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A Note on the Scaling Limit of a Complete Open Surface

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1. Introduction.

It is interesting to study the geometric meaning of total curvature of complete open surfaces. The influence of the total curvature of a Riemannian plane on the Lebesgue measure of rays were investigated first by M. Maeda [3], [4], K. Shiga [5] and later by K. Shiohama, T. Shioya and M. Tanaka [6], etc. The author proved in [2] that a pointed Hausdorff approximation map between connected, complete and noncompact Riemannian 2-manifolds with finite total curvature has a natural continuous extension to their ideal boundaries with the Tits metrics. In view of the above results it is natural to expect that the scaling limit of such an M will be a flat cone generated by the ideal boundary $M(\infty)$ of M equipped with the Tits metric d_∞.

Let M be a connected, complete and noncompact Riemannian 2-manifold with a finite total curvature. The Huber theorem implies that M is finitely connected. A compact set $C \subset M$ is by definition a core of M iff $M \setminus \mathrm{Int}(C)$ consists of k tubes $U_1, \cdots, U_k$ such that each U_i is homeomorphic to $S^1 \times [0, \infty)$ and such that each ∂U_i is a piecewise smooth simple closed curve. If $\kappa(\partial U_i)$ is the total geodesic curvature of ∂U_i, then the Gauss-Bonnet theorem implies $c(C) + \sum_{i=1}^{k} \kappa(\partial U_i) = 2\pi\chi(M)$. Moreover

$$s_i := \kappa(\partial U_i) - c(U_i)$$

is nonnegative and independent of the choice of tubes having the same end as U_i and

$$2\pi\chi(M) - c(M) = \sum_{i=1}^{k} s_i .$$

In [9] T. Shioya proved that M admits an ideal boundary $M(\infty)$ with the Tits metric d_∞ such that $(M(\infty), d_\infty)$ is the union of circles with lengths $s_1, \cdots, s_k$.

Let d be the distance function induced from the Riemannian metric of M. We denote by $(M_t; o)$ for an arbitrary fixed point $o \in M$ and for $t > 0$ the scaling by t of the

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pointed metric space $(M, d; o)$, and we write

$$(M_t; o) := (M, d/t; o).$$

Our result is stated as

THEOREM 1.1. *The pointed Hausdorff limit of $(M_t; o)$ as $t \to \infty$ is isometric to the flat cones $K(M(\infty), d_\infty; o^*)$ having the same vertices at o^* and generated by the ideal boundary of M.*

Here $K(M(\infty), d_\infty; o^*)$ is the union of k flat cones $K(U_1(\infty), d_\infty; o^*), \cdots, K(U_k(\infty), d_\infty; o^*)$ such that each $K(U_i(\infty), d_\infty; o^*)$ is generated by $(U_i(\infty), d_\infty)$ which is the circle of length s_i and has its vertex at o^*.

Theorem 1.1 provides simple and intuitive consequences which have been proved in [7] and [8]. Let $B(p; t)$ be the metric t-ball around $p \in M$ and $S(p; t) := \{x \in M : d(x, p) = t\}$. Let $A(t)$ and $L(t)$ be the area and the length of $B(p; t)$ and $S(p; t)$ respectively. Theorem 1.1 implies that the scaling limits of $B(p; t)$ and $S(p; t)$ are the unit ball and unit circle around o^* of $K(M(\infty), d_\infty; o^*)$. Let $S_p(1) \subset T_pM$ be the unit circle and μ the Lebesgue measure of $S_p(1)$. Let $A_p \subset S_p(1)$ be the set of all unit vectors tangent to rays from p. Noticing that both $\lim_{t\to\infty} L(t)^2/A(t)$ and $\mu(A_p)$ are scaling invariant, we see that the following Corollary 1.2 is direct from Theorem 1.1.

COROLLARY 1.2. *Let M be as in Theorem 1.1. Then*

$$\lim_{t\to\infty} \frac{L(t)^2}{A(t)} = 2(2\pi\chi(M) - c(M))$$

and

$$\lim_{j\to\infty} \mu(A_{p_j}) = s_i$$

for all divergent sequence $\{p_j\} \subset U_i$.

For the notion of (pointed) Hausdorff limit, see [1].

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2. Preliminaries.

If M is as in our Theorem 1.1 we observe, by taking the scaling limit, that a core C shrinks to a point, say, o^*. The pointed Hausdorff limit of $(M_t; o)$ at $t \to \infty$ is obtained by taking the limit $t \to \infty$ in the scaling by t of the pointed metric space $(M, d; o)$. We want to show that the Hausdorff limit of each U_i is the flat cone generated by $(U_i(\infty), d_\infty)$, which is a circle of length s_i. Because each U_i can be embedded isometrically into a Riemannian plane having total curvature $2\pi - s_i$, we only need to consider a Riemannian plane M with finite total curvature.

From now on let M be a Riemannian plane with finite total curvature. We define the ideal boundary $M(\infty)$ of a Riemannian plane M and the Tits metric d_∞ of $M(\infty)$. Let $\gamma, \sigma: [0, \infty) \to M$ be arbitrary rays and $D(\gamma, \sigma) \subset M$ be the half plane bounded by γ, σ and a piecewise smooth curve c joining points on γ and σ such that c intersects orthogonally to γ and σ. Then $D(\sigma, \gamma) = M \setminus D(\gamma, \sigma)$. We put

$$L(\gamma, \sigma) := -c(D(\gamma, \sigma)) - k(\partial D(\gamma, \sigma)) \tag{2.1}$$

where $c(D(\gamma, \sigma))$ is the total curvature of $D(\gamma, \sigma)$ and $\kappa(\partial D(\gamma, \sigma))$ is the total geodesic curvature of c. Notice that $L(\gamma, \sigma)$ does not depend on the choice of the curve c. We also define $L(\sigma, \gamma)$ by the same way. It is proved in [10] that if γ is asymptotic to σ, then $L(\gamma, \sigma)=0$. Two rays γ and σ are called equivalent if $L(\gamma, \sigma)=0$ or $L(\sigma, \gamma)=0$. We denote the equivalent class of a ray γ by $\gamma(\infty)$ and the set of all equivalent classes by $M(\infty)$ which is called the ideal boundary of M. The Tits metric d_∞ of $M(\infty)$ is given

$$d_\infty(x, y) = \min\{L(\gamma, \sigma), L(\sigma, \gamma)\}, \qquad x, y \in M(\infty)$$

such that $\gamma(\infty)=x$ and $\sigma(\infty)=y$ respectively. The following facts proved by T. Shioya [10] and used here will be prepared. These facts are valid not only for Riemannian planes but for more general Riemannian 2-manifolds. Let M be a finitely connected compact complete noncompact Riemannian 2-manifold having finite total curvature with one end.

FACT 1. *$(M(\infty), d_\infty)$ is isometric to a circle of the total length $2\pi\chi(M) - c(M)$. In particular, $M(\infty)$ is a single point if $c(M) = 2\pi\chi(M)$.*

FACT 2.

$$\lim_{t \to \infty} \frac{L(S(p, t) \cap D(\gamma, \sigma))}{t} = L(\gamma, \sigma).$$

FACT 3. *If $D(\gamma, \sigma)$ dose not have any ray emanating from p, then*

$$\lim_{t \to \infty} \frac{L(S(p, t) \cap D(\gamma, \sigma))}{t} = 0.$$

FACT 4.

$$d_\infty(\gamma(\infty), \sigma(\infty)) = \min\left\{\lim_{t \to \infty} \frac{L(S(p, t) \cap D(\gamma, \sigma))}{t}, \lim_{t \to \infty} \frac{L(S(p, t) \cap D(\sigma, \gamma))}{t}\right\}.$$

3. Proof of Theorem 1.1.

As stated at the beginning of Preliminaries, we only need for the proof of Theorem 1.1 to show that the Hausdorff limit of U_i is the cone $K(U_i(\infty), d_\infty; o^*)$. This is equivalent to show that a Riemannian plane M with finite total curvature has its scaling limit $K(M(\infty), d_{\infty}; o^*)$. Rays on M are still rays on M_t for all $t>0$, and A_p for every fixed $p \in M$ leaves invariant under the scaling of metrics. Metrics ρ_t on A_p are induced in Lemma 3.1 such that $\lim_{t\to\infty}(A_p, \rho_t)$ is isometric to $(M(\infty), d_{\infty})$. We then conclude the proof of Theorem 1.1 by showing in Proposition 3.2 that the pointed Hausdorff limit of $(M_t; o)$ at $t\to\infty$ is isometric to $K(A_p, \rho_{\infty}; p)$. We induce a metric ρ_t on A_p by

$$\rho_t(\dot{\gamma}(0), \dot{\sigma}(0)) := \min\left\{\frac{L(S(p, t) \cap D(\gamma, \sigma))}{t}, \frac{L(S(p, t) \cap D(\sigma, \gamma))}{t}\right\}$$

where γ and σ are rays emanating from p.

LEMMA 3.1. *The limit (A_p, ρ_{∞}) of (A_p, ρ_t) as $t\to\infty$ is isometric to $(M(\infty), d_{\infty})$.*

PROOF. From Fact 2, we see that (A_p, ρ_t) has a limit as $t\to\infty$. We have a natural correspondence between A_p and $M(\infty)$ by assigning $u \in A_p$ to $\gamma(\infty)$, where γ is a ray from p with $\dot{\gamma}(0)=u$. For $x, y \in M(\infty)$, let $\gamma(\infty)=x$ and $\sigma(\infty)=y$. From Fact 4, we get

$$\rho_{\infty}(\dot{\gamma}(0), \dot{\sigma}(0)) = \min\left\{\lim_{t\to\infty} \frac{L(S(p, t) \cap D(\gamma, \sigma))}{t}, \lim_{t\to\infty} \frac{L(S(p, t) \cap D(\sigma, \gamma))}{t}\right\}$$
$$= d_{\infty}(\gamma(\infty), \sigma(\infty)) = d_{\infty}(x, y).$$

PROPOSITION 3.2. *For a base point $o \in M$ and for an arbitrary fixed point p, the pointed Hausdorff limit of $(M_t; o)$ as $t\to\infty$ is isometric to the cone $K(A_p, \rho_{\infty}; p)$ with the vertex at p generated by (A_p, ρ_{∞}).*

PROOF. For arbitrary points $x, y \in K(A_p, \rho_{\infty}; p)$, there exist $u, v \in A_p$ and $a, b>0$ such that $x=au$ and $y=bv$ respectively. On the cone $K(A_p, \rho_{\infty}; p)$ we have

$$\rho_{\infty}(x, y)^2 = a^2 + b^2 - 2ab\cos\rho_{\infty}(u, v).$$

On the other hand, for sufficiently large $t>0$ we take rays γ and σ emanating from p such that $\dot{\gamma}(0)=u$ and $\dot{\sigma}(0)=v$ on M_t. Let τ_t be a minimizing geodesic joining $\gamma(ta)$ and $\sigma(tb)$, where we assume $a<b$. Let D_t be a disk bounded by the triangle whose vertices are at p, $\gamma(ta)$ and $\sigma(ta)$. If

$$\alpha_t := \angle(p, \gamma(ta), \sigma(ta)) \quad \text{and} \quad \beta_t := \angle(p, \sigma(ta), \gamma(ta)),$$

then $\lim_{t\to\infty}\alpha_t = \lim_{t\to\infty}\beta_t$ holds, see T. Shioya [10]. From Gauss-Bonnet theorem for D_t we get

$$\alpha_t + \beta_t + \angle(u, v) - \pi = c(D_t).$$

Setting $\lim_{t\to\infty} D_t = D_{\infty}$, (2.1) gives

$$I(\gamma, \sigma) = -c(D_{\infty}) + \angle(u, v),$$

and we obtain

$$\omega := \lim_{t\to\infty} \alpha_t = \lim_{t\to\infty} \beta_t = \tfrac{1}{2}\{\pi - (\angle(u, v) - c(D_\infty))\} = \tfrac{1}{2}(\pi - \rho_\infty(u, v)) .$$

Moreover, we have

$$\lim_{t\to\infty} \frac{d(\gamma(ta), \sigma(ta))}{t} = 2a\cos\omega = 2a\sin\frac{\rho_\infty(u, v)}{2} .$$

The triangle $\Delta(\gamma(ta), \sigma(ta), \sigma(tb))$ on M_t converges as $t\to\infty$ to a plane triangle with two edge lengths $b-a$, $2a\sin\rho_\infty(u,v)/2$ making an angle $\pi-\omega$ between them. Thus we get

$$\lim_{t\to\infty} d_t(\gamma(ta), \sigma(tb))^2 = (b-a)^2 + 4a^2\sin^2\frac{\rho_\infty(u, v)}{2} - 4a(b-a)\sin\frac{\rho_\infty(u, v)}{2}\cos(\pi-\omega)$$

$$= a^2 + b^2 - 2ab\cos\rho_\infty(u, v) .$$

Noticing that for an arbitrary fixed point $p\in M$ $\lim_{t\to\infty}(1/t)d(o, p) = 0$, we complete the proof.

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