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Compact Space-Like m-Submanifolds in a Pseudo-Riemannian Sphere $S_p^{m+p}(c)$

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Dedicated to Professor Tsunero Takahashi on his 60th birthday

Introduction.

In this paper, we shall consider the problem whether or not there exists a compact space-like m-dimensional submanifold in a pseudo-Riemannian sphere $S_p^{m+p}(c)$ with parallel mean curvature vector which is not totally umbilic.

A pseudo-Riemannian sphere $S_p^{m+p}(c)$ is an $(m+p)$-dimensional indefinite Riemannian space of index p and with constant curvature $c>0$, which is constructed in a pseudo-Euclidean space R_p^{m+1+p} as follows. First, a pseudo-Euclidean space R_p^{m+p+1} is of real $(m+p+1)$-tuples $x=(x_1, \cdots, x_{m+p+1})$ with scalar product

$$\langle x, y\rangle=\sum_{i=1}^{m+1} x_i y_i-\sum_{\alpha=m+2}^{m+p+1} x_\alpha y_\alpha .$$

Then

$$S_p^{m+p}(c)=\{x\in R_p^{m+p+1} \mid \langle x, x\rangle=1/c\} .$$

In the special case $p=1$, we call $S_1^{m+1}(c)$ a de Sitter space.

Let us consider M a compact space-like m-dimensional submanifold in $S_p^{m+p}(c)$. Then M is diffeomorphic to a Riemannian sphere S^m. (See Lemma 1 in §1). Here, M is totally umbilic if and only if M is a space-like $(m+1)$-plane section in $S_p^{m+p}(c)$, and then, M is congruent to a Riemannian sphere $S^m(c')$ of constant curvature c' where $c\geq c'>0$.

Montiel [9] has proved that a compact space-like hypersurface M in a de Sitter space $S_1^{m+1}(c)$ is totally umbilic if the mean curvature H of M is constant.

So we have been considering the higher codimensional case, and gotten the following.

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THEOREM. *Let M be a compact space-like m-dimensional submanifold in a pseudo-Riemannian sphere $S_p^{m+p}(c)$ with parallel mean curvature vector. If the normal connection of M is flat, then M is totally umbilic.*

It follows from this theorem that *if there exists a compact space-like m-dimensional submanifold M in $S_p^{m+p}(c)$ with parallel mean curvature vector which is not totally umbilic, then $m \geq 3$, $p \geq 3$ and M is not non-negatively curved.* (see Corollary 6, Corollary 9 and Theorem 11.)

Judging from the view mentioned later, I guessed that the answer to our problem is nonexistence. Recently, Alias and Romero [3] has also considered this problem by use of their new method. In fact, our Corollary 9 are independently obtained by them. But the problem remains unsettled.

Pseudo-Riemannian space form $N_p^{m+p}(c)$ with constant curvature c is the generic notation for pseudo-Riemannian sphere $S_p^{m+p}(c)$ $(c>0)$, pseudo-Euclidean space R_p^{m+p} $(c=0)$ and pseudo-hyperbolic space $H_p^{m+p}(c)$ $(c<0)$. Here $H_p^{m+p}(c)$ $(c<0)$ is constructed by the connected component of $\{x \in R_{p+1}^{m+p+1} \mid \langle x, x \rangle = 1/c\}$.

In a pseudo-Riemannian space form, space-like submanifolds with parallel mean curvature vector have been studied by many mathematicians, since Calabi [4] and S. Y. Cheng and Yau [8] proved the famous Bernstein type theorem in a Minkowski space R_1^{m+1}. The Bernstein type theorem in pseudo-Riemannian spheres asserts that a complete maximal space-like m-submanifold in $S_p^{m+p}(c)$ is totally geodesic. (See Ishihara [9]). Here "maximal" means that the mean curvature vanishes identically.

Let M be a complete space-like hypersurface with constant mean curvature H in $S_1^{m+1}(c)$. It is known that there exist some noncompact nonmaximal examples of M which is not totally umbilic. (See Akutagawa [2]). However, when $m=2$ and $H^2 \leq c$ or when $m>2$ and $H^2 < 4(m-1)c/m^2$, it has been proved by Akutagawa [2] that M is totally umbilic. (Ramanathan [10] has independently proved the case $m=2$.) Furthermore, Q. M. Cheng [7] has proved that the Akutagawa's theorem holds in the case of higher codimension, that is, if M is a complete space-like m-dimensional submanifold in $S_p^{m+p}(c)$ with parallel mean curvature vector $\vec{H}$, M is totally umbilic when $m=2$ and $|\vec{H}|^2 \leq c$ or when $m>2$ and $|\vec{H}|^2 < 4(m-1)c/m^2$.

On the other hand, a part of the Akutagawa's theorem in $S_1^{m+1}(c)$ is contained in Montiel's result. In fact, the condition $H^2 < 4(m-1)c/m^2$ indicates the compactness of M by virtue of the Myers theorem combined with the calculus of the Ricci curvature.

At the end of this section, we remark that there exist no compact space-like m-dimensional submanifolds in a pseudo-Riemannian space form $N_p^{m+p}(c)$ with constant curvature $c \leq 0$. (See, for example, Aiyama [1].)

§1. An integral equality for compact space-like m-submanifolds in $S_p^{m+p}(c)$ and its applications.

Let X: $M \to S_p^{m+p}(c)$ be a compact space-like m-dimensional submanifold immersed into a pseudo-Riemannian sphere.

In this section, we introduce an integral equality for the immersion X, and give our main result as its application. This integral equality is gotten by expanding Montiel's one in [10] into a higher codimensional case after the method similar to Reilly [12].

First of all, we remark that M is orientable. In fact, M is diffeomorphic to a Riemannian sphere as follows.

LEMMA 1. *There exists a diffeomorphism* $\varphi : S^m \to M$ *such that* $X \circ \varphi : S^m \to S_p^{m+p}(c)$ *is an embedding prescribed below by* (1.1).

PROOF. We can define a diffeomorphism F: $S^m(1) \times H^p(-c) \to S_p^{m+p}(c)$ by

$$F(x, y) = (y_{p+1}x_1, \cdots, y_{p+1}x_{m+1}, y_1, \cdots, y_p),$$

where $x = (x_1, \cdots, x_{m+1}) \in S^m \subset \boldsymbol{R}^{m+1}$ and $y = (y_1, \cdots, y_{p+1})$ is an element of a hyperbolic space $H^p(-c) = \{y \in \boldsymbol{R}_1^{p+1} \mid \langle y, y \rangle = -1/c,\ y_{p+1} > 0\}$. Here let ϖ: $S^m(1) \times H^p(-c) \to S^m(1)$ be the projection. Since X is space-like, the composition $\varpi \circ F^{-1} \circ X$: $M \to S^m(1)$ is a local diffeomorphism. Furthermore, by the compactness of M, it must be a diffeomorphism ψ. Put $\varphi = \psi^{-1}$. Accordingly, there is a smooth mapping $u = (u_1, \cdots, u_{p+1})$: $S^m(1) \to H^p(-c)$ such that

$$X \circ \varphi(x) = F(x, u(x)) = (u_{p+1}(x)x_1, \cdots, u_{p+1}(x)x_{m+1}, u_1(x), \cdots, u_p(x)) . \tag{1.1}$$ □

Our local calculations are done relative to an adapted positively oriented orthonormal frame field $\{e_1, \cdots, e_{m+p}\}$ on $S_p^{m+p}(c)$, that is $e_1, \cdots, e_m$ are space-like orthonormal local vector fields tangent to $X(M)$ and positively oriented to M. We use the following convention on the range of indices:

$$i, j, \cdots = 1, \cdots, m; \qquad \alpha, \beta, \cdots = m+1, \cdots, m+p .$$

We denote by h_{ij}^α the components of the second fundamental form II relative to e_i, e_j and e_α, that is, $h_{ij}^\alpha = \langle \nabla_{e_i}^E e_j, e_\alpha \rangle$ where ∇^E is the Levi-Civita connection of $E = \boldsymbol{R}_p^{m+1+p}$. Then the mean curvature vector $\vec{H}$, its length H and the square of the length S of the second fundamental form are respectively given below;

$$\vec{H} = -\frac{1}{m}\sum_{\alpha,i} h_{ii}^\alpha e_\alpha , \quad H = \frac{1}{m}\left[\sum_\alpha \left(\sum_i h_{ii}^\alpha\right)^2\right]^{1/2} \quad \text{and} \quad S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 .$$

We denote by ∇ and $\nabla^\perp$ the Levi-Civita connection on M and the normal connection of M in $S_p^{m+p}(c)$, respectively. The components of the covariant derivative $\nabla\Pi$ of Π are denoted by h_{ijk}^α.

For the $(m+1+p)$-dimensional vector space $E = \boldsymbol{R}_p^{m+1+p}$, let Λ be its exterior algebra, and Λ^p the subspace spanned by p-planes $\mathfrak{v}=v_1\wedge\cdots\wedge v_p$ (where $v_1,\cdots,v_p$ are p linearly independent vectors in E). It is known that the scalar product $\langle\ ,\ \rangle$ on Λ^p can be induced by the one on E as follows:

$$\langle\mathfrak{v},\mathfrak{w}\rangle:=\det((\langle v_a,w_b\rangle)_{1\le a,b\le p})$$

for any $\mathfrak{v}=v_1\wedge\cdots\wedge v_p$ and $\mathfrak{w}=w_1\wedge\cdots\wedge w_p\in\Lambda^p$.

Set $N=e_{m+1}\wedge\cdots\wedge e_{m+p}$. This means that N is globally defined on M as the smooth field of oriented unit normal (time-like) p-planes of M in $S_p^{m+p}(c)$. Let $A_{m+1},\cdots,A_{m+p}$ be p orthonormal time-like vectors in E, and set $A=A_{m+1}\wedge\cdots\wedge A_{m+p}\in\Lambda^p$. For the fixed element A of Λ^p, we define the smooth function U on M by $U=\langle N,A\rangle$. Furthermore, set

$$V_\alpha=\langle e_{m+1}\wedge\cdots\wedge e_{\alpha-1}\wedge X\wedge e_{\alpha+1}\wedge\cdots\wedge e_{m+p},A\rangle\,,$$

$$U_{\alpha i}=\langle e_{m+1}\wedge\cdots\wedge e_{\alpha-1}\wedge e_i\wedge e_{\alpha+1}\wedge\cdots\wedge e_{m+p},A\rangle\,,$$

$$U_{\alpha\beta ij}=\begin{cases}\langle e_{m+1}\wedge\cdots\wedge e_{\alpha-1}\wedge e_i\wedge e_{\alpha+1}\wedge\cdots \\ \qquad\wedge e_{\beta-1}\wedge e_j\wedge e_{\beta+1}\wedge\cdots\wedge e_{m+p},A\rangle & \text{if}\quad \alpha\neq\beta\,,\\ 0 & \text{if}\quad \alpha=\beta\,.\end{cases}$$

Here we note that $U_{\alpha i}$ and $U_{\alpha\beta ij}$ depend on the choice of local frame fields and that $U_{\alpha\beta ij}=-U_{\beta\alpha ij}=-U_{\alpha\beta ji}$.

PROPOSITION 2. *In the notation introduced above, we have the following integral equality*:

$$(1.2)\quad 0=m\int_M(S-mH^2)UdM-(m-1)\int_M\sum_{i,j,\alpha}h^\alpha_{iij}U_{\alpha j}dM+m\int_M\sum_{i,j,k}\sum_{\alpha\neq\beta}h^\alpha_{ij}h^\beta_{ik}U_{\alpha\beta jk}dM\,,$$

where dM is the Riemannian measure of M.

PROOF. Define a vector field W on M by the formula $W=\sum_i W_ie_i$, where

$$W_i=\sum_{j,\alpha}\left(\sum_k h^\alpha_{kk}\delta_{ij}-mh^\alpha_{ij}\right)U_{\alpha j}\,.$$

Here it is immediately proved that W does not depend on the choice of orthonormal frame fields. This integral equality follows by computing $\mathrm{div}(W)$ and applying Stokes' theorem $\int_M\mathrm{div}(W)=0$.

By choosing an adapted orthonormal frame field such that $\nabla_{e_i}e_j=\nabla^\perp_{e_i}e_\alpha=0$ for any i, j and α at a point q in M, the computation of $\mathrm{div}(W)$ becomes easier, that is, $\mathrm{div}(W)=\sum_i e_i(W_i)$ at q. By using the Codazzi equation $h^\alpha_{ijk}=h^\alpha_{ikj}$, the symmetry of h^α_{ij} in i and j, and the above skew-symmetry of $U_{\alpha\beta ij}$, $\mathrm{div}(W)$ is calculated as appearing in (1.2);

$$\begin{aligned}\operatorname{div}(W)=&\sum_{i,j,\alpha}\left(\sum_k h^{\alpha}_{kki}\delta_{ij}-mh^{\alpha}_{iji}\right)U_{\alpha j}\\&+\sum_{i,j,\alpha}\left(\sum_k h^{\alpha}_{kk}\delta_{ij}-mh^{\alpha}_{ij}\right)\Big(\langle e_{m+1}\wedge\cdots\wedge\overset{\alpha\text{th}}{\nabla^E_{e_i}e_j}\wedge\cdots\wedge e_{m+p},A\rangle\\&\qquad+\sum_{\beta\neq\alpha}\langle e_{m+1}\wedge\cdots\wedge\overset{\alpha\text{th}}{e_j}\wedge\cdots\wedge\overset{\beta\text{th}}{\nabla^E_{e_i}e_\beta}\wedge\cdots\wedge e_{m+p},A\rangle\Big)\\=&-(m-1)\sum_{i,j}h^{\alpha}_{iij}U_{\alpha j}\\&+\sum_{i,j,\alpha}\left(\sum_k h^{\alpha}_{kk}\delta_{ij}-mh^{\alpha}_{ij}\right)\left(-\sum_{\beta\neq\alpha}\sum_k h^{\beta}_{ik}U_{\alpha\beta jk}-h^{\alpha}_{ij}U-c^{-1}\delta_{ij}V_\alpha\right)\\=&-(m-1)\sum_{i,j,\alpha}h^{\alpha}_{iij}U_{\alpha j}+m\sum_{i,j,k}\sum_{\alpha\neq\beta}h^{\alpha}_{ij}h^{\beta}_{ik}U_{\alpha\beta jk}+m(S-mH^2)U.\end{aligned}$$

□

As an application of the integral equality, we can explain our main

THEOREM 3. *Let M be a compact space-like m-dimensional submanifold in a pseudo-Riemannian sphere $S^{m+p}_p(c)$ with parallel mean curvature vector. If the normal connection of M in $S^{m+p}_p(c)$ is flat, then M is totally umbilic.*

In order to prove this theorem, we first prepare the following Lemma 4.

LEMMA 4. *$U>0$ on all M or $U<0$ on all M.*

PROOF. Since U is the determinant of the $p\times p$-matrix $(\langle e_\alpha, A_\beta\rangle)$, $U=0$ if and only if there exists a time-like vector A_* on the p-plane spanned by $\{A_{m+1},\cdots,A_{m+p}\}$ which is perpendicular to all e_α $(m+1\leq\alpha\leq m+p)$. However, all vectors perpendicular to the p-plane spanned by $\{e_{m+1},\cdots,e_{m+p}\}$ are space-like. Thus the smooth function U never vanishes. □

REMARK. In fact, the smooth function U on M satisfies $|U|\geq 1$. This is proved, for example, by using "angles" between two space-like $(m+1)$-planes in R^{m+p+1}_p (cf. [1]).

PROOF OF THEOREM 3. Parallelism of the mean curvature vector asserts that $\sum_i h^{\alpha}_{iij}=0$ for all j and α. Furthermore, it is well known that the normal connection of a space-like submanifold in a pseudo-Riemannian space form is flat if and only if $\sum_k h^{\alpha}_{ik}h^{\beta}_{kj}=\sum_k h^{\beta}_{ik}h^{\alpha}_{kj}$ for all i, j, α and β. From the integral equality (1.2) combined with these assumptions and the skew-symmetry of $U_{\alpha\beta ij}$, it follows that $\int_M(S-mH^2)U\,dM=0$. Moreover, $S\geq mH^2$ from Schwarz's inequality, and the equality holds only when M is totally umbilic. Therefore, by virtue of Lemma 4, we have completed the proof of the theorem. □

At the end of this section, we mention a trivial case when the normal connection is flat.

LAMMA 5. *Let M be a submanifold in a semi-Riemannian manifold N with non-null and non-zero parallel mean curvature vector. If the codimension is less than* 3, *then the normal connection of M in N is flat.*

REMARK. When the direction normal to a submanifold M in a semi-Riemannian manifold N is not definite, a normal vector field η may be null (i.e. $\langle\eta, \eta\rangle=0$ and $\eta\neq 0$) at some points of M. In our proof of this lemma, we need to assume that the mean curvature vector is not null everywhere.

PROOF. The following property is well known: The normal connection of an m-dimensional submanifold in an $(m+p)$-dimensional semi-Riemannian manifold is flat if and only if there exist locally p orthonormal parallel normal vector fields. If $p=2$ and the non-null and non-zero mean curvature vector $\vec{H}$ is parallel, then the unit normal vector field perpendicular to $\vec{H}$ also is parallel. Then the normal connection is flat. □

Therefore, we immediately get the following corollary of Theorem 3.

COROLLARY 6. *Let M be a compact space-like m-dimensional submanifold in a pseudo-Riemannian sphere* $S_p^{m+p}(c)$ *with parallel mean curvature vector. If the codimension p is less than* 3, *then M is totally umbilic.*

§2. Space-like surfaces with parallel mean curvature vector in a pseudo-Riemannian space form.

In this section, we explain that the answer to our problem in the case $m=2$ is nonexistence. This result is proved as the corollary of Theorem 3 in the previous section, by virtue of the following Lemma 7 and Proposition 8. The method in this section is similar to Chen's one in [5].

LEMMA 7. *Let M be a space-like surface in a semi-Riemannian space form N with parallel non-null mean curvature vector* $\vec{H}$. *If M is neither minimal* (*i.e., maximal*) *nor pseudo-umbilic, then the normal connection of M in N is flat.*

PROOF. Let $\{e_i, e_\alpha\}$ $(1\leq i\leq m=2, 3\leq\alpha\leq n)$ be a local orthonormal frame field such that, at each point of M, e_i are tangent to M and $e_3=\vec{H}/H$. We denote the components of the normal curvature of M in N by $R_{\alpha\beta ij}$. It follows from the equation of Ricci combined with the parallelism of e_3 that

$$0=R_{3\alpha ij}=\sum_k h^\alpha_{ik}h^3_{kj}-\sum_k h^3_{ik}h^\alpha_{kj}\,. \tag{2.1}$$

We can choose a local frame field $\{e_1, e_2\}$ such that $h^3_{ij}=\lambda_i\delta_{ij}$. Then the equality (2.1) indicates that

$$(2.2) \qquad (\lambda_j - \lambda_i)h^\alpha_{ij} = 0 \qquad \text{for any } i, j \text{ and } \alpha$$

at the points of M where $\lambda_1 \neq \lambda_2$. That is, at not pseudo-umbilic points of M, the normal curvature of M in N vanishes.

On the other hand, the points of M which are umbilic with respect to a normal direction in N are isolated. This is proved by applying the fact that a complex analytic function φ on a Riemann surface has only isolated zero points unless φ is identically zero. In fact, on the Riemann surface M with complex isothermal coordinate $z = x_1 + ix_2$, the complex valued function $\varphi = (h^3_{11} - h^3_{22})/2 - ih^3_{12}$ (where $e_i = \partial/\partial x_i$) is complex analytic (from the Coddazi equation and the parallelizm of e_3), and the zero points of φ are umbilic with respect to the normal direction e_3.

Accordingly, the normal curvature is identically zero, that is, the normal connection is flat. □

PROPOSITION 8. (Chen [6]) *Let M be a submanifold in a pseudo-Riemannian space form $N^n_q(c)$ with non-null parallel mean curvature vector $\vec{H}$. If M is pseudo-umbilic, then M is a minimal (i.e., maximal) submanifold of a totally umbilic hypersurface $N^{n-1}_{q'}(c')$ in $N^n_q(c)$, where q' is q when $\vec{H}$ is space-like or $q-1$ when $\vec{H}$ is time-like.*

COROLLARY 9. *Only compact space-like surfaces in a pseudo-Riemannian sphere $S^{2+p}_p(c)$ with parallel mean curvature vector are totally umbilical ones.*

PROOF. Let M be a compact space-like surface in $S^{2+p}_p(c)$ with parallel mean curvature vector.

If M is neither maximal nor pseudo-umbilic, since the normal connection of M in $S^{2+p}_p(c)$ is flat by virtue of Lemma 7, the proof is obtained by Theorem 3 in §1.

Then we first consider the maximal case. In this case, by the Ishihara's theorem in [9], we know that M is totally geodesic. Next, suppose that M is pseudo-umbilic. Using Proposition 8, we can assert that M is a maximal surface in a pseudo-Riemannian space form $N^{p+1}_{p-1}(c')$ with constant curvature c'. If $c' \geq 0$, by applying the Ishihara's theorem again, it immediately follows that M is a totally umbilic surface in $S^{2+p}_p(c)$. Furthermore, in the case $c' < 0$, we know that there exist no compact space-like surfaces in $N^{p+1}_{p-1}(c')$.

This completes the proof of this corollary. □

Furthermore, we remark that the following theorem analogous to the Chen and Yau's one explained in [5] holds.

THEOREM 10. *Suppose that M is a space-like surface in a pseudo-Riemannian space form $N^{2+p}_p(c)$ with parallel mean curvature vector. Then M is one of the following surfaces*:

(1) *maximal space-like surfaces of $N^{2+p}_p(c)$,*

(2) *maximal space-like surfaces of a totally umbilic hypersurface $N^{p+1}_{p-1}(c')$ in*

$N_p^{2+p}(c)$,

(3) *space-like surfaces with constant mean curvature of a totally umbilic 3-dimensional submanifold* $N_1^3(c')$ *in* $N_p^{2+p}(c)$.

§3. Non-negatively curved space-like *m*-submanifolds with parallel mean curvature vector in $S_p^{m+p}(c)$.

In this last section, we assert that flatness of the normal connection is implied in non-negativity of the sectional curvatures on compact space-like m-submanifold with parallel mean curvature vector in $S_p^{m+p}(c)$. Then we get the following theorem as the corollary of Theorem 3.

THEOREM 11. *Let M be a compact space-like m-dimensional submanifold in a pseudo-Riemannian sphere* $S_p^{m+p}(c)$ *with parallel mean curvature vector. If the sectional curvature of M is non-negative, then M is totally umbilic.*

PROOF. We may prove only for $p \geqq 2$.

Let $\{e_i, e_\alpha\}$ $(i=1, \cdots, m, \alpha=m+1, \cdots, m+p)$ be any local orthonormal frame field on M such that e_i are tangent to M and e_α are normal to M in $S_p^{m+p}(c)$. Put $S_\alpha=\sum_{i,j}(h_{ij}^\alpha)^2$, that is, S_α is the square norm of the second fundamental form II directed to e_α. Furthermore, put $S=\sum_\alpha S_\alpha$. We remark that each S_α is a locally defined function, but S is defined on all M.

The Laplacian of S_α is calculated from the Codazzi equation, the Ricci formula for the second fundamental form and parallelism of the mean curvature vector as follows:

$$\begin{aligned}
\tfrac{1}{2}\Delta S_\alpha &= \sum_{i,j,k}(h_{ijk}^\alpha)^2+\sum_{i,j,k}h_{ij}^\alpha h_{ijkk}^\alpha=\sum_{i,j,k}(h_{ijk}^\alpha)^2+\sum_{i,j,k}h_{ij}^\alpha h_{kijk}^\alpha \\
&= \sum_{i,j,k}(h_{ijk}^\alpha)^2+\sum_{i,j}h_{ij}^\alpha\Big\{\sum_k h_{kikj}^\alpha-\sum_{k,l}(h_{kl}^\alpha R_{lijk}+h_{li}^\alpha R_{lkjk})+\sum_{\substack{k\\ \beta\neq\alpha}}h_{ki}^\beta R_{\alpha\beta jk}\Big\} \\
&= \sum_{i,j,k}(h_{ijk}^\alpha)^2-\sum_{i,j,k,l}h_{ij}^\alpha(h_{kl}^\alpha R_{lijk}+h_{li}^\alpha R_{lkjk})-\sum_{\substack{i,j,k,l\\ \beta\neq\alpha}}h_{ij}^\alpha h_{ki}^\beta(h_{jl}^\beta h_{kl}^\alpha-h_{kl}^\beta h_{jl}^\alpha)\,,
\end{aligned}$$

where h_{ijk}^α (resp. h_{ijkl}^α) are the components of the covariant derivative $\nabla\Pi$ (resp. $\nabla\nabla\Pi$) of the second fundamental form Π, and R_{ijkl} and $R_{\alpha\beta ij}$ are the components of the Riemannian curvature tensor and the normal curvature tensor of M in $S_p^{m+p}(c)$, respectively.

If, for a fixed α, we choose a local frame field $\{e_i\}$ as $h_{ij}^\alpha=\lambda_i^\alpha\delta_{ij}$, the above equation is rewritten as follows:

$$\Delta S_\alpha=2\sum_{i,j,k}(h_{ijk}^\alpha)^2+\sum_{i,k}(\lambda_i^\alpha-\lambda_k^\alpha)^2R_{kiik}+\sum_{\substack{i,k\\ \beta\neq\alpha}}(\lambda_i^\alpha-\lambda_k^\alpha)^2(h_{ik}^\beta)^2\,. \tag{3.1}$$

Then non-negativity of the sectional curvatures R_{ijji} implies that $\Delta S_\alpha \geqq 0$ for any α and $\Delta S \geqq 0$ (on M). It follows from compactness of M that $\Delta S = 0$. It means that $\Delta S_\alpha = 0$ for any α.

Now we choose orthonormal tangent vectors e_i at a point x in M as $h_{ij}^{m+1} = \lambda_i^{m+1}\delta_{ij}$. It follows from (3.1) and $\Delta S_\alpha = 0$ that

$$(\lambda_i^{m+1} - \lambda_j^{m+1})^2 (h_{ij}^\beta)^2 = 0 \qquad \text{for any } i, j \text{ and } \beta \neq m+1 .$$

So $h_{ij}^\beta = 0$ for any triple $\{\beta, i, j\}$ such that $\beta \neq m+1$ and $\lambda_i^{m+1} \neq \lambda_j^{m+1}$. This implies that the $m \times m$-matrices (h_{ij}^{m+1}) and (h_{ij}^{m+2}) are simultaneously diagonalizable, that is, we can choose orthonormal tangent vectors e_i at x as $h_{ij}^{m+1} = \lambda_i^{m+1}\delta_{ij}$, $h_{ij}^{m+2} = \lambda_i^{m+2}\delta_{ij}$. Again from (3.1) and $\Delta S_\alpha = 0$ it follows that $h_{ij}^\beta = 0$ for any triple $\{\beta, i, j\}$ such that $\beta \neq m+1$, $m+2$ and, either $\lambda_i^{m+1} \neq \lambda_j^{m+1}$ or $\lambda_i^{m+2} \neq \lambda_j^{m+2}$. Then also (h_{ij}^{m+1}), (h_{ij}^{m+2}) and (h_{ij}^{m+3}) are simultaneously diagonalizable. Iterating this procedure, we can prove that the all $m \times m$-matrices (h_{ij}^α) are simultaneously diagonalizable.

This means that for any local orthonormal frame field $\{e_i\}$,

$$\sum_k h_{ik}^\alpha h_{kj}^\beta = \sum_k h_{ik}^\beta h_{kj}^\alpha \qquad \text{for any } i, j \text{ and } \alpha, \beta ,$$

and then, the normal connection of M in $S_p^{m+p}(c)$ is flat. Using this fact in Theorem 3, we obtain that M is totally umbilic. □

As mentioned in §1, we can regard an immersion from a compact space-like m-dimensional submanifold into a semi-Riemannian sphere $S_p^{m+p}(c)$ as an embedding of S^m in $S_p^{m+p}(c)$. This proposition includes the following: *If the mean curvature vector of an isometric immersions from $S^m(c')$ into $S_p^{m+p}(c)$ is parallel, the immersion is totally umbilic.*

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