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A Sufficient Condition for the Existence of Periodic Points of Homeomorphisms on Surfaces

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Abstract. Let $\rho_1, \rho_2, \cdots, \rho_{2g+1}$ be rotation vectors for periodic points of a homeomorphism on an orientable surface of genus $g>1$. Assume that the convex hull of the set $\{\rho_1, \rho_2, \cdots, \rho_{2g+1}\}$, $\mathrm{Conv}(\rho_1, \rho_2, \cdots, \rho_{2g+1})$, has nonempty interior. We will give a sufficient condition for the existence of a dense subset of $\mathrm{Conv}(\rho_1, \rho_2, \cdots, \rho_{2g+1})$ that is realized by periodic points.

0. Introduction.

Last several years, rotation sets for homeomorphisms on surfaces have been intensively investigated. Especially for tori and annuli, this investigation has given us valuable knowledge on chaotic behavior of homeomorphisms. In [7] Pollicott introduced a concept of rotation sets in higher genus case, and obtained a sufficient condition in terms of this rotation set for a homeomorphism to have positive topological entropy. In the case of tori, it has been shown that for any rational point in the interior of a rotation set, there exists a periodic point of which rotation vector is this rational point [3], however, in the case of a surface of genus greater than 1, there is no corresponding result.

In this paper, we will show a result on the relation between rational rotation vectors and periodic points. Here as a concept of rotation sets, we adopt the one introduced by Pollicott. To state our main theorem, we need some preparation. Let N be a surface of genus greater than 1, and let $g: N \to N$ be a homeomorphism of N isotopic to the identity. Let $P=\{y_1, \cdots, y_n\}$ be a set of fixed points of g. Isotoping g to leave P fixed, if necessary, let us assume that g is continuously differentiable and normal at P. Then as in [1], to blow up N at P, one obtains a surface N_P and an induced homeomorphism $\hat{g}: N_P \to N_P$. We will use the similar symbols for another surface M, a homeomorphism f of M and its fixed point set Q, *i.e.* M_Q, $\hat{f}$. For a set of periodic points P, let $k(P)$ denote the least common period of y_j, $j=1, \cdots, n$, and for $B \subset \boldsymbol{R}^m$, let $\mathrm{Conv}\,B$ denote the convex hull of B. Let us call two periodic points of f *m-Nielsen equivalent* if they

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are Nielsen equivalent as fixed points of f^m for a positive integer m, and let us define an *m-Nielsen class* of periodic points by an equivalence class under this equivalence relation, and for distinct two homeomorphisms f, g, let us also call an f-periodic point and a g-periodic point *m-Nielsen equivalent* if they are Nielsen equivalent as fixed points of f^m and g^m. Now we state the main theorem.

THEOREM. *Let M be a connected orientable closed surface of genus $g>1$, and let $f: M\to M$ be a homeomorphism isotopic to the identity. Let $x_1, x_2, \cdots, x_N$ be periodic points of f, where $N=2g+1$, and let $\rho_1, \rho_2, \cdots, \rho_N$ be rotation vectors for these periodic points. Set $P=\{x_1, x_2, \cdots, x_N\}$.*

Assume that $\mathrm{Conv}(\rho_1, \rho_2, \cdots, \rho_N)$ *has an interior point ρ_0 corresponding to a periodic point of the blown up homeomorphism $\hat{f}$ belonging to an $mk(P)$-Nielsen class of non-zero index for some $m>0$. Then*

i) f is isotopic to a generalized pseudo-Anosov homeomorphism,

ii) there exists a dense subset of $\mathrm{Conv}(\rho_1, \rho_2, \cdots, \rho_N)$ *that consists of rotation vectors for periodic points.*

In §1, we give definitions and preliminary argument, and then two lemmas are shown. §2 is dovoted to prove the main theorem.

1. Preliminaries.

Let M be a connected orientable closed surface, and let $f: M\to M$ be a homeomorphism isotopic to the identity. We will give a definition of rotation sets as in [7], but in a slightly modified way. To make the definition, let us give some preparation. Gluing $M\times\{1\}\subset M\times[0, 1]$ to $M\times\{0\}$ by identifying $(x, 1)$ with $(f(x), 0)$, the mapping torus V_f is obtained, and under this identification, the suspension flow f_t on V_f with respect to f is defined, *i.e.* $f_t(x, s)=(f^{[t+s]}(x), t+s-[t+s])$. Let $C^0(M, S^1)$ be the set of continuous maps from M to $S^1=\{z\in \boldsymbol{C} \mid |z|=1\}$. Then the set of homotopy equivalence classes of $C^0(M, S^1)$ is naturally identified with $H^1(M; \boldsymbol{Z})$ as follows. For any closed path $\gamma: S^1\to M$ and $[\alpha]\in C^0(M, S^1)/\sim$, set $[\alpha]([\gamma])=\deg(\alpha\circ\gamma)$, where $[\gamma]$ denotes an element of $H_1(M; \boldsymbol{Z})$ represented by γ. Similarly we may regard $H^1(V_f; \boldsymbol{Z})$ as $C^0(V_f, S^1)/\sim$.

Set $\pi(t)=e^{2\pi it}$ for $t\in\boldsymbol{R}$, then one obtains a universal cover of S^1. For any $\alpha\in C^0(V_f, S^1)$ and $x\in M$, set $\alpha_x(t)=\alpha\circ f_t(x, 0)$ for $t\geq 0$. Lifting $\alpha_x: [0, +\infty)\to S^1$ to $\boldsymbol{R}$ with respect to π, one obtains a function $\tilde{\alpha}_x$, and set

$$\Lambda_{x,T}(\alpha)=\frac{1}{T}(\tilde{\alpha}_x(T)-\tilde{\alpha}_x(0)) \qquad \text{for} \quad T>0\,.$$

Thus to assign α to $\Lambda_{x,T}(\alpha)$, one obtains functionals $\Lambda_{x,T}$ on $C^0(V_f, S^1)$. Let F_x be the set of limit points of $\{\Lambda_{x,T}\}_{T>0}$. As in [7], for each x, $\Lambda_x\in F_x$ takes constant values on homotopy equivalence classes. Hence we may consider $\Lambda_x \in H_1(V_f; \boldsymbol{R})$. Here, since f is isotopic to the identity, V_f is homeomorphic to $M \times S^1$. According to Hamstrom [4], the identity component of the group of the homeomorphisms of an oriented closed surface of genus >1 is contractible. Therefore any two isotopies joining the identity to f are mutually homotopic. Equivalently, the identification of V_f with $M \times S^1$ is unique up to homotopy. Especially the identification $H_1(V_f; \boldsymbol{R}) \cong H_1(M; \boldsymbol{R}) \oplus \boldsymbol{R}$ is unique, and an easy calculation shows that

$$\Lambda_x = (\rho_x, 1) \in H_1(M; \boldsymbol{R}) \oplus \boldsymbol{R} .$$

The argument in the last paragraph implies that ρ_x depends only on the homeomorphism f, and hence, set

$$\rho(x, f) = \{\rho_x \mid \Lambda_x = (\rho_x, 1) \in F_x\} ,$$

and let us call $\rho(x, f)$ *the rotation set of f at x*. Set $\rho(f) = \bigcup_{x \in M} \rho(x, f)$, and let us call this *the rotation set of f*. By definition, if x is a periodic point, then $\rho(x, f)$ consists of only one point, and thus let us call this point *a rotation vector*. Note that for the 2-torus case, the Hamstrom's theorem stated above does not hold, and thus in order to define rotation sets, one needs to specify the lift of a homeomorphism to the universal covering (cf. [3]).

We will show two lemmas. To do this, we need a preliminary argument. Let us give a homeomorphism from V_f to $M \times S^1$. Let $H_f : M \times I \to M$ be an isotopy from the identity to f, *i.e.* $H_f(x, 0) = x$ and $H_f(x, 1) = f(x)$, and for $(x, t) \in M \times I$, set

$$\tilde{H}_f(x, t) = (H_f(x, t), t) .$$

Then clearly $\tilde{H}_f : M \times I \to M \times I$ is a homeomorphism, and this induces a homeomorphism $\overline{H_f} : V_f \to M \times S^1$, because $\tilde{H}_f(f(x), 0) = (f(x), 0)$ and $\tilde{H}_f(x, 1) = (f(x), 1)$. The suspension flow f_t induces the flow on $M \times S^1$ that is equivalent to f_t under $\overline{H_f}$. We identify the flow f_t on V_f with this induced flow on $M \times S^1$, and denote this by the same symbol. Thus we may deal with suspension flows for distinct homeomorphisms, both of which are isotopic to the identity, as flows on the same ambient manifold $M \times S^1$, and by [4] again, we may culculate rotation sets by using these induced flows on $M \times S^1$ instead of suspension flows originally defined on mapping tori.

Let us give the first lemma.

LEMMA 1. *Let x be a periodic point of f with period n. Then*

$$\rho(x, f^n) = n\rho(x, f) .$$

The proof of this lemma is an easy exercise. The second lemma is as follows.

LEMMA 2. *Let f and g be homeomorphisms on M isotopic to the identity, and let x and y be periodic points of f and g with the same least period n. Assume that the f-orbit* *of x is globally shadowed by the g-orbit of y. Then $\rho(x, f)=\rho(y, g)$.*

PROOF. Let us recall that x and y are n-Nielsen equivalent [5]. By Lemma 1, it is sufficient to show that $\rho(x, f^n)=\rho(y, g^n)$.

Let H_{f^n} and H_{g^n} be isotopies from the identity to f^n and g^n. As in the argument previous to Lemma 1, one obtains homeomorphisms

$$\overline{H_{f^n}}: V_{f^n}\to M\times S^1 , \qquad \overline{H_{g^n}}: V_{g^n}\to M\times S^1 .$$

Under these homeomorphisms, the closed orbits $f^n_{[0,1]}(x, 0)$ and $g^n_{[0,1]}(y, 0)$ are regarded as closed paths $C_f, C_g: I\to M\times S^1$ defined by $C_f(s)=(H_{f^n}(x, s), s)$ and $C_g(s)=(H_{g^n}(x, s), s)$. Then we will show that C_f is homotopic to C_g. By the argument previous to Lemma 1, this implies the lemma.

To do this, we will define paths in M from $H_{f^n}(x, t)$ to $H_{g^n}(y, t)$. Let γ be a path in M from x to y, and let us define a homotopy from f^n to g^n by

$$G(z, s)=\begin{cases} H_{f^n}(z, 1-2s) & \text{if } \ 0\leq s\leq 1/2 , \\ H_{g^n}(z, 2s-1) & \text{if } \ 1/2\leq s\leq 1 . \end{cases}$$

Then the desired paths are given by

$$p_t(s)=\begin{cases} H_{f^n}(\gamma(s), t) & \text{for } \ 0\leq s\leq (1-t)/2 , \\ G(\gamma(s), s) & \text{for } \ (1-t)/2\leq s\leq (1+t)/2 , \\ H_{g^n}(\gamma(s), t) & \text{for } \ (1+t)/2\leq s\leq 1 . \end{cases}$$

Since $H_{f^n}(z, t)=G(z, (1-t)/2)$ and $H_{g^n}(z, t)=G(z, (1+t)/2))$, p_t is well-defined, and moreover p_t continuously depends on t.

Now let us define a continuous map $A: I\times I\to M\times S^1$ by $A(t, s)=(p_t(s), t)$. Then recalling that $H_{f^n}(z, 0)=H_{g^n}(z, 0)=z$, we have $A(0, s)=(\gamma(s), 0)$ and $A(1, s)=(G(\gamma(s), s), 1)$. Since x and y are n-Nielsen equivalent, and since G is a homotopy from f^n to g^n, γ is homotopic to $G(\gamma(\cdot), \cdot)$ leaving end points fixed. This implies that the two orbits C_f, C_g are homotopic, and hence, completes the proof. □

REMARK. In [6], Jiang shows a similar result. Lemma 2 is a generalization of this Jiang's result.

2. Proof of Theorem.

As stated in the paragraph previous to Lemma 1, we may calculate rotation sets by using a flow f_t on $M\times S^1$. Here, let us show the calculation. Take $[\alpha]\in H^1(M\times S^1; \boldsymbol{Z})$, and choose a representative $\alpha^*(x, s)=\alpha_1(x)\alpha_2(s)\in C^0(M\times S^1, S^1)$. Set $\alpha^1_x(t)=\alpha_1\circ pr_1\circ f_t(x, 0)$ for $t\geq 0$, where pr_1 denotes the projection $M\times S^1\to M$. Note that in §1, we use α^* to define the rotation set at x. Let us take a lift $\tilde{\alpha}^1_x$ of α^1_x to $\boldsymbol{R}$ with respect to π. Then one obtains functionals

$$A^1_{x,T}(\alpha)=\frac{1}{T}(\tilde{\alpha}^1_x(T)-\tilde{\alpha}^1_x(0))$$

on $C^0(M, S^1)$, and as the set of limit points of $\{A^1_{x,T}\}_{T>0}$, which is denoted by F^1_x, one obtains the rotation set at x, *i.e.* $\rho(x, f)=F^1_x$.

Now we have done all preparation, and so we start a proof of the main result.

PROOF OF THEOREM. Let us denote the periodic point with rotation vector ρ_0 by $\bar{z}$, and set $g=f^{mk(P)}$. As in §0, one obtains $\hat{g}=\hat{f}^{mk(P)}: M_P\to M_P$. Let $\hat{h}: M_P\to M_P$ be the Thurston canonical form for $\hat{g}$. Collapsing each boundary component of M_P to a point, M is again obtained, and $\hat{h}$ induces a homeomorphism $h: M\to M$ with $h|M-P=\hat{h}|\mathrm{Int}\, M_P$ under natural identification.

Let us show that h is generalized pseudo-Anosov. Since $h\simeq f^{mk(P)}$, this implies that f is isotopic to a generalized pseudo-Anosov homeomorphism. The argument in [7] shows that $\hat{h}$ is either pseudo-Anosov or reducible, so let us show that $\hat{h}$ is not reducible. Suppose by contradiction that $\hat{h}$ is reducible. Let γ be one of reducing curves. Then as in [7], γ separates M into two components M_1, M_2. Since h maps M_1 and M_2 into themselves respectively, we have

$$\rho(h)=(\rho(h)\cap H_1(M_1, \partial M_1; \boldsymbol{R}))\cup(\rho(h)\cap H_1(M_2, \partial M_2; \boldsymbol{R})) \qquad (*)$$

where $H_1(M_i, \partial M_i; \boldsymbol{R})$, $i=1, 2$, are regarded as subgroups of $H_1(M; \boldsymbol{R})$. Since the Nielsen class, with respect to $\hat{g}$, including $\bar{z}$ has non-zero index, there exists a fixed point w of $\hat{h}$ to be Nielsen equivalent to $\bar{z}$ (cf. Theorem 3 in Chapter IV, E [2]). Then by Lemma 2, $\rho(g, \bar{z})=\rho(h, w)$, and by Lemma 1, $\rho(g, \bar{z})=\{mk(P)\rho_0\}$. Hence we have $\rho(h, w)=\{mk(P)\rho_0\}$. Since h is isotopic to $f^{mk(P)}$ leaving P fixed, it is easy to show that $\rho(x_i, f^{mk(P)})=\rho(x_i, h)$ with $i=1, 2, \cdots, N$. By Lemma 1 again, $\rho(x_i, h)=\{mk(P)\rho_i\}$. This and the assumption that $\rho_0\in\mathrm{Int\,Conv}(\rho_1, \rho_2, \cdots, \rho_N)$ imply that $mk(P)\rho_0\in\mathrm{Int\,Conv}\,\rho(h)\cap\rho(h)$, but this contradicts the equality $(*)$. This completes the proof of i).

We will show the assersion ii). It is sufficient to show that h has a set of periodic points with set of rotation vectors dense in $\mathrm{Conv}(\rho'_1, \rho'_2, \cdots, \rho'_N)$, where $\rho'_i=mk(P)\rho_i$, $i=1, 2, \cdots, N$. Because by Handel's result [5], there exist periodic points of $f^{mk(P)}$ that are globally shadowed by periodic points of h, and then Lemmas 1 and 2 imply the assertion ii).

Let us show that for any vector ρ with rational coordinate $\in\mathrm{Int\,Conv}(\rho'_1, \rho'_2, \cdots, \rho'_N)$, there exists a convergent sequence $\{v_n\}$ to ρ such that there exist periodic points y_n of h with rotation vectors v_n. Since $\rho\in\mathrm{Int\,Conv}(\rho'_1, \rho'_2, \cdots, \rho'_N)$, there exist positive integers $m_1, m_2, \cdots, m_N$ with

$$\rho=\frac{1}{\sum_{i=1}^N m_i}\sum_{j=1}^N m_j\rho_j\,.$$

Let us take a Markov partition for the pseudo-Anosov map $\hat{h}: M_P\to M_P$. Recalling that components of ∂M_P correspond to the fixed points $x_1, x_2, \cdots, x_N$, and that $\hat{h}$ preserves components of ∂M_P, for each x_j there exists a finite symbol sequence C_j such that $C_j^{\infty} = \cdots C_j C_j C_j \cdots$ corresponds to x_j under the composite of the semi-conjugacy, that is between the symbolic dynamics and $\hat{h}$, and the natural projection from M_P to M. Let us denote the m times repeat of C_j by C_j^m. By the transitivity of pseudo-Anosov homeomorphisms, there exist symbol sequences D_i such that $C_i D_i C_{i+1}$ for $i=1, 2, \cdots, N-1$ and $C_N D_N C_1$ are admissible. Let us take infinitely repeated sequences

$$E_r = (C_1^{rm_1} D_1 C_2^{rm_2} D_2 \cdots D_{N-1} C_N^{rm_N} D_N)^{\infty},$$

and let us take periodic points y_r of h that correspond to the sequences E_r. Then we assert that the sequence of the rotation vectors v_r for y_r converges to ρ. This assertion implies ii) as stated above.

Let c_i and d_i denote the length of C_i and D_i. For each i, let us consider an admissible sequence $C_i^{pm_i} C_i^{(r-2p)m_i} C_i^{pm_i} F$, where F is any positively infinite sequence, and let us take a point $z \in M$ corresponding to this sequence. Then for any $\varepsilon > 0$, there exists a positive integer p_0 such that for any $p \geq p_0$,

$$d(h^n(z), h^n(x_i)) < \varepsilon \qquad \text{for} \quad pm_i c_i \leq n \leq (r-p)m_i c_i .$$

Let us take $[\alpha] \in H^1(M \times S^1; \mathbf{Z})$ and choose a representative $\alpha^*(x, s) = \alpha_1(x)\alpha_2(s) \in C^0(M \times S^1, S^1)$. As stated in the first paragraph of this section, let us define a map α_x^1 and its lift $\tilde{\alpha}_x^1$. Then there exists a constant $K > 0$ such that for any $x \in M$,

$$|\tilde{\alpha}_x^1(1) - \tilde{\alpha}_x^1(0)| < K,$$

and for any $\varepsilon > 0$, by the argument in the previous paragraph, one can choose a positive integer p_1 such that for any $p > p_1$,

$$|\tilde{\alpha}_z^1(n) - \tilde{\alpha}_{x_i}^1(n)| < \varepsilon \qquad \text{for} \quad pm_i c_i \leq n \leq (r-p)m_i c_i$$

for z as taken in the previous paragraph. Thus, recalling that

$$\sum_{j=1}^{N} m_j \rho = \sum_{j=1}^{N} m_j \rho'_j$$

and

$$\rho'_j([\alpha]) = \tilde{\alpha}_{x_j}^1(m) - \tilde{\alpha}_{x_j}^1(m-1) \qquad \text{for any } m,$$

then, for $r > 2p_1$ and $n = k\sum_{j=1}^{N}(rm_j c_j + d_j)$, we have

$$\frac{1}{n}|(\tilde{\alpha}_{y_r}^1(n) - \tilde{\alpha}_{y_r}^1(0)) - n\rho([\alpha])|$$

$$< \frac{2k(\sum_{j=1}^{N}(r-2p_1)m_j c_j)}{n}\varepsilon + \frac{k(\sum_{j=1}^{N}(2p_1 m_j c_j + d_j))}{n}(K + |\rho([\alpha])|)$$

$$<2\varepsilon+\frac{(\sum_{j=1}^{N}(2p_1m_jc_j+d_j))}{\sum_{j=1}^{N}(rm_jc_j+d_j)}(K+|\rho([\alpha])|).$$

This inequality implies that $v_r \to \rho$ as $r \to \infty$. This completes the proof. $\square$

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