f, g)(z) = \sum_{n=0}^{\infty} z^n \int f(x) g(F^n(x)) dx$$
$$= \int \{(I - zP)^{-1} f(x)\} g(x) dx ,$$

and construct a renewal equation of $(f, g)(z)$. By this renewal equation we define a Fredholm matrix $\Phi(z)$. The spectrum problem of P becomes first a problem of singularity of $(f, g)(z)$, then by renewal equation it turns out to be an eigenvalue problem of the Fredholm matrix, that is, if 1 is the eigenvalue of $\Phi(z)$, then this shows $z^{-1} \in Spec(F)$. This is the main tool to prove Theorem A. These results will be summarized in §3.

2. Notations.

We will first state the notations and the conditions which the mappings F_t must satisfy. Let I be a bounded interval and each F_t $(t = 1, 2, \cdots, \infty)$ be a mapping from I into itself. There exists a finite set A which is totally ordered and the mappings $\{F_t\}$ satisfy the following assumptions.

ASSUMPTION (I) Each F_t $(t = 1, 2, \cdots, \infty)$ is piecewise C^2: More precisely, for each F_t, there exists a partition $\{\langle a \rangle_t\}_{a \in A}$ of I into subintervals with the index set A such that

(1) for $a, b \in A$ such that $a < b$, an inequality $x < y$ holds for any $x \in \langle a \rangle_t$ and $y \in \langle b \rangle_t$,

(2) F_t is monotone on each $\langle a\rangle_t$,

(3) F_t can extend to $cl\langle a\rangle_t$ as C^2 function,

where clJ stands for the closure of a set J.

ASSUMPTION (II) The lower Lyapunov number ξ_t corresponding to F_t $(t=1, 2, \cdots, \infty)$ is positive:

$$\xi_t = \liminf_{n\to\infty} \operatorname*{ess\,inf}_{x\in I} \frac{1}{n} \log |(F_t^n)'(x)| > 0 .$$

ASSUMPTION (III) F_t converges to F_∞ in piecewise C^1:

(1) The partition $\{\langle a\rangle_t\}$ converges to $\{\langle a\rangle_\infty\}$, that is, for any $a\in A$

$$\lim_{t\to\infty} \inf\{x\in\langle a\rangle_t\} = \inf\{x\in\langle a\rangle_\infty\} ,$$

$$\lim_{t\to\infty} \sup\{x\in\langle a\rangle_t\} = \sup\{x\in\langle a\rangle_\infty\} .$$

(2) When x is not a division point of the partition $\{\langle a\rangle_\infty\}_{a\in A}$,

$$\lim_{t\to\infty} F_t(x) = F_\infty(x) ,$$

$$\lim_{t\to\infty} F_t'(x) = F_\infty'(x) .$$

(3) F_t'' is uniformly bounded:

$$\sup_{0\le t\le\infty} \operatorname*{ess\,sup}_{x\in I} |F_t''(x)| < \infty .$$

Note that by Assumption (II), we get for $t=1, 2, \cdots, \infty$

$$\operatorname*{ess\,inf}_{x\in I} |F_t'(x)| > 0 .$$

Moreover, to avoid the notational confusion, we only treat, in this article, the cases that $F_t^n(x)$ $(n\ge 0)$ is not a division point (i.e. the endpoint of some $\langle a\rangle_t$, $a\in A$) for each division point x of the partition corresponding to F_t. Even for other cases, we can also prove Theorem A just in a similar way. Note that the assumptions (I)–(II) corresponds to the conditions (A1)–(A3) in [10], therefore each F_t satisfy the theorems in [10], which we will summarize in the next section.

Hereafter we fix ρ $(e^{-\xi_\infty} < \rho < 1)$ and consider z which satisfies $|z| < 1/\rho$.

The following lemma says that for sufficiently large t each P_t has only isolated eigenvalues in the domain $\{\lambda : |\lambda| > e^{-\xi_\infty}\}$ (cf. [3]).

LEMMA 2.1. (1) *There exist $K_1 = K_1(\rho)$ and t_0 such that for any $t\ge t_0$ and any n*

$$\operatorname*{ess\,sup}_{x\in I} |(F_t^n)'(x)|^{-1} < K_1\rho^n .$$

(2) *We also have*

$$\liminf_{t\to\infty} \xi_t \geq \xi_\infty .$$

PROOF. Take $\delta>0$ so small that $\rho-2\delta>e^{-\xi_\infty}$. Then from the definition of ξ_∞, we can take $K_1(\infty)$ such that for any n

$$|(F^n_\infty)'|^{-1}<K_1(\infty)(\rho-2\delta)^n .$$

Take N sufficiently large and ε sufficiently small such that

$$K_1(\infty)e^{\varepsilon}(\rho-2\delta)^N<(\rho-\delta)^N .$$

Then for any $n>N$, we get

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in I} \log|(F^n_t)'(x)| &\geq \operatorname*{ess\,inf}_{x\in I} \log|(F^N_t)'(x)| + \operatorname*{ess\,inf}_{x\in I} \log|(F^{n-N}_t)'(x)| \\ &\geq \left[\frac{n}{N}\right] \operatorname*{ess\,inf}_{x\in I} \log|(F^N_t)'(x)| + \inf_{1\leq m\leq N-1} \operatorname*{ess\,inf}_{x\in I} \log|(F^m_t)'(x)| .\end{aligned}$$

Therefore

$$\frac{1}{n}\operatorname*{ess\,inf}_{x\in I} \log|(F^n_t)'(x)| \geq \frac{1}{N}\operatorname*{ess\,inf}_{x\in I} \log|(F^N_t)'(x)| + R(n) ,$$

where

$$R(n)=\frac{1}{n}\inf_t\left\{\left(\left[\frac{n}{N}\right]-\frac{n}{N}\right)\operatorname*{ess\,inf}_{x\in I} \log|(F^N_t)'(x)| + \inf_{1\leq m\leq N-1} \operatorname*{ess\,inf}_{x\in I} \log|(F^m_t)'(x)|\right\} .$$

Hence for sufficiently large t, we get

$$\begin{aligned}\frac{1}{n}\operatorname*{ess\,inf}_{x\in I} \log|(F^n_t)'(x)| &\geq \frac{1}{N}\left\{\operatorname*{ess\,inf}_{x\in I} \log|(F^N_\infty)'(x)| - \varepsilon\right\} + R(n) \\ &\geq -\frac{1}{N}\{\log[K_1(\infty)(\rho-2\delta)^N]+\varepsilon\} + R(n) \\ &\geq -\log(\rho-\delta) + R(n) .\end{aligned}$$

Since $R(n)$ converges to 0 as $n\to\infty$, there exists a constant K_1 such that

$$\operatorname*{ess\,sup}_{x\in I} |(F^n_t)'(x)|^{-1} < K_1\rho^n .$$

This proves (1). The assertion (2) follows from (1).

LEMMA 2.2. *The following constant K_2 is bounded*:

$$K_2 = \limsup_{t\to\infty} \frac{\operatorname*{ess\,sup}_{x\in I} |F_t''(x)|}{\left(\operatorname*{ess\,inf}_{x\in I} |F_t'(x)|\right)^2} .$$

PROOF. The boundedness of K_2 follows from the C^1 convergence of F_t to F_∞ and the uniform boundedness of $|F_t''(x)|$.

2.1. Alphabets, words and sentences. We will define several notations which are almost the same as in [9], [10]. We call each element $a \in A$ an alphabet. For an alphabet a, we set

$$\begin{aligned} sgn\, a &= sgn\, F_t'|_{int\langle a\rangle_t} \\ &= \begin{cases} + & \text{if } F_t'(x)>0 \text{ for } x\in int\langle a\rangle_t , \\ - & \text{if } F_t'(x)<0 \text{ for } x\in int\langle a\rangle_t , \end{cases} \end{aligned}$$

where $int\, J$ is the interior of a set J. This definition does not depend on the parameter t.

A finite sequence of alphabets will be called a word and for a word $w = a_1 \cdots a_N$ $(a_i \in A)$ we denote

$$\begin{aligned} &|w| = N \qquad \text{(the length of } w), \\ &w[K] = a_K \qquad (1 \le K \le N), \\ &[w]_M = a_1 \cdots a_M \qquad (1 \le M \le N), \\ &sgn\, w = \prod_{i=1}^{N} sgn\, a_i , \\ &\theta_w = a_2 \cdots a_N . \end{aligned}$$

We denote the empty word by ε and define $|\varepsilon| = 0$, and $sgn\, \varepsilon = +$.

We call an infinite sequence of alphabets $\alpha = a_1 a_2 \cdots$ a sentence and denote the N-th coordinate by

$$\alpha[N] = a_N ,$$

the initial N-word by

$$[\alpha]_N = a_1 \cdots a_N ,$$

and the shifted sequence by

$$\theta\alpha = a_2 a_3 \cdots .$$

For words $u = a_1 \cdots a_N$, $v = b_1 \cdots b_M$, and a sentence $\alpha = c_1 c_2 \cdots$, we denote $u \cdot v = a_1 \cdots a_N b_1 \cdots b_M$ and $u \cdot \alpha = a_1 \cdots a_N c_1 c_2 \cdots$.

We introduce orders in the following way. For $x, y \in I$, by the expression $x <_\sigma y$ $(\sigma \in \{+, -\})$ we mean $x < y$ if $\sigma = +$ and $x > y$ if $\sigma = -$. We also use this expression for alphabets, words and sentences in a natural way.

(i) For words w_1, w_2,

$$w_1 < w_2 \qquad \text{if } [w_1]_N = [w_2]_N \text{ and } w_1[N+1] <_\sigma w_2[N+1] \text{ for some } N\,,$$

where $\sigma = sgn\,[w_1]_N$.

(ii) For sentences α_1, α_2,

$$\alpha_1 < \alpha_2 \qquad \text{if } [\alpha_1]_N < [\alpha_2]_N \text{ for some } N\,.$$

Until now, the notations mentioned above do not depend on the parameter t. Now let for a word $w = a_1 \cdots a_N$

$$\langle w \rangle_t = \bigcap_{i=1}^{N} F_t^{-i+1}(\langle a_i \rangle_t)$$

and for a sentence α

$$\{\alpha\}_t = \bigcap_{N=1}^{\infty} cl\langle [\alpha]_N \rangle_t\,.$$

Thus $\langle w \rangle_t$ (resp. $\{\alpha\}_t$) is the subinterval (resp. the point) corresponding to a word w (resp. a sentence α) with respect to the mapping F_t.

We denote by $W_N = A^N$ the set of all words with length N and set $W = \bigcup_{N=0}^{\infty} W_N$, where $W_0 = \{\varepsilon\}$. We denote by S the set of all sentences. By the assumption $\xi_t > 0$ the set $\{\alpha\}_t$ consists of exactly one point if $\{\alpha\}_t \neq \emptyset$. In [10], we restrict W_N, W and S the set of words or sentences for which $\langle w \rangle_t \neq \emptyset$ or $\{\alpha\}_t \neq \emptyset$. But in our situation, we need to consider w or α for which $\langle w \rangle_t = \emptyset$ and $\{\alpha\}_t = \emptyset$, because the symbolic dynamics may change with F_t.

2.2. Plus and minus expansions. As we discussed in [9], [10], the structure of the dynamics becomes much clearer if it is considered on the signed symbolic dynamics. In [10], we did not treat signed words $\tilde{\alpha}$ for which $\{\tilde{\alpha}\} = \emptyset$. But for later use, we need to define such signed sentences. Thus we slightly change the definitions, nevertheless the results in [10] still hold in our new definitions.

For each $x \in I$, we define a sentence $\alpha_t^x = a_1^x a_2^x \cdots \in S$, called the expansion of x, by the condition $F_t^{i-1}(x) \in \langle a_i^x \rangle_t$ for all i. Then, $x = \{\alpha_t^x\}_t$ since $\xi_t > 0$. For a sentence α we consider signed sentences α^+, α^- and denote by $\tilde{S}$ the set of all signed sentences. We can consider F as a shift operator on $\tilde{S}$. We define for a sentence $\alpha \in S$ and $\sigma \in \{+, -\}$:

(1)
$$\{\alpha^\sigma\}_t = \begin{cases} \sup\{x \in \langle \alpha[1] \rangle_t\} & \text{if } \alpha > \sup\{\alpha_t^x : x \in \langle \alpha[1] \rangle_t\}\,, \\ \inf\{x \in \langle \alpha[1] \rangle_t\} & \text{if } \alpha < \inf\{\alpha_t^x : x \in \langle \alpha[1] \rangle_t\}\,, \end{cases}$$

(2) otherwise

$$\{\alpha^\sigma\}_t = \begin{cases} \sup\{x \in I : \alpha_t^x < \alpha^\sigma,\ \alpha_t^x[1] = \alpha[1]\} & \text{if } \sigma = +\,, \\ \inf\{x \in I : \alpha_t^x > \alpha^\sigma,\ \alpha_t^x[1] = \alpha[1]\} & \text{if } \sigma = -\,, \end{cases}$$

where we consider the topology on S induced from the order. Note that if $\{\alpha\}_t \neq \emptyset$, then $\{\alpha^\sigma\}_t$ equals $\{\alpha\}_t$ as an expansion.

We can define F'_t on $\tilde{S}$:

$$(3)\qquad F'_t(\alpha^\sigma)=\begin{cases} \lim\limits_{x\uparrow \sup\langle\alpha[1]\rangle_t} F'_t(x) & \text{if } \alpha>\sup\{\alpha_t^x : x\in\langle\alpha[1]\rangle_t\}\,, \\ \lim\limits_{x\downarrow \inf\langle\alpha[1]\rangle_t} F'_t(x) & \text{if } \alpha<\inf\{\alpha_t^x : x\in\langle\alpha[1]\rangle_t\}\,, \end{cases}$$

(4) otherwise

$$F'_t(\alpha^\sigma)=\begin{cases} \lim\limits_{x\uparrow\{\alpha^\sigma\}_t} F'_t(x) & \text{if } \sigma=+\,, \\ \lim\limits_{x\downarrow\{\alpha^\sigma\}_t} F'_t(x) & \text{if } \sigma=-\,. \end{cases}$$

We define order on $\tilde{S}$ by

(1) if $\alpha<\beta$, then $\alpha^\sigma<\beta^\tau$ ($\sigma, \tau\in\{+, -\}$), and

(2) $\alpha^+<\alpha^-$.

The condition (2) may seem unnatural, but as we explained in [10], when α^+ (resp. α^-) is realized by some F_t, then α^+ (resp. α^-) is the limit of the expansion of the points from below (resp. upper).

We also consider w^+ and w^- for $w\in\bigcup_{N=1}^\infty W_N$ and we define $|w^+|=|w^-|=|w|$. We denote

$$\tilde{W}_N=\{w^\sigma : |w|=N, \sigma\in\{+, -\}\}\,,$$

$$\tilde{W}=\bigcup_{N=1}^{\infty}\tilde{W}_N\,,$$

$$\tilde{A}=\tilde{W}_1=\{a^\sigma : a\in A, \sigma\in\{+, -\}\}\,,$$

and by $\tilde{w}\in\tilde{W}$ ($\tilde{\alpha}\in\tilde{S}$) we denote w^+ or w^- (α^+ or α^-), respectively. We can naturally identify w^+ with $\sup\{\alpha^+\in\tilde{S} : [\alpha]_{|w|}=w\}$, and w^- with $\inf\{\alpha^-\in\tilde{S} : [\alpha]_{|w|}=w\}$. Let

$$\varepsilon(w^\sigma)=\varepsilon(\alpha^\sigma)=\sigma \qquad \sigma\in\{+, -\}\,,$$

and we call $\varepsilon(w^\sigma)$ and $\varepsilon(\alpha^\sigma)$ the sign of w^σ and α^σ. We also use the convention $\varepsilon(\theta^n\alpha^\sigma)=\varepsilon(\theta^n w^\sigma)=\sigma$, when such an expression appears in below.

LEMMA 2.3. *$F'_t(\tilde{\alpha})$ converges uniformly to $F'_\infty(\tilde{\alpha})$.*

PROOF. For any $w\in W$, we extend the map $F_t^{|w|}\downarrow_w^{-1}$, which is the inverse map of $F_t^{|w|}$ restricted to $\langle w\rangle_t$, as follows:

(1) If there exists $y\in\langle w\rangle_t$ such that $F_t^{|w|}(y)=x$, then we put $F_t^{|w|}\downarrow_w^{-1}(x)=y$.

(2) Otherwise,

$$F_t^{|w|}\downarrow_w^{-1}(x)=\begin{cases}\{w^+\}_t & \text{if } F_t^{|w|}(w^+)<_{sgn\,w}x\,,\\ \{w^-\}_t & \text{if } F_t^{|w|}(w^-)>_{sgn\,w}x\,.\end{cases}$$

Then for any $x, x'\in I$ and a word $w=a_1\cdots a_n$

$$|F_t^n\downarrow_w^{-1}(x')-F_\infty^n\downarrow_w^{-1}(x)|=|F_t^n\downarrow_w^{-1}(x')-F_t^n\downarrow_w^{-1}(x)|+|F_t^n\downarrow_w^{-1}(x)-F_\infty^n\downarrow_w^{-1}(x)|$$

$$\leq \sup_{y\in\langle w\rangle_t}|(F_t^n)'(y)|^{-1}|x-x'|+|F_t^{n-1}\downarrow_{[w]_{n-1}}^{-1}(F_t\downarrow_{a_n}^{-1}(x))-F_\infty^{n-1}\downarrow_{[w]_{n-1}}^{-1}(F_\infty\downarrow_{a_n}^{-1}(x))|$$

$$\leq K_1\rho^n|x-x'|+|F_t^{n-1}\downarrow_{[w]_{n-1}}^{-1}(x_1')-F_\infty^{n-1}\downarrow_{[w]_{n-1}}^{-1}(x_1)|\,,$$

where $x_1=F_\infty\downarrow_{a_n}^{-1}(x)$ and $x_1'=F_t\downarrow_{a_n}^{-1}(x)$. Hence we inductively get

$$|F_t^n\downarrow_w^{-1}(x')-F_\infty^n\downarrow_w^{-1}(x)|\leq K_1\rho^n|x-x'|+\frac{K_1}{1-\rho}\Delta(t)\,,$$

where

$$\Delta(t)=\sup_{a\in A}\sup_{y\in I}|F_t\downarrow_a^{-1}(y)-F_\infty\downarrow_a^{-1}(y)|\,.$$

Since for $\tilde{\alpha}=a_1a_2\cdots$

$$\{\tilde{\alpha}\}_t\in\bigcup_{p\in\langle a_{n+1}\rangle_t}\{y:\alpha_t^y=a_1\cdots a_n\alpha_t^p\}\,,$$

and $\Delta(t)$ converges to 0 as $t\to\infty$, this shows that $\{\tilde{\alpha}\}_t$ converges uniformly to $\{\tilde{\alpha}\}_\infty$ as $t\to\infty$. Then by continuity, it follows, if $\{\tilde{\alpha}\}_\infty\in\langle\tilde{\alpha}[1]\rangle_t$

$$|F_t'(\tilde{\alpha})-F_\infty'(\tilde{\alpha})|=|F_t'(\{\tilde{\alpha}\}_t)-F_\infty'(\{\tilde{\alpha}\}_\infty)|$$

$$\leq|F_t'(\{\tilde{\alpha}\}_t)-F_t'(\{\tilde{\alpha}\}_\infty)|+|F_t'(\{\tilde{\alpha}\}_\infty)-F_\infty'(\{\tilde{\alpha}\}_\infty)|$$

$$\leq\sup_{x\in I}|F_t''(x)|\,|\{\tilde{\alpha}\}_t-\{\tilde{\alpha}\}_\infty|+|F_t'(\{\tilde{\alpha}\}_\infty)-F_\infty'(\{\tilde{\alpha}\}_\infty)|\,,$$

and other cases can be shown in a similar way. This proves the lemma.

3. Fredholm matrices and Perron-Frobenius operators.

In this section, we fix t ($t=1, 2, \cdots, \infty$) and omit the suffix t and by F, P, we express F_t, P_t and so on.

In [10], we characterize the spectrum $Spec(F)$ of the Perron-Frobenius operator P associated with the piecewise C^2 mapping F by eigenvalues of the Fredholm matrix $\Phi(z)$, and using this result, we also characterize $Spec(F)$ by singularities of the zeta function $\zeta(z)$. These results are the main tools in the next section. The outline of [10] is as follows (for details, refer to [10]):

For a word $w\in W$, we consider a generating function for $g\in L^\infty$

$$s_g^w(z)=\sum_{n=0}^{\infty} z^n \int 1_w(x)g(F^n(x))dx\,,$$

where 1_w is the indicator function of $\langle w\rangle$. Then by expressing this on the signed symbolic dynamics, we get the renewal equation of the following form (the precise definitions will be given afterwards):

$$(I-\Phi(z))s_g(z)=\chi_g(z) \qquad (g\in L^\infty)\,.$$

Hence, the spectrum problem of the Perron-Frobenius operator now turns into the spectrum problem of the Fredholm matrix $\Phi(z)$ and we get the results which we mentioned in the introduction (Theorems B and C).

Now we will pick up the notations in [10] which we need in this article.

(1) Definition of $\Phi(z)$: Set for $\tilde{\alpha}\in\tilde{S}$ or $\tilde{\alpha}\in\tilde{W}$, and $\tilde{v}\in\tilde{W}$

$$\phi(\tilde{\alpha},\tilde{v})=\{(F'_{|v|})^{-1}-(F'_{|v|-1})^{-1}\delta[|v|>l(\tilde{\alpha})]\}(\tilde{\alpha}[1]\cdot\tilde{v})\{\delta[\tilde{v}\le_{\varepsilon(\tilde{\alpha})}\theta\tilde{\alpha}]-1/2\}\,,$$

where F'_N is a formal "derivative" which depends on first N coordinates defined by

$$F'_N(\tilde{\alpha})=\frac{F([\tilde{\alpha}]_N^+)-F([\tilde{\alpha}]_N^-)}{Lebes\langle[\tilde{\alpha}]_N\rangle}\,,$$

where $Lebes\,J$ is the Lebesgue measure of a set J and for a statement L

$$\delta[L]=\begin{cases}1 & \text{if } L \text{ is true}\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

and

$$l(\tilde{\alpha})=\begin{cases}1 & \tilde{\alpha}\in\tilde{S}\\ \min\{K:\{\tilde{\alpha}\}=\{\tilde{w}\},\ \tilde{w}\in\tilde{W}_K\} & \tilde{\alpha}\in\tilde{W}\,.\end{cases}$$

Then tracing the orbits of the division points of the partition, we define an infinite dimensional matrix $\Phi(z)=\Phi(z;F)$ on $\tilde{W}=\bigcup_{k=1}^{\infty}\tilde{W}_k$ corresponding to the mapping F by

$$\Phi(z)_{\tilde{u},\tilde{v}}=\begin{cases}\sum_{n=0}^{\infty} z^{n+1}F^{n\prime}(\tilde{u})^{-1}\phi(\theta^n\tilde{u},\tilde{v}) & \text{if } \{\tilde{u}\} \text{ is a division point}\,,\\ z\phi(\tilde{u},\tilde{v}) & \text{otherwise}\,.\end{cases}$$

We express by $\Phi_N(z)$ the truncation of $\Phi(z)$ to the index set $\bigcup_{k=1}^{N}\tilde{W}_k$.

(2) Definition of $s_g(z)$: Let for $g\in L^\infty$ and $\tilde{\alpha}\in\tilde{S}$ or $\tilde{\alpha}\in\tilde{W}$

$$\begin{aligned}s_g^{\tilde{\alpha}}&=s_g^{\tilde{\alpha}}(z\,;F)\\ &=\int dx\,g(x)\sum_{\substack{w\in W\\ \langle w\rangle\neq\varnothing}} z^{|w|}\,|(F^{|w|})'(w\cdot x)|^{-1}\varepsilon(\tilde{\alpha})\{\delta[w\cdot x<\{\tilde{\alpha}\}]-1/2\}\\ &\quad\cdot\delta[w[1]=\tilde{\alpha}[1],\ \{(\theta w)\cdot\alpha^x\}\neq\varnothing]\,,\end{aligned}$$

and $s_g(z)=(s_g^{\tilde{u}})_{\tilde{u}\in\tilde{W}}$.

(3) Definition of $\chi_g(z)$: Let for $g\in L^\infty$ and $\tilde{\alpha}\in\tilde{S}$ or $\tilde{\alpha}\in\tilde{W}$

$$\chi(\tilde{\alpha}, x)=\varepsilon(\tilde{\alpha})\{\delta[x<\{\tilde{\alpha}\}]-1/2\},$$

$$\chi_g^{\tilde{\alpha}}(z)=\begin{cases}\sum_{n=0}^{\infty} z^n((F^n)'(\tilde{\alpha}))^{-1}\int dx\, g(x)\chi\{\theta^n\tilde{\alpha}, x) & \text{if } \{\tilde{\alpha}\} \text{ is a division point}, \\ \int dx\, g(x)\chi(\tilde{\alpha}, x) & \text{otherwise},\end{cases}$$

and $\chi_g(z)=(\chi_g^{\tilde{u}}(z))_{\tilde{u}\in\tilde{W}}$.

LEMMA 3.1. (1) *For a word $u\in W$, we get*

$$s_g^u(z)=s_g^{u^+}+s_g^{u^-}.$$

(2) *For u such that the both $\{u^\sigma\}$ are not division points for $\sigma\in\{+,-\}$, we get*

$$\chi_g^{u^+}(z)+\chi_g^{u^-}(z)=\int_{\langle u\rangle} g(x)dx.$$

PROOF. The proofs of (1) and (2) are found in Lemma 3.1 and (3.3) of [10], respectively.

(4) Definition of $\mathscr{B}(z;F)$: We denote by $\mathscr{B}$ the space of vectors $s=(s^{\tilde{v}})_{\tilde{v}\in\tilde{W}}$ $(s^{\tilde{v}}\in \boldsymbol{C})$ which satisfies the following (i)–(iii).

(i) The components of s satisfy the relations

$$\varepsilon(\tilde{u})s^{\tilde{u}}=\varepsilon(\tilde{v})s^{\tilde{v}}$$

whenever $\{\tilde{u}\}=\{\tilde{v}\}$ and $\tilde{u}[1]=\tilde{v}[1]$, that is, $\tilde{u}$ and $\tilde{v}$ express the same point with same first alphabet.

(ii) The following limit exists for $\tilde{\alpha}\in\tilde{S}$, and coincides with $s^{\tilde{u}}$ if $\tilde{\alpha}=\tilde{u}\in\tilde{W}$:

$$s^{\tilde{\alpha}}=\lim_{N\to\infty}\varepsilon(\alpha)s^{[\tilde{\alpha}]_N^+}$$
$$=-\lim_{N\to\infty}\varepsilon(\alpha)s^{[\tilde{\alpha}]_N^-},$$

where $[\tilde{\alpha}]_N^\sigma=\{[\tilde{\alpha}]_N\}^\sigma$.

(iii)
$$\|s\|=\|s\|_\infty+\|s\|_v<\infty,$$

where

$$\|s\|_\infty=\sup_{\tilde{w}\in\tilde{W}}|s^{\tilde{w}}|,$$

$$\|s\|_v=\sup_{V(f)=1}|\langle f, s\rangle|,$$

$$\langle f, s\rangle = \limsup_{N\to\infty} \sum_{u\in W_N} \frac{\int_{\langle u\rangle} f dx}{Lebes\langle u\rangle} s^u .$$

We also use the following norm for $0<r<1$

$$\| s \|_r = \sup_{N\geq 1} \sup_{\tilde{\alpha},\, \tilde{\beta}\in\tilde{S}} \{ | s^{\tilde{\alpha}} + s^{\tilde{\beta}} | r^{-N} : [\tilde{\alpha}]_N = [\tilde{\beta}]_N \text{ and } \varepsilon(\tilde{\alpha})\varepsilon(\tilde{\beta}) = - \} .$$

We get the relations of the norm:

LEMMA 3.2 (Lemma 4.2 in [10]). (1) *If* $\| s_g(z) \|_v < \infty$ *for* $g\in L^\infty$, *then* $\| s_g(z) \|_\infty < \infty$.
(2) *If* $\| s \|_r < \infty$ *for some* $0<r<1$, *then* $\| s \|_v < \infty$.

Now let $\mathscr{B}(z\,;F)$ be the set of $s=(s^{\tilde{u}})\in\mathscr{B}$ which satisfies:

(iv) $$\sup_{u\in W} \frac{| s^u - z\, | F'_{|u|}(u) |^{-1} s^{\theta u} |}{Lebes\langle u\rangle} < \infty ,$$

where $s^u = s^{u^+} + s^{u^-}$ and $s^{\theta u} = s^{(\theta u)^+} + s^{(\theta u)^-}$.

(5) Definition of $\mathscr{X}$: Let

$$\mathscr{X} = \mathscr{X}(z\,;F) = \{ \chi_g(z) : g\in L^\infty \} .$$

For a vector $s=\chi_g(z)\in\mathscr{X}$, we define norm by

$$\| s \|_\infty = \| \chi_g(z) \|_\infty = \| g \|_\infty = \operatorname*{ess\,sup}_{x\in I} | g(x) | .$$

Then we get from the definitions:

THEOREM 3.3 ([10]). *$(I-\Phi(z))$ is a bounded operator from $\mathscr{B}(z\,;F)$ to $\mathscr{X}(z\,;F)$ if $|z|<r/\rho$.*

The proofs are found in Proposition 4.4 [10].

Then the spectrum problem of the Perron–Frobenius operator can be expressed in terms of the Fredholm matrix, and one of the aim of [10] is to prove the following theorem.

THEOREM 3.4 (Theorem 6.3 in [10]). *For $|z|<e^\xi$, the following statements are equivalent*:

(1) $z^{-1}\in Spec(F)$,
(2) $(I-\Phi_N(z))^{-1}$ *is unbounded.*

To prove this theorem, we need the following lemmas.

LEMMA 3.5 (Lemma 6.2 in [10]). *Suppose that $z^{-1}\notin Spec(F)$ and there exists $s\in\mathscr{B}(z\,;F)$ such that $(I-\Phi(z))s=\chi_g(z)$, then $s^{u^+}+s^{u^-}=s^u_g(z)$ for any $u\in W$.*

LEMMA 3.6 (Lemma 6.4 in [10]). *If* $(I-\Phi_N(z))^{-1}$ *is unbounded, then there exists* $s\in\mathscr{B}(z;F)$ *such that* $\|s\|=1$ *and* $(I-\Phi(z))s=0$.

LEMMA 3.7 (cf. the proof of Lemma 6.5 in [10]). *Suppose that* $(I-\Phi_N(z))^{-1}$ *is bounded, then there exists* $s\in\mathscr{B}(z;F)$ *such that* $\|s\|_r<\infty$ *for some* $0<r<1$ *and* $(I-\Phi(z))s=\chi_g(z)$.

LEMMA 3.8. (1) *For* $s\in\beta$,

$$\{\Phi(z)s\}^{\tilde{u}} = \begin{cases} z\lim\limits_{N\to\infty}\sum\limits_{\tilde{v}\in\tilde{W}_N} F_N'(u[1]\tilde{v})^{-1}\{\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}\{s^{v^+}+s^{v^-}\} & \tilde{u}\notin\tilde{A}\,, \\ \sum\limits_{n=1}^{\infty}\lim\limits_{N\to\infty}\sum\limits_{\tilde{v}\in\tilde{W}_N}(F^{n-1})'(\tilde{u})^{-1}F_N'(u[n]\tilde{v})^{-1}\{\delta[\tilde{v}\leq_{\varepsilon(\tilde{u})}\theta^n\tilde{u}]-1/2\}s^{\tilde{v}} & \tilde{u}\in\tilde{A}\,. \end{cases}$$

(2) *Let* $\hat{s}=(I-\Phi(z))s$ *for* $s\in\mathscr{B}$. *Then* $\hat{s}\in\mathscr{S}$ *and for* $\tilde{a}\in\tilde{A}$

$$\hat{s}^{\tilde{a}}=\sum_{n=0}^{\infty}z^n\varepsilon(\tilde{a})F^{n\prime}(\tilde{a})^{-1}\lim_{\tilde{u}\to\theta^n\tilde{a}}\hat{s}^{\tilde{u}}\,,$$

where for $\tilde{u}$ *which appears in* $\lim_{\tilde{u}\to\theta^n\tilde{a}}$ $\{\tilde{u}\}$ *is not a division point.*

(3) *For* s *such that* $\|s\|_r<\infty$ *and* $|z|<r/\rho$,

$$\|(I-\Phi(z))s\|_\infty\leq K_1K_2(1-|z|\rho)^{-1}(1-r)^{-1}\|s\|_r/\rho\,.$$

PROOF. (1) For $\tilde{u}\in\tilde{W}$ such that $\{\tilde{u}\}$ is not a division point,

$$\begin{aligned}\{\Phi(z)s\}^{\tilde{u}} &= z\sum_{\tilde{v}}\phi(\tilde{u},\tilde{v})s^{\tilde{v}} \\ &= z\lim_{N\to\infty}\sum_{k=1}^{N}\sum_{\tilde{v}\in\tilde{W}_k}\phi(\tilde{u},\tilde{v})s^{\tilde{v}} \\ &= z\lim_{N\to\infty}\sum_{k=l(\tilde{u})}^{N}\sum_{\tilde{v}\in\tilde{W}_k}\{(F_{|v|}')^{-1}-(F_{|v|-1}')^{-1}\delta[|v|>l(\tilde{u})]\}(\tilde{u}[1]\cdot\tilde{v}) \\ &\qquad\cdot\{\delta[\tilde{v}\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}s^{\tilde{v}} \\ &= z\lim_{N\to\infty}\sum_{\tilde{v}\in\tilde{W}_N}(F_N')^{-1}(\tilde{u}[1]\cdot\tilde{v})\{\delta[\tilde{v}\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}s^{\tilde{v}}\,.\end{aligned}$$

Since $\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]=\delta[v^-\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]$ if $|\tilde{u}|\leq N$, we get the proof of (1) for the case $\tilde{u}$ which is not a division point. The proof of another case is almost the same.

(2) For $\tilde{u}$ which is not a division point and $\varepsilon(\tilde{u})=\varepsilon(\tilde{a})$,

$$\sum_{n=0}^{\infty}z^n(F^n)'(\tilde{a})^{-1}\lim_{\tilde{u}\to\theta^n\tilde{a}}\hat{s}^{\tilde{u}}$$

$$= \sum_{n=0}^{\infty} z^n (F^n)'(\tilde{a})^{-1} \lim_{\tilde{u} \to \theta^n \tilde{a}} \left(s^{\tilde{u}} - z \lim_{N \to \infty} \sum_{\tilde{v} \in \tilde{W}_N} F'_N(u[1]\tilde{v})^{-1} \{ \delta[\tilde{v} \leq_{\varepsilon(\tilde{u})} \theta \tilde{u}] - 1/2 \} s^{\tilde{v}} \right)$$

$$= \sum_{n=0}^{\infty} z^n (F^n)'(\tilde{a})^{-1} \lim_{\tilde{u} \to \theta^n \tilde{a}} \left(s^{\tilde{u}} - z \lim_{N \to \infty} \sum_{\tilde{v} \in \tilde{W}_N} \varepsilon(\tilde{a}) F'_N(u[1]\tilde{v})^{-1} \{ \delta[\tilde{v} < \theta \tilde{u}] - 1/2 \} s^{\tilde{v}} - z F'(\tilde{u})^{-1} s^{\theta \tilde{u}} \right)$$

$$= \sum_{n=0}^{\infty} z^n (F^n)'(\tilde{a})^{-1} \left(s^{\theta^n \tilde{a}} - \lim_{N \to \infty} \sum_{\tilde{v} \in \tilde{W}_N} \varepsilon(\tilde{a}) F'_N(a[n]\tilde{v})^{-1} \{ \delta[\tilde{v} < \theta^{n+1} \tilde{a}] - 1/2 \} s^{\tilde{v}} \right) - \sum_{n=1}^{\infty} z^n \varepsilon(\tilde{a}) (F^n)'(\tilde{a})^{-1} s^{\theta^n \tilde{a}}$$

$$= s^{\tilde{a}} - \{\Phi(z)s\}^{\tilde{a}} = \hat{s}^{\tilde{a}} .$$

Thus we get the proof of (2).

(3) For $\tilde{u}$ such that $\{\tilde{u}\}$ is not a division point, we get from (1)

$$\{\Phi(z)s\}^{\tilde{u}} = \lim_{N \to \infty} z \sum_{v \in W_N} F'_N(u[1]v)^{-1} \{ \delta[v^+ \leq_{\varepsilon(\tilde{u})} \theta \tilde{u}] - 1/2 \} (s^{v^+} + s^{v^-})$$

$$= \lim_{N \to \infty} z \sum_{v \in W_N} F'_N(u[1]v)^{-1} \{ \delta[v^+ \leq_{\varepsilon(\tilde{u})} \theta \tilde{u}] - 1/2 \} s^{v}$$

$$= \lim_{N \to \infty} x[N] .$$

Then, we get by Lemma 2.2

$$| x[N] - x[N-1] | = \left| \sum_{v \in W_N} z \{ (F'_N)^{-1} - (F'_{N-1})^{-1} \} (u[1] \cdot v) \{ \delta[v \leq_{\varepsilon(\tilde{u})} \theta \tilde{u}] - 1/2 \} s^{v} \right|$$

$$\leq |z| \sum_{v \in W_N} K_2 Lebes\langle v \rangle |s^v|$$

$$\leq K_2 r^N \| s \|_r / \rho ,$$

and this shows

$$| \{\Phi(z)s\}^{\tilde{u}} | \leq K_2 (1-r)^{-1} \| s \|_r / \rho .$$

In a similar way, using Lemma 2.1, we get $| z^n ((F^n_t)')^{-1} | < K_1 r^n$. Therefore by (2) we get the evaluation for $\tilde{u}$ when $\{\tilde{u}\}$ is a division point. This proves (3).

The outline of the proof of Theorem 3.4 is as follows:

If $(I - \Phi_N(z))^{-1}$ is unbounded, then by Lemma 3.6, there exists $s \in \mathscr{B}(z\,;\,F)$ such that

$\| s \| = 1$ and $(I - \Phi(z))s = 0$. Suppose that $z^{-1} \notin Spec(F)$. Then by Lemma 3.5, $s^{u^+} + s^{u^-} = 0$ for all $u \in W$. Therefore by Lemma 3.8 (1)

$$\{(I - \Phi(z))s\}_{\bar{u}} = s^{\bar{u}} - (\Phi(z)s)_{\bar{u}} = s^{\bar{u}} .$$

Hence we get $s = 0$. This is the contradiction. Hence if $(I - \Phi_N(z))^{-1}$ is unbounded, then $z^{-1} \in Spec(F)$. On the contrary, if $(I - \Phi_N(z))^{-1}$ is bounded, then by Lemma 3.7, there exists $s \in \mathscr{B}(z ; F)$ such that $\| s \|_r < \infty$ and $(I - \Phi(z))s = \chi_g(z)$. Since the eigenvalues of the Perron-Frobenius operator is isolated in $|z| > e^{-\xi}$([3]), $\mathscr{C} = \{z : (I - \Phi_N(z))^{-1}$ is unbounded$\}$ is also isolated. At the same time, $s_g(z ; F_N)$ converges to $s_g(z ; F)$ in $|z| < 1$, this shows the existence of $s_g(z ; F) \in \mathscr{B}(z ; F)$ in $z \in \mathscr{C}$. Therefore, if $(I - \Phi_N(z))^{-1}$ is bounded, for any $f \in BV$, there exists

$$\int \{(I - zP)^{-1} f(x)\} g(x) dx = \sum_{u \in W} C_u s_g^u(z ; F) ,$$

where $f(x) = \sum_{u \in W} C_u 1_u(x)$ and $\sum_{u \in W} | C_u | r^{|u|} < \infty$. This proves the theorem.

4. The proof of Theorem A.

We need the following theorem for the proof of Theorem A.

THEOREM 4.1.

$$n(z ; F) = \limsup_{t \to \infty} \inf_{0 < r < 1} \sup_{\| g \| = 1} \| s_g(z ; F_t) \|_r$$

is bounded in some neighborhood of $z_\infty (| z_\infty | < e^{\xi})$, *if and only if* $z_\infty^{-1} \notin Spec(F_\infty)$.

We will prove Theorem 4.1 after the proof of Theorem A.

PROOF OF THEOREM A. Since the eigenvalues of the Perron-Frobenius operator P_∞ has no accumulation point in $|z| < e^{\xi_\infty}$ (cf. [3]), $n(z ; F)$ is uniformly bounded in wider sense in the domain $\mathscr{D} = \{z : |z| < e^{\xi_\infty}, z^{-1} \notin Spec(F_\infty)\}$. Moreover $s_g(z ; F_t)$ converges to $s_g(z ; F_\infty)$ as $t \to \infty$ in the unit disk, therefore the above convergence still holds with respect to $\| \cdot \|_r$ for some r in the domain $\mathscr{D}$. Let for $f \in BV$

$$\begin{aligned} (f, g)(z ; F_t) &= \sum_{n=0}^{\infty} z^n \int_I f(x) g(F_t^n(x)) dx \\ &= \int_I \{(I - zP_t)^{-1} f(x)\} g(x) dx \\ &= \sum_{w \in W} C_w s_g^w(z ; F_t) , \end{aligned}$$

where $f = \sum_{w \in W} C_w 1_w$. Note that for $f \in BV$ there exists a decomposition such that $f = \sum_{w \in W} C_w 1_w$ satisfies $\sum_{w \in W} | C_w | r^{|w|} < \infty$ for any $0 < r < 1$ (cf. [10]). Therefore

$(f, g)(z; F_\infty)$ is bounded in wider sense and $(f, g)(z; F_t)$ converges to $(f, g)(z; F_\infty)$ as $t \to \infty$ in the domain $\mathscr{D}$. Therefore by Rouché's theorem, for any U such that $(f, g)(z; F_\infty) \neq 0$ on ∂U the number of singularities of $(f, g)(z; F_t)$ in U equals that of $(f, g)(z; F_\infty)$ for sufficiently large t. Now suppose that $z_\infty^{-1} \in Spec(F_\infty)$. Then there exists $f \in BV$ such that $(f, g)(z; F_\infty)$ has a singularity at z_∞ for some $g \in L^\infty$. Therefore in any neighborhood U of z_∞, there exists a singularity of $(f, g)(z; F_t)$ for sufficiently large t. This proves the existence of a sequence $\{z_t\}$ such that z_t converges to z_∞ and $z_t^{-1} \in Spec(F_t)$. On the contrary, if $z_\infty^{-1} \notin Spec(F_\infty)$, then $\| s_g(z; F_\infty) \|_r$ is bounded in some neighborhood U of z_∞ for some r. Therefore $(f, g)(z; F_\infty)$ has no singularity in U for any $f \in BV$ and $(f, g)(z; F_t)$ has also no singularity in U for sufficiently large t. Hence there exists no $z_t \in U$ such that $z_t^{-1} \in Spec(F_t)$. This completes the proof of Theorem A.

PROOF OF THEOREM 4.1. We only need to show:

(1) if $n(z; F)$ is unbounded in any neighborhood of z_∞, then there exists $s \in \mathscr{B}(z; F_\infty)$ such that $s \neq 0$ and $(I - \Phi(z; F_\infty))s = 0$,

(2) if $n(z; F)$ is bounded in some neighborhood of z_∞, then for any $g \in L^\infty$ there exists $s_g(z; F_\infty) \in \mathscr{B}(z; F_\infty)$.

Because if there exists $s \in \mathscr{B}(z; F_\infty)$ such that $s \neq 0$ and $(I - \Phi(z; F_\infty))s = 0$, and if we assume that $z^{-1} \notin Spec(F)$, this contradicts Lemma 3.5. Therefore if $n(z; F)$ is unbounded, then $z^{-1} \in Spec(F_\infty)$. On the contrary, if $s_g(z; F_\infty) \in \mathscr{B}(z; F_\infty)$, then $(f, g)(z)$ exists for any $f \in BV$. Therefore $z^{-1} \notin Spec(F_\infty)$.

Now we will prove (1). From the assumption, there exist sequences $\{z_t\}$ and $\{g_t\}$ such that

(i) $\lim_{t \to \infty} z_t = z_\infty$,

(ii) $g_t \in L^\infty$, $\lim_{t \to \infty} \| g_t \|_\infty = 0$ and $\inf_{0<r<1} \| s_{g_t}(z_t; F_t) \|_r = 1$.

Then there exists a subsequence, which we also denote with suffix t, such that there exists a limit

$$s_\infty^{\tilde{w}} = \lim_{t \to \infty} s_{g_t}^{\tilde{w}}(z; F_t) \qquad \text{for any } \tilde{w} \in \tilde{W},$$

and by (ii) it follows $s_\infty \neq 0$. It also holds $\| s_\infty \|_r \leq 1$, because for α^-, β^+ such that $\alpha^- < \beta^+$ and $[\alpha^-]_N = [\beta^+]_N$

$$|s_\infty^{\alpha^-} + s_\infty^{\beta^+}| = \lim_{t \to \infty} |s_{g_t}^{\alpha^-}(z; F_t) + s_{g_t}^{\beta^+}(z; F_t)|$$

$$\leq r^N \lim_{t \to \infty} \| s_{g_t}(z; F_t) \|_r .$$

Then by Lemma 3.2, we get $\| s_\infty \|_\infty < \infty$. Thus we only need to prove

$$(I-\Phi(z_\infty\,;\,F_\infty))s_\infty=0\,.$$

Let $s\in\mathscr{B}(z\,;\,F)$ such that $\|s\|_r<\infty$. Then for $\tilde{u}$ such that $\{\tilde{u}\}_\infty$ is not a division point of the partition, since $\{\tilde{u}\}_t$ is not also a division point for the partition corresponding to F_t for sufficiently large t, we get

$$(\sharp)\qquad \{[\Phi(z_t\,;\,F_t)-\Phi(z_\infty\,;\,F_\infty)]s\}^{\tilde{u}}$$

$$=\lim_{N\to\infty}\sum_{|v|=N}[z_tF'_{t,N}(u[1]\cdot v)^{-1}-z_\infty F'_{\infty,N}(u[1]\cdot v)^{-1}]$$

$$\cdot\{\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}(s^{v^+}+s^{v^-})$$

$$=\lim_{N\to\infty}\sum_{|v|=N}(z_t-z_\infty)F'_{t,N}(u[1]\cdot v)^{-1}\{\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}(s^{v^+}+s^{v^-})$$

$$+\lim_{N\to\infty}\sum_{|v|=N}z_\infty[F'_{t,N}(u[1]\cdot v)^{-1}-F'_{\infty,N}(u[1]\cdot v)^{-1}]$$

$$\cdot\{\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}(s^{v^+}+s^{v^-})\,.$$

Now we will show ($\sharp$) tends to 0 as $t\to\infty$. By Lemma 3.8 (3),

$$|\text{the first term of the right hand term of }(\sharp)|$$

$$=\left|\frac{z_t-z_\infty}{z_t}\{\Phi(z_t\,;\,F_t)s\}^{\tilde{u}}\right|\leq\left|\frac{z_t-z_\infty}{z_t}\right|K_1K_2\|s\|_r(1-r)^{-1}\,.$$

Therefore this term converges to zero as $t\to\infty$. To prove the second term of the right hand term of ($\sharp$) converges to 0, let

$$y[N]=\sum_{|v|=N}z_\infty[|F'_{t,N}(u[1]\cdot v)|^{-1}-|F'_{\infty,N}(u[1]\cdot v)|^{-1}]$$

$$\cdot\{\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}(s^{v^+}+s^{v^-})\,.$$

Then the second term of ($\sharp$) equals $\lim_{N\to\infty}y[N]$. We get

$$y[N+1]-y[N]$$

$$=\sum_{|v|=N+1}z_\infty\{\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}(s^{v^+}+s^{v^-})$$

$$\cdot[(|F'_{t,N+1}(u[1]\cdot v)|^{-1}-|F'_{t,N}(u[1]\cdot v)|^{-1})$$

$$-(|F'_{\infty,N+1}(u[1]\cdot v)|^{-1}-|F'_{\infty,N}(u[1]\cdot v)|^{-1})]$$

$$=\sum_{|v|=N+1}z_\infty\{\delta[v^+\leq_{\varepsilon(\tilde{u})}\theta\tilde{u}]-1/2\}(s^{v^+}+s^{v^-})$$

$$\cdot|(F'_{t,N+1}F'_{t,N}F'_{\infty,N+1}F'_{\infty,N})(u[1]\cdot v)|^{-1}$$

$$\cdot|(\{(F_\infty-F_t)'_{N+1}-(F_\infty-F_t)'_N\}F'_{t,N}F'_{\infty,N})(u[1]\cdot v)$$

$$+((F'_{\infty,N}-F'_{t,N})\{(F'_{t,N}-F'_{t,N+1})F'_{\infty,N} + F'_{t,N+1}(F'_{\infty,N}-F'_{\infty,N+1})\})(u[1]\cdot v)|\,.$$

Then $|(s^{v^+}+s^{v^-})|\le\|s\|_r r^N$ and noticing for example

$$|F'_{\infty,N}-F'_{\infty,N+1}|(u[1]\cdot v)\le \sup_{x\in I}|F''_{\infty}(x)|\,Lebes\langle u[1]\cdot v\rangle\,,$$

we get $\lim_{t\to\infty}$(the second term of (♯))$=0$. Therefore (♯) tends to 0 as $t\to\infty$. Now

$$\begin{aligned}(I-\Phi(z_\infty\,;\,F_\infty))s_\infty &= \lim_{t\to\infty}\{(I-\Phi(z_t\,;\,F_t))s_{g_t}(z_t\,;\,F_t)+(\Phi(z_t\,;\,F_t)-\Phi(z_\infty\,;\,F_\infty))s_{g_t}(z_t\,;\,F_t) \\ &\qquad + \Phi(z_\infty\,;\,F_\infty)(s_{g_t}(z_t\,;\,F_t)-s_\infty)\} \\ &=0\,.\end{aligned}$$

By Lemma 3.8 (2), we can prove (1) for $\tilde{u}$ which is not a division point. This proves (1).

Since $Spec(F_\infty)$ has no accumulation point in $|z|<e^{\xi_\infty}$, the set of z which satisfies the condition (1) has also no accumulation point. Therefore, as in the proof of Theorem 3.4, since $n(z\,;\,F)$ is uniformly bounded in wider sense in $\mathscr{D}$ and $s_g^{\tilde{\alpha}}(z\,;\,F_t)$ converges to $s_g^{\tilde{\alpha}}(z\,;\,F_\infty)$ in the unit disk for any $\tilde{\alpha}\in\tilde{S}$, we can prove

(i) $s_g^{\tilde{\alpha}}(z\,;\,F_t)$ converges to $s_g^{\tilde{\alpha}}(z\,;\,F_\infty)$ in $\mathscr{D}$,

(ii) for any $\tilde{\alpha}$, $\tilde{\beta}$ which satisfies $(\tilde{\alpha},\tilde{\beta})=\varnothing$ we get

$$\varepsilon(\tilde{\alpha})s_g^{\tilde{\alpha}}(z\,;\,F_\infty)=\varepsilon(\tilde{\beta})s_g^{\tilde{\beta}}(z\,;\,F_\infty)\,,$$

(iii) $\|s_g(z\,;\,F_\infty)\|_r$ is bounded for some $0<r<1$.

Thus $s_g(z\,;\,F_\infty)\in\mathscr{B}(z\,;\,F_\infty)$. This proves (2), hence the theorem is proved.

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