TOKYO J. MATH.
VOL. 18, NO. 1, 1995

On Certain Multiple Series with Functional Equation in a Totally Real Number Field I

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§1. Introduction.

In the analytic theory of partition function, the double series

$$f(\tau)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m}e^{-2\pi mn\tau}\qquad(\mathrm{Re}\,\tau>0) \tag{1.1}$$

plays an important role. It is well-known that $f(\tau)$ satisfies the functional equation:

$$f(\tau)-\frac{\pi}{12\tau}-\frac{1}{4}\log\tau=f\left(\frac{1}{\tau}\right)-\frac{\pi}{12}\tau-\frac{1}{4}\log\frac{1}{\tau}. \tag{1.2}$$

This remarkable equation has been proved by various methods (cf. Chandrasekharan [1, p. 170] or Schoenfeld [5]).

In this paper, we shall consider a multiple series that is a generalization of (1.1) in a totally real number field and prove that it satisfies a functional equation.

Let K be a totally real number field of degree n, $K^{(q)}$ $(q=1,\cdots,n)$ the conjugates of K. Let $\mathfrak{d}$ be the differente ideal of K, $D=N(\mathfrak{d})$ (norm of $\mathfrak{d}$) the absolute value of the discriminant of K, and R the regulator of K.

If μ is a number of K, then we denote by $\mu^{(q)}$ the conjugates of μ in $K^{(q)}$ $(q=1,\cdots,n)$. We define n-dimensional vector $\mu=(\mu^{(1)},\cdots,\mu^{(n)})$. More generally, we shall often use n-dimensional complex vector $\xi=(\xi_1,\cdots,\xi_n)$. For such ξ we put

$$S(\xi)=\sum_{q=1}^{n}\xi_q\,,\qquad N(\xi)=\prod_{q=1}^{n}\xi_q\,.$$

Let $\tau_1,\cdots,\tau_n$ be complex numbers with positive real parts. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the fractional ideals of K. For such $\mathfrak{a}$, $\mathfrak{b}$ and $\tau_1,\cdots,\tau_n$, we define the series $M(\tau\,;\,\mathfrak{a},\mathfrak{b})$ as follows:

$$M(\tau\,;\,\mathfrak{a},\mathfrak{b})=\sum_{\substack{(\mu)\subset\mathfrak{a}\\(\mu)\neq 0}}\frac{1}{|N(\mu)|}\sum_{\substack{\nu\subset\mathfrak{b}\\\nu\neq 0}}\exp\{-2\pi S(|\mu\nu|\tau)\}\,, \tag{1.3}$$

Received June 8, 1993

where the outer sum is taken over all non-zero principal ideals (μ) contained in $\mathfrak{a}$ and the inner sum is taken over all non-zero numbers of $\mathfrak{b}$. $M(\tau : \mathfrak{a}, \mathfrak{b})$ is well-defined, since the inner sum of (1.3) is independent of the choice of the generators of the ideal (μ).

To state our result we need another series:

$$\zeta(s, \mathfrak{a}) = \sum_{\substack{(\mu)\subset\mathfrak{a} \\ (\mu)\neq 0}} \frac{1}{|N(\mu)|^s} \qquad (s=\sigma+it\,;\ \sigma>1)\,,$$

where the sum has the same meaning as the outer sum in (1.3). This series $\zeta(s, \mathfrak{a})$ has the analytic continuation over the whole s-plane (see Lemma 2.2 below).

The purpose of this paper is to prove the following

THEOREM. *If we put*

$$\Phi(\tau\,;\ \mathfrak{a}, \mathfrak{b}) = M(\tau\,;\ \mathfrak{a}, \mathfrak{b}) - \frac{\zeta(2, \mathfrak{a})}{\pi^n N(\mathfrak{b})\sqrt{D}}(\tau_1 \cdots \tau_n)^{-1}$$
$$- \frac{2^{n-2}R}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1 \cdots \tau_n) - \frac{2^n\zeta^{(n)}(0, \mathfrak{b})}{n!\,N(\mathfrak{a})\sqrt{D}}\,,$$

then we have

$$N(\mathfrak{a}\mathfrak{b})^{1/2}\Phi(\tau\,;\ \mathfrak{a}, \mathfrak{b}) = N(\mathfrak{a}^*\mathfrak{b}^*)^{1/2}\Phi(\tau^{-1}\,;\ \mathfrak{b}^*, \mathfrak{a}^*)\,, \tag{1.4}$$

where $\mathfrak{a}^* = (\mathfrak{a}\mathfrak{d})^{-1}$ *and* $\mathfrak{b}^* = (\mathfrak{b}\mathfrak{d})^{-1}$.

Before proving our Theorem, we shall consider, in §2, the functions $\zeta(s, \lambda\,;\ \mathfrak{a})$, which are slightly different from the zeta functions $\zeta(s, \lambda\,;\ C)$ studied by Hecke in [2]. We shall state some properties of the $\zeta(s, \lambda\,;\ \mathfrak{a})$ in Lemmas 2.2, 2.3 and 2.4, which will be used for the proof of our Theorem.

In §3, we shall begin by applying the transformation formula of Hecke-Rademacher to $M(\tau\,;\ \mathfrak{a}, \mathfrak{b})$ and we shall obtain the representation of $M(\tau\,;\ \mathfrak{a}, \mathfrak{b})$ as the series of the complex integrals:

$$M(\tau\,;\ \mathfrak{a}, \mathfrak{b}) = \sum_{\lambda} \frac{1}{2\pi i}\int_{(2)} H_\lambda(s, \tau\,;\ \mathfrak{a}, \mathfrak{b})ds\,. \tag{1.5}$$

The integrands $H_\lambda(s, \tau\,;\ \mathfrak{a}, \mathfrak{b})$ are the products of the gamma function, the $\zeta(s, \lambda\,;\ \mathfrak{a})$ and some elementary functions (see (3.7) below). Using Lemmas 2.2 and 2.4, we shall have the estimate of $H_\lambda(s, \tau\,;\ \mathfrak{a}, \mathfrak{b})$, by which we shall be able to change the path of integration in (1.5). Then the functional equation satisfied by $H_\lambda(s, \tau\,;\ \mathfrak{a}, \mathfrak{b})$ (Lemma 3.3) will give the equation as follows:

$$M(\tau\,;\ \mathfrak{a}, \mathfrak{b}) = (DN(\mathfrak{a}\mathfrak{b}))^{-1}M(\tau^{-1}\,;\ \mathfrak{b}^*, \mathfrak{a}^*) + R(\tau\,;\ \mathfrak{a}, \mathfrak{b})\,,$$

where $R(\tau\,;\ \mathfrak{a}, \mathfrak{b})$ is the sum of the residues of $H_1(s, \tau\,;\ \mathfrak{a}, \mathfrak{b})$. Finally we shall calculate $R(\tau\,;\ \mathfrak{a}, \mathfrak{b})$ and then we shall complete the proof of Theorem.

§2. Zeta functions with Grössencharacters.

Let $\varepsilon_1, \cdots, \varepsilon_{n-1}$ be the fundamental units of K. Let $e_q^{(j)}$ $(q=1, \cdots, n; j=1, \cdots, n-1)$ the numbers satisfying the following equations:

$$\sum_{q=1}^{n} e_q^{(j)}=0 \qquad (j=1, \cdots, n-1),$$

$$\sum_{q=1}^{n} e_q^{(j)} \log|\varepsilon_k^{(q)}|=\begin{cases}1 & (j=k)\\ 0 & (j\neq k)\end{cases} \qquad (j, k=1, \cdots, n-1).$$

For rational integers $m_1, \cdots, m_{n-1}$ we put

(2.1) $$v_q=v_q(m_1, \cdots, m_{n-1})=2\pi \sum_{j=1}^{n-1} e_q^{(j)} m_j \qquad (q=1, \cdots, n).$$

Here we note that

(2.2) $$\sum_{q=1}^{n} v_q=0.$$

Now we define the Grössencharacter λ to be the function over complex vectors $\xi=(\xi_1, \cdots, \xi_n)$:

$$\lambda(\xi)=\prod_{q=1}^{n} |\xi_q|^{-iv_q}.$$

Let C be a class of the ideal numbers. Following Hecke [2], we put

$$\zeta(s, \lambda; C)=\sum_{\substack{(\nu)\\ 0\neq\nu\in C}} \frac{\lambda(\nu)}{|N(\nu)|^s} \qquad (\sigma>1),$$

where the sum is taken over non-zero integral ideal numbers ν in C not associated with each other. (For the details of Grössencharacters and ideal numbers, see Hecke [2] or Rademacher [4]).

We quote from Hecke [2] some properties of $\zeta(s, \lambda; C)$:

LEMMA 2.1. (1) *$\zeta(s, \lambda; C)$ has the analytic continuation over the whole s-plane and satisfies the functional equation as follows*:

$$\Gamma(s; \lambda)(D\pi^{-n})^{s/2}\zeta(s, \lambda; C)=\lambda(\delta)\Gamma(1-s; \bar{\lambda})(D\pi^{-n})^{(1-s)/2}\zeta(1-s, \bar{\lambda}; C'),$$

where C' is the class of ideal numbers such that $CC'\ni(\delta)=\mathfrak{d}$ and $\Gamma(s; \lambda)$ is the product of the gamma function:

$$\Gamma(s; \lambda)=\prod_{q=1}^{n} \Gamma\left(\frac{s+iv_q}{2}\right).$$

(2) *If* $\lambda\neq 1$, *then*

$$\Gamma(s;\lambda)\zeta(s,\lambda;C)$$

is an entire function.

(3) *If* $\lambda=1$, *then*

$$\Gamma(s;1)\zeta(s,1;C)$$

is a meromorphic function with two simple poles at $s=0$ *and* 1.

(4) $\zeta(s,1;C)$ *is regular in the whole s-plane except at* $s=1$, *where* $\zeta(s,1;C)$ *has a simple pole with the residue*

$$\operatorname*{Res}_{s=1}\zeta(s,1;C)=\frac{2^{n-1}R}{\sqrt{D}}.$$

PROOF. (Hecke [2].) □

Now we consider the series

$$\zeta(s,\lambda;\mathfrak{a})=\sum_{0\neq(\mu)\subset\mathfrak{a}}\frac{\lambda(\mu)}{|N(\mu)|^s}\qquad(\sigma>1),$$

where the sum is taken over all non-zero principal ideals (μ) contained in $\mathfrak{a}$. This series is well-defined, since $\lambda(\varepsilon)=1$ for any unit ε of K.

Let $C=C(\mathfrak{a}^{-1})$ be the class of ideal numbers containing $\mathfrak{a}^{-1}=(\hat{\alpha})^{-1}$. Since $\hat{\nu}$ in C is an integer of K if and only if $\hat{\nu}$ is the product of $\hat{\alpha}^{-1}$ and a number μ in $\mathfrak{a}$, we have

$$\sum_{\substack{(\hat{\nu})\\ 0\neq\hat{\nu}\in C}}\frac{\lambda(\hat{\nu})}{|N(\hat{\nu})|^s}=\lambda(\hat{\alpha})^{-1}N(\mathfrak{a})^s\sum_{0\neq(\mu)\subset\mathfrak{a}}\frac{\lambda(\mu)}{|N(\mu)|^s}\qquad(\sigma>1).$$

Hence the equation

$$\zeta(s,\lambda;C(\mathfrak{a}^{-1}))=\lambda(\hat{\alpha})^{-1}N(\mathfrak{a})^s\zeta(s,\lambda;\mathfrak{a})\tag{2.3}$$

holds in the whole s-plane. Moreover, replacing $\mathfrak{a}$ in (2.3) by $\mathfrak{a}^*=(\mathfrak{a}\mathfrak{d})^{-1}$, we have

$$\zeta(s,\lambda;C(\mathfrak{a}\mathfrak{d}))=\frac{\lambda(\hat{\alpha}\hat{\delta})}{D^sN(\mathfrak{a})^s}\zeta(s,\lambda;\mathfrak{a}^*).\tag{2.4}$$

By (2.3), (2.4) and Lemma 2.1, we easily obtain the following

LEMMA 2.2. (1) $\zeta(s,\lambda;\mathfrak{a})$ *has the analytic continuation over the whole s-plane and satisfies the functional equation as follows*:

$$\Gamma(s;\lambda)\zeta(s,\lambda;\mathfrak{a})=\frac{\pi^{n(s-1/2)}}{N(\mathfrak{a})\sqrt{D}}\Gamma(1-s;\bar{\lambda})\zeta(1-s,\bar{\lambda};\mathfrak{a}^*).\tag{2.5}$$

(2) *If* $\lambda\neq 1$, *then*

$$\Gamma(s\,;\lambda)\zeta(s,\lambda\,;\mathfrak{a})$$

is an entire function.

(3) *If* $\lambda=1$, *then*

$$\Gamma(s\,;1)\zeta(s,1\,;\mathfrak{a})$$

is a meromorphic function with two simple poles at $s=0$ *and* 1.

(4) $\zeta(s,1\,;\mathfrak{a})$ *is regular in the whole s-plane except at* $s=1$, *where* $\zeta(s,1\,;\mathfrak{a})$ *has a simple pole with the residue*

$$\operatorname*{Res}_{s=1}\zeta(s,1\,;\mathfrak{a})=\frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}\,.$$

From now on, we write

$$\zeta(s,\mathfrak{a})=\zeta(s,1\,;\mathfrak{a})\,,$$

which is the function stated in §1 above.

Further we note that $\zeta(s,\mathfrak{a})$ has a zero point of order $n-1$ at $s=0$.

LEMMA 2.3. (1) *We have*

$$\zeta^{(n-1)}(0,\mathfrak{a})=-(n-1)!\,R/2\,. \tag{2.6}$$

(2) *If we expand* $\zeta(1+s,\mathfrak{a})$ *for small s as follows*:

$$\zeta(1+s,\mathfrak{a})=\frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}\,\frac{1}{s}+c(\mathfrak{a})+O(|s|)\,, \tag{2.7}$$

then

$$c(\mathfrak{a})=\frac{2^{n}}{N(\mathfrak{a})\sqrt{D}}\left\{\frac{nR}{2}(\log(2\pi)+\gamma)+\frac{1}{n!}\zeta^{(n)}(0,\mathfrak{a}^{*})\right\}, \tag{2.8}$$

where γ *is Euler's constant.*

PROOF. The functional equation (2.5) gives

$$\zeta(1+s,\mathfrak{a})=\frac{\pi^{n(s+1/2)}}{N(\mathfrak{a})\sqrt{D}}\,\frac{\Gamma(-s/2)^{n}}{\Gamma((1+s)/2)^{n}}\zeta(-s,\mathfrak{a}^{*})\,. \tag{2.9}$$

For small s we have the following expansions of the functions in the right-hand side of (2.9):

$$\pi^{ns}=1+ns\log\pi+\cdots,$$

$$\Gamma\left(-\frac{s}{2}\right)^{n}=\frac{(-2)^{n}}{s^{n}}\left(1+\frac{n}{2}\gamma s+\cdots\right),$$

$$\Gamma\left(\frac{1+s}{2}\right)^{-n}=\pi^{-n/2}\left(1-\frac{n}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)s+\cdots\right),$$

$$\zeta(-s,\mathfrak{a}^{*})=\frac{(-1)^{n-1}}{(n-1)!}\zeta^{(n-1)}(0,\mathfrak{a}^{*})s^{n-1}+\frac{(-1)^{n}}{n!}\zeta^{(n)}(0,\mathfrak{a}^{*})s^{n}+\cdots.$$

Hence

$$(2.10)\qquad \begin{aligned}\zeta(1+s,\mathfrak{a})=&\frac{2^{n}}{N(\mathfrak{a})\sqrt{D}}\frac{-1}{(n-1)!}\zeta^{(n-1)}(0,\mathfrak{a}^{*})\frac{1}{s}\\&+\frac{2^{n}}{N(\mathfrak{a})\sqrt{D}}\left\{\frac{-1}{(n-1)!}\zeta^{(n-1)}(0,\mathfrak{a}^{*})\left[n\log\pi+\frac{n}{2}\gamma\right.\right.\\&\left.\left.-\frac{n}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)\right]+\frac{1}{n!}\zeta^{(n)}(0,\mathfrak{a}^{*})\right\}+O(|s|).\end{aligned}$$

Comparing (2.10) with (2.7), we have

$$(2.11)\qquad \zeta^{(n-1)}(0,\mathfrak{a}^{*})=-(n-1)!\,R/2,$$

which gives (2.6) since the right-hand side of (2.11) is independent of the choice of the ideal $\mathfrak{a}$. From (2.11), (2.10) and the formula

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)=-\log 4-\gamma,$$

(2.8) follows at once. □

LEMMA 2.4. *In the strip* $-1/2\leqq\sigma\leqq 3$, *we have*

$$\zeta(s,\lambda;\mathfrak{a})(s-1)^{e(\lambda)}\ll(1+|t|)^{2n},$$

where

$$e(\lambda)=\begin{cases}1 & \text{if } \lambda=1,\\ 0 & \text{if } \lambda\neq 1\end{cases}$$

and the constants implied in this estimation depend on λ *and* $\mathfrak{a}$.

PROOF. This lemma is proved in the same way as that of [4, Hilfssatz 15], so we omit the proof. □

§3. Proof of Theorem.

Let $\varepsilon_1, \cdots, \varepsilon_{n-1}$ be the fundamental units of K. We rewrite the inner sum of (1.3) as follows:

$$(3.1)\qquad \sum_{\substack{\nu\in\mathfrak{b}\\ \nu\neq 0}} \exp\{-2\pi S(|\mu\nu|\tau)\}$$
$$=2\sum_{\substack{(\nu)\subset\mathfrak{b}\\ (\nu)\neq 0}}\ \sum_{a_1,\cdots,a_{n-1}=-\infty}^{\infty} \exp\{-2\pi S(|\mu\nu\varepsilon_1^{a_1}\cdots\varepsilon_{n-1}^{a_{n-1}}|\tau)\},$$

where $a_1, \cdots, a_{n-1}$ run through all rational integers and the outer sum is taken over all non-zero principal ideals (ν) contained in $\mathfrak{b}$.

Now we quote the transformation formula of Hecke-Rademacher from Rademacher [4] as a lemma:

LEMMA 3.1. *Let $W_1, \cdots, W_n$ be complex numbers with positive real parts. Then we have*

$$(3.2)\qquad \sum_{a_1,\cdots,a_{n-1}=-\infty}^{\infty} \exp\{-2\pi S(|\varepsilon_1^{a_1}\cdots\varepsilon_{n-1}^{a_{n-1}}|W)\}$$
$$=\frac{1}{R}\sum_{m_1,\cdots,m_{n-1}=-\infty}^{\infty}\frac{1}{2\pi i}\int_{(2)}\prod_{q=1}^{n}\frac{\Gamma(s+iv_q)}{(2\pi W_q)^{s+iv_q}}ds,$$

where $m_1, \cdots, m_{n-1}$ run through all rational integers, the v_q are the values defined by (2.1) *and the integral in* (3.2) *is the complex integral taken along the vertical line $\sigma=2$.*

PROOF. ([4, Hilfssatz 14].) □

Applying this Lemma with $W_q=|\nu^{(q)}\mu^{(q)}|\tau_q$ $(q=1, \cdots, n)$ to the sum in the right-hand side of (3.1), we have

$$(3.3)\qquad M(\tau\,;\,\mathfrak{a},\mathfrak{b})=\frac{2}{R}\sum_{0\neq(\mu)\subset\mathfrak{a}}\frac{1}{|N(\mu)|}$$
$$\times\sum_{\substack{(\nu)\subset\mathfrak{b}\\ (\nu)\neq 0}}\ \sum_{m_1,\cdots,m_{n-1}=-\infty}^{\infty}\frac{1}{2\pi i}\int_{(2)}\prod_{q=1}^{n}\frac{\Gamma(s+iv_q)}{(2\pi\tau_q|\nu^{(q)}\mu^{(q)}|)^{s+iv_q}}ds.$$

By the well-known estimation of the gamma function, we have for $\sigma=2$

$$(3.4)\qquad \prod_{q=1}^{n}\frac{\Gamma(s+iv_q)}{(2\pi\tau_q)^{s+iv_q}}\ll\prod_{q=1}^{n}(1+|t+v_q|)^{3/2}\exp(-\alpha|t+v_q|),$$

where

(3.5)
$$\alpha = \min_{1 \leqq q \leqq n} (\pi/2 - |\arg \tau_q|) \quad (>0)\,.$$

(3.4) easily gives the estimate of $M(\tau\,;\,\mathfrak{a},\mathfrak{b})$ as follows:

$$M(\tau\,;\,\mathfrak{a},\mathfrak{b}) \ll \sum_{m_1,\cdots,m_{n-1}=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2}\sum_{q=1}^{n}|t+v_q|\right)dt\,.$$

Since this series is convergent ([3, p. 206]), we see that the series in the right-hand side of (3.3) is absolutely convergent. Therefore we can change, in (3.3), the order of the summations over (ν), (μ) and $m_1,\cdots,m_{n-1}$. Moreover, we can invert the order of the summations over (ν), (μ) and the integration.

Thus we have

$$M(\tau\,;\,\mathfrak{a},\mathfrak{b}) = \frac{2}{R}\sum_{m_1,\cdots,m_{n-1}=-\infty}^{\infty} \frac{1}{2\pi i}\int_{(2)}\prod_{q=1}^{n}\frac{\Gamma(s+iv_q)}{(2\pi\tau_q)^{s+iv_q}}$$
$$\times \sum_{\substack{(\mu)\subset\mathfrak{a}\\(\mu)\neq 0}}\frac{\lambda(\mu)}{|N(\mu)|^{1+s}}\sum_{\substack{(\nu)\subset\mathfrak{b}\\(\nu)\neq 0}}\frac{\lambda(\nu)}{|N(\nu)|^{s}}ds\,,$$

where the λ are the Grössencharacters. The sum of this right-hand side over rational integers $m_1,\cdots,m_{n-1}$ can be regarded as the sum $\sum_\lambda$ over all Grössencharacters λ. Therefore we obtain the expression of $M(\tau\,;\,\mathfrak{a},\mathfrak{b})$ as follows:

(3.6)
$$M(\tau\,;\,\mathfrak{a},\mathfrak{b}) = \frac{2}{R}\sum_{\lambda}\frac{1}{2\pi i}\int_{(2)}\prod_{q=1}^{n}\frac{\Gamma(s+iv_q)}{(2\pi\tau_q)^{s+iv_q}}\zeta(s,\lambda\,;\,\mathfrak{b})\zeta(1+s,\lambda\,;\,\mathfrak{a})ds\,.$$

Now we shall put

(3.7)
$$H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b}) = \frac{2}{R}\prod_{q=1}^{n}\frac{\Gamma(s+iv_q)}{(2\pi\tau_q)^{s+iv_q}}\zeta(s,\lambda\,;\,\mathfrak{b})\zeta(1+s,\lambda\,;\,\mathfrak{a})$$

and prove the following three lemmas on the properties of $H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})$.

LEMMA 3.2. (1) *If* $\lambda\neq 1$, *then* $H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})$ *is an entire function.*

(2) $H_1(s,\tau\,;\,\mathfrak{a},\mathfrak{b})$ *has three poles, that is, two simple poles at* $s=1$ *and* -1 *and one double pole at* $s=0$.

PROOF. We apply the duplication formula of the gamma function to the factors in the right-hand side of (3.7). Then

$$H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b}) = \frac{2}{R}\prod_{q=1}^{n}\left\{\frac{1}{(2\pi\tau_q)^{s+iv_q}}\frac{2^{s+iv_q-1}}{\sqrt{\pi}}\Gamma\left(\frac{s+iv_q}{2}\right)\Gamma\left(\frac{s+iv_q+1}{2}\right)\right\}$$
$$\times\zeta(s,\tau\,;\,\mathfrak{b})\zeta(1+s,\lambda\,;\,\mathfrak{a})\,,$$

or, by the definition of $\Gamma(s\,;\,\lambda)$,

$$(3.8)\qquad H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})=\frac{2}{R}\frac{1}{(2\sqrt{\pi})^n}\prod_{q=1}^{n}\frac{1}{(\pi\tau_q)^{s+iv_q}}$$
$$\times\Gamma(s\,;\,\lambda)\zeta(s,\lambda\,;\,\mathfrak{b})\Gamma(1+s\,;\,\lambda)\zeta(1+s,\lambda\,;\,\mathfrak{a})\,.$$

If $\lambda\neq 1$, then it follows from Lemma 2.2, (2) that $\Gamma(s\,;\,\lambda)\zeta(s,\lambda\,;\,\mathfrak{b})$ and $\Gamma(1+s\,;\,\lambda)\zeta(1+s,\lambda\,;\,\mathfrak{a})$ are entire functions. Therefore $H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})$ is an entire function.

The assertion (2) also follows from Lemma 2.2, (3) at once. □

LEMMA 3.3. *$H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})$ satisfies the functional equation as follows*:

$$H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})=(DN(\mathfrak{a}\mathfrak{b}))^{-1}H_{\bar\lambda}(-s,\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)\,,$$

where $\mathfrak{a}^*=(\mathfrak{a}\mathfrak{d})^{-1}$ *and* $\mathfrak{b}^*=(\mathfrak{b}\mathfrak{d})^{-1}$.

PROOF. We apply the functional equation (2.5) to (3.9). Then we have

$$H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})=\frac{2}{R}\frac{1}{DN(\mathfrak{a}\mathfrak{b})}\frac{1}{(2\sqrt{\pi})^n}\prod_{q=1}^{n}\left(\frac{\pi}{\tau_q}\right)^{s+iv_q}$$
$$\times\Gamma(1-s\,;\,\bar\lambda)\zeta(1-s,\bar\lambda\,;\,\mathfrak{b}^*)\Gamma(-s\,;\,\bar\lambda)\zeta(-s,\bar\lambda\,;\,\mathfrak{a}^*)$$

(here we use (2.2)). Comparing this expression with (3.8), we obtain the lemma immediately. □

LEMMA 3.4. *For $-3/2\leqq\sigma\leqq 2$, we have*

$$H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})\ll\exp\left(-\frac{\alpha}{2}|t|\right),$$

where α is the number defined by (3.5). *The constants implied in this estimate depend on λ, τ, $\mathfrak{a}$ and $\mathfrak{b}$.*

PROOF. In view of Lemma 3.3, it is sufficient to prove the lemma under the assumption $0\leqq\sigma\leqq 2$. Then it follows from (3.4), (3.7) and Lemma 2.4 that

$$H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})\ll(1+|t|)^{4n+3n/2}\exp\left(-\alpha\sum_{q=1}^{n}|t+v_q|\right)\ll\exp\left(-\frac{\alpha}{2}|t|\right).\qquad\square$$

By Lemma 3.4 it is clear that

$$\int_{2+iT}^{-3/2+iT}H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})ds\to 0\qquad(|T|\to\infty)\,,$$

where the integral is taken along the horizontal line from $2+iT$ to $-3/2+iT$. Therefore by Lemma 3.2 and Cauchy's formula,

$$(3.9)\qquad \frac{1}{2\pi i}\int_{(2)} H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})ds = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{(-3/2)} H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})ds & (\text{if } \lambda\neq 1)\,, \\ \dfrac{1}{2\pi i}\displaystyle\int_{(-3/2)} H_1(s,\tau\,;\,\mathfrak{a},\mathfrak{b})ds + R(\tau\,;\,\mathfrak{a},\mathfrak{b}) & (\text{if } \lambda=1)\,, \end{cases}$$

where

$$R(\tau\,;\,\mathfrak{a},\mathfrak{b}) = \operatorname*{Res}_{s=1} H_1 + \operatorname*{Res}_{s=0} H_1 + \operatorname*{Res}_{s=-1} H_1$$

is the sum of the residues of $H_1(s,\tau\,;\,\mathfrak{a},\mathfrak{b})$. Combining (3.6), (3.7) and (3.9), we have

$$(3.10)\qquad \begin{aligned} M(\tau\,;\,\mathfrak{a},\mathfrak{b}) &= \sum_\lambda \frac{1}{2\pi i}\int_{(2)} H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})ds \\ &= \sum_\lambda \frac{1}{2\pi i}\int_{(-3/2)} H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})ds + R(\tau\,;\,\mathfrak{a},\mathfrak{b})\,. \end{aligned}$$

On the other hand, we see from Lemma 3.3 that

$$\sum_\lambda \frac{1}{2\pi i}\int_{(-3/2)} H_\lambda(s,\tau\,;\,\mathfrak{a},\mathfrak{b})ds = (DN(\mathfrak{a}\mathfrak{b}))^{-1}\sum_\lambda \frac{1}{2\pi i}\int_{(3/2)} H_{\bar\lambda}(s,\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)ds\,.$$

Since $\bar\lambda$ runs through all Grössencharacters, the last sum is equal to

$$\sum_\lambda \frac{1}{2\pi i}\int_{(3/2)} H_\lambda(s,\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)ds = M(\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)\,.$$

Hence (3.10) gives

$$(3.11)\qquad M(\tau\,;\,\mathfrak{a},\mathfrak{b}) = (DN(\mathfrak{a}\mathfrak{b}))^{-1}M(\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*) + R(\tau\,;\,\mathfrak{a},\mathfrak{b})\,.$$

Now we shall calculate $R(\tau\,;\,\mathfrak{a},\mathfrak{b})$.

First we easily obtain from the expression

$$(3.12)\qquad H_1(s,\tau\,;\,\mathfrak{a},\mathfrak{b}) = \frac{2}{R}\,\frac{\Gamma(s)^n}{((2\pi)^n\tau_1\cdots\tau_n)^s}\,\zeta(s,\mathfrak{b})\zeta(1+s,\mathfrak{a})$$

that

$$(3.13)\qquad \operatorname*{Res}_{s=1} H_1 = \frac{2}{R}\,\frac{\zeta(2,\mathfrak{a})}{(2\pi)^n}(\tau_1\cdots\tau_n)^{-1}\operatorname*{Res}_{s=1}\zeta(s,\mathfrak{b}) = \frac{\zeta(2,\mathfrak{a})}{\pi^n N(\mathfrak{b})\sqrt{D}}(\tau_1\cdots\tau_n)^{-1}\,.$$

As for $\operatorname{Res}_{s=-1} H_1$, it follows from Lemma 3.3 and (3.13) that

(3.14)
$$\begin{aligned}\operatorname*{Res}_{s=-1} H_1 &= \lim_{s\to-1}(s+1)H_1(s,\tau\,;\,\mathfrak{a},\mathfrak{b})\\ &=(DN(\mathfrak{ab}))^{-1}\lim_{s\to-1}(s+1)H_1(-s,\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)\\ &=-(DN(\mathfrak{ab}))^{-1}\lim_{s\to1}(s-1)H_1(s,\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)\\ &=-\frac{\zeta(2,\mathfrak{b}^*)}{\pi^nN(\mathfrak{b})\sqrt{D}}\tau_1\cdots\tau_n\,.\end{aligned}$$

In order to compute the residue at $s=0$, we expand the functions in the right-hand side of (3.12) as follows:

$$\Gamma(s)^n=\frac{1}{s^n}(1-n\gamma s+\cdots)\,,$$

$$((2\pi)^n\tau_1\cdots\tau_n)^{-s}=1-s\log((2\pi)^n\tau_1\cdots\tau_n)+\cdots\,,$$

$$\zeta(s,\mathfrak{b})=-\frac{R}{2}s^{n-1}+\frac{1}{n!}\zeta^{(n)}(s,\mathfrak{b})s^n+\cdots\,,$$

$$\zeta(1+s,\mathfrak{a})=\frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}\frac{1}{s}+c(\mathfrak{a})+\cdots\,.$$

Then

(3.15)
$$\begin{aligned}\operatorname*{Res}_{s=0} H_1 &= n\gamma\frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}-c(\mathfrak{a})+\frac{2^{n-1}nR}{N(\mathfrak{a})\sqrt{D}}\log(2\pi)\\ &\quad+\frac{2^n\zeta^{(n)}(0,\mathfrak{b})}{n!N(\mathfrak{a})\sqrt{D}}+\frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n)\\ &=\frac{2^n}{n!N(\mathfrak{a})\sqrt{D}}\{\zeta^{(n)}(0,\mathfrak{b})-\zeta^{(n)}(0,\mathfrak{a}^*)\}+\frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n)\,.\end{aligned}$$

Collecting the values of residues (3.13), (3.14) and (3.15), and putting them into (3.11), we have

(3.16)
$$\begin{aligned}M(\tau\,;\,\mathfrak{a},\mathfrak{b}) &= (DN(\mathfrak{ab}))^{-1}M(\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)\\ &\quad+\frac{\zeta(2,\mathfrak{a})}{\pi^nN(\mathfrak{b})\sqrt{D}}(\tau_1\cdots\tau_n)^{-1}-\frac{\zeta(2,\mathfrak{b}^*)}{\pi^nN(\mathfrak{b})\sqrt{D}}\tau_1\cdots\tau_n\\ &\quad+\frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n)+\frac{2^n}{n!N(\mathfrak{a})\sqrt{D}}\{\zeta^{(n)}(0,\mathfrak{b})-\zeta^{(n)}(0,\mathfrak{a}^*)\}\,.\end{aligned}$$

If we put

$$\Phi(\tau\,;\,\mathfrak{a},\mathfrak{b})=M(\tau\,;\,\mathfrak{a},\mathfrak{b})-\frac{\zeta(2,\mathfrak{a})}{\pi^n N(\mathfrak{b})\sqrt{D}}(\tau_1\cdots\tau_n)^{-1}$$

$$-\frac{2^{n-2}R}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n)-\frac{2^n\zeta^{(n)}(0,\mathfrak{b})}{n!\,N(\mathfrak{a})\sqrt{D}},$$

then we can rewrite (3.16) in a simple form:

$$\Phi(\tau\,;\,\mathfrak{a},\mathfrak{b})=(DN(\mathfrak{a}\mathfrak{b}))^{-1}\Phi(\tau^{-1}\,;\,\mathfrak{b}^*,\mathfrak{a}^*)\,,$$

which gives (1.4). Thus we have completed the proof of Theorem.

As a special case of Theorem, we easily obtain

COROLLARY. *If we put*

$$\Phi(\tau\,;\,\mathfrak{a})=M(\tau\,;\,\mathfrak{a},\mathfrak{a}^*)-\pi^{-n}N(\mathfrak{a})\sqrt{D}\,\zeta(2,\mathfrak{a})(\tau_1\cdots\tau_n)^{-1}-\frac{2^{n-2}R}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n)\,,$$

then

$$\Phi(\tau\,;\,\mathfrak{a})=\Phi(\tau^{-1}\,;\,\mathfrak{a})\,.$$

In the case $K=\boldsymbol{Q}$ and $\mathfrak{a}=\boldsymbol{Z}$, we see that

$$\Phi(\tau\,;\,\boldsymbol{Z})=2f(\tau)-\frac{\pi}{6\tau}-\frac{1}{2}\log\tau\,,$$

where $f(\tau)$ is the double series (1.1). Hence Corollary gives the functional equation (1.2).

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