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On the Schur Indices of $SU_{l+1}(F_q)$ and $Spin_{2l}^{-}(F_q)$

Zyozyu OHMORI

Hokkaido University of Education

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Dedicated to Professor Tosiro Tsuzuku

Introduction.

Let G be a connected, reductive algebraic group defined over a finite field F_q with q elements of characteristic p and let F be the corresponding Frobenius endomorphism of G. As usual, G^F denotes the finite group of F-fixed points of G (G^F = the group of F_q-rational points). Let B be an F-stable Borel subgroup of G and U be its unipotent radical. Then U is also F-stable and U^F is a Sylow p-subgroup of G^F.

In [4] Gel'fand and Graev found that when $G=SL_n$ (or GL_n) any irreducible character of G^F occurs with non-zero multiplicity in some induced characters λ^{G^F} where λ runs over all the linear characters of U^F and if λ is "in general position" then λ^{G^F} is multiplicity-free. The latter "multiplicity-one theorem" holds for a general G (Yokonuma [19], Steinberg [16]) but the former fact does not hold for a general G (e.g. Sp_4). However it seems that almost all the irreducible characters of G^F occur in $\sum_\lambda \lambda^{G^F}$.

R. Gow has initiated to investigate the rationality-properties of the characters λ^{G^F} in order to get informations about the Schur indices of the irreducible characters of G^F ([5, 6], also cf. [7]).

In the rest of this introduction we assume that p is not a bad prime for G for the sake of simplicity. In [13] we studied the rationality of the λ^{G^F} generally and saw that any λ^{G^F} takes values in $k=\mathbf{Q}(\sqrt{(-1)^{(p-1)/2}p})$ (we assume that $p\neq 2$) and is realizable in k_v for any finite place v of k. From this it follows that if χ is an irreducible character of G^F such that $\langle \chi, \lambda^{G^F}\rangle_{G^F}=1$ for some λ or $p\nmid \chi(1)$ then the Schur index $m_{\mathbf{Q}}(\chi)$ of χ with respect to $\mathbf{Q}$ is at most two. In [14] we announced some more detailed results when G is a simple algebraic group. Main purpose of this paper is to give their proofs when G is a twisted group. For the sake of simplicity we shall assume that G is simply-connected. As the cases $G={}^3D_4$, ${}^2E_6$ are treated in [13] we assume here that $G=SU_{l+1}$ or $Spin_{2l}^{-}$.

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First we give some sufficient conditions subject for that all the λ^{G^F} are realizable in Q; Theorem 1 (in §5) is a consequence of these results. In some cases (e.g. q is square) all the λ^{G^F} are realizable in Q_r for any prime number $r \neq p$. Theorem 2 is a consequence of this fact. A large part of Theorems 1, 2 are already proved in [13]. In Theorem 3 we treat the group $G^F = SU_{l+1}(F_q)$ when q is square.

Let G be general. As we have recalled above the Schur indices of many irreducible characters of G^F are at most two. (The author does not know examples of characters of G^F with the indices $\geqq 3$.) In certain cases (e.g. $G = GL_n$, SL_{2n+1}, CSp_4, SO_5, G_2) all the indices are equal to one. But, generally, there are characters of the index equal to two (e.g. SL_2, Sp_4, U_n). In [6] Gow gave some sufficient conditions subject for the existence of characters of the Schur index equal to two when $G = SL_{2n}$. In Theorem 4 of our paper we shall give similar results when $G = SU_{l+1}$, $Spin_{2l}^-$. Our method is slightly different from Gow's one. Characters with the index equal to two will be found in some λ^{G^F} where λ is in general position.

In §1 we compute the inner products $\langle \lambda^{G^F}, \lambda^{G^F} \rangle_{G^F}$ where λ are in general position. The result will be used in the proof of Theorem 4. In §2, in order to study the rationality of the λ^{G^F} we study the rationality of the λ^{B^F}. Naturally we follow the method described in Yamada [18], §3. Then certain subgroups L, M of B^F will be introduced and the rationality of the λ^{B^F} will be reduced to that of the λ^L. We have $L = MU^F$ (semidirect product). In §3, we determine the structure of M completely. In §4 we calculate the Hasse invariants of the Schur algebras associated with the irreducible components of λ^L. The main results (Theorems 1, 2, 3, 4) are stated and proved in §5.

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§1. Gelfand-Graev characters.

Let K be an algebraically closed field of characteristic $p > 0$. Let G be a simple algebraic group (over K) defined over F_q with the Frobenius endomorphism F. We fix an F-stable Borel subgroup B of G with the unipotent radical U and an F-stable maximal torus T of B. Let R, R^+ and Δ be respectively the set of roots of G with respect to T, the set of positive roots determined by B and the set of simple roots. For $\alpha \in R$, U_α denotes the root subgroup of G associated with α. Let ρ be the permutation on R defined by $FU_\alpha = U_{\rho\alpha}$ $(\alpha \in R)$; then ρ fixes R^+ and Δ. Let I be the set of orbits of ρ on Δ. For each $i \in I$, let $U_i = \prod_{\alpha \in i} U_\alpha$ (direct product). Let $U.$ be the normal subgroup of U generated by the U_α, $\alpha \in R^+ - \Delta$. Then we have $U^F/U^F_. = (U/U.)^F = \prod_{i \in I} U_i^F$. For each $i \in I$, we fix a root γ_i in i and we put $q_i = q^{|i|}$. Then, for each i, there is an isomorphism ϕ_i of U_i^F with the additive group of F_{q_i} such that $\phi_i(tut^{-1}) = \gamma_i(t)\phi_i(u)$ for $t \in T^F$ and $u \in U_i^F$. Hence there is an isomorphism ϕ of $U^F/U^F_. = \prod_{i \in I} U_i^F$ with $\prod_{i \in I} F_{q_i}$ such that, for $u \in U^F/U^F_.$ and $t \in T^F$, we have

$$\phi(tut^{-1})=(\gamma_i(t)\phi_i(u_i))_{i\in I} \qquad (u=(u_i)_{i\in I},\ u_i\in U_i^F\ (i\in I))\,.$$

Now let Λ be the set of irreducible characters λ of U^F such that $\lambda|U_\cdot^F=1$ and let Λ_0 be the set of λ in Λ such that $\lambda|U_i^F\neq 1$ for all $i\in I$. For $\lambda\in\Lambda$, put $\Gamma_\lambda=\lambda^{G^F}$. Then the following lemma is well known:

LEMMA 1 (Gel'fand-Graev [4], Yokonuma [19], Steinberg [16]). *If $\lambda\in\Lambda_0$, then Γ_λ is multiplicity-free.*

Let us compute the inner products $\langle\Gamma_\lambda, \Gamma_\lambda\rangle_{G^F}$ for $\lambda\in\Lambda_0$. For a subset J of Δ, put

$$T_J=\bigcap_{\alpha\in J}\operatorname{Ker}\alpha$$

(we put $T_\varnothing=T$). If J is ρ-stable, then T_J is an F-stable subgroup of T (perhaps disconnected). By induction on $|J|$, we see that $\dim T_J=|\Delta|-|J|$. By a method similar to the proof of the main theorem of [19], we can prove:

LEMMA 2 (cf. Yokonuma [19], Steinberg [16]). *If $\lambda\in\Lambda_0$, then there is a set S of ρ-stable subsets J of Δ such that*

$$\langle\Gamma_\lambda, \Gamma_\lambda\rangle_{G^F}=\sum_{J\in S}|T_J^F|\,,$$

where S contains $\varnothing$ and Δ.

Let Z be the centre of G and put $c=|Z^F|$. (Note: $Z=T_\Delta$.)

PROPOSITION 1. *If $\lambda\in\Lambda_0$, then we have*

$$\langle\Gamma_\lambda, \Gamma_\lambda\rangle_{G^F}=r(q-1)+c$$

for some positive integer r.

PROOF. It suffices to prove that if J is a ρ-stable subset of Δ and $J\neq\Delta$, then $q-1$ divides $|T_J^F|$. Let J be such a set, and let T_J^0 be the identity-component of T_J. Then T_J^0 is an F-stable subtorus of T and $\dim T_J^0=|\Delta|-|J|$. Let $X=\operatorname{Mor}(T, K^\times)$ and $X'=\operatorname{Mor}(T_J^0, K^\times)$. Then we have an exact sequence of modules:

$$0\longrightarrow X''\longrightarrow X\xrightarrow{\text{res}} X'\longrightarrow 0\,,$$
$$\chi\longmapsto \chi|T_J^0$$

where res is the restriction map and X'' is the kernel of res. We note that J is contained in X''. From this sequence we get an exact sequence of vector spaces over $\boldsymbol{R}$:

$$0\to X''\otimes\boldsymbol{R}\to X\otimes\boldsymbol{R}\to X'\otimes\boldsymbol{R}\to 0\,.$$

Here we have $X\otimes\boldsymbol{R}=\langle\Delta\rangle_{\boldsymbol{R}}$ (the vector space over $\boldsymbol{R}$ spanned by Δ); and, as $\dim_{\boldsymbol{R}}X''\otimes\boldsymbol{R}=\dim_{\boldsymbol{R}}X\otimes\boldsymbol{R}-\dim_{\boldsymbol{R}}X'\otimes\boldsymbol{R}=|\Delta|-(|\Delta|-|J|)=|J|$ and $J\subset X''$, we have

$X'' \otimes \boldsymbol{R} = \langle J \rangle_{\boldsymbol{R}}$. F acts on X (resp. on X') by $F(\chi) = x \circ F$ for $\chi \in X$ (resp. for $\chi \in X'$); this action can be extended linearly to that of F on $X \otimes \boldsymbol{R}$ (resp. on $X' \otimes \boldsymbol{R}$). Then we have:

$$|(T_J^0)^F| = |\det_{X' \otimes \boldsymbol{R}}(F-1)| = \frac{|\det_{X \otimes \boldsymbol{R}}(F-1)|}{|\det_{X'' \otimes \boldsymbol{R}}(F-1)|}$$

$$= \frac{\prod_{i \in I} (q_i - 1)}{\prod_{\substack{i \in I \\ i \subset J}} (q_i - 1)} \qquad \text{(cf. [17], 11.10)}$$

$$= \prod_{\substack{i \in I \\ i \subset \Delta - J}} (q_i - 1) .$$

Hence:

$$|T_J^F| = (T_J^F : (T_J^0)^F) |(T_J^0)^F| = (T_J^F : (T_J^0)^F) \prod_{i \subset \Delta - J} (q_i - 1) .$$

Hence $q-1$ divides $|T_J^F|$ (cf. $J \neq \Delta$).

Let $\lambda \in \Lambda_0$. Let $\eta_1, \cdots, \eta_c$ be all the irreducible characters of Z^F. For $1 \leqq i \leqq c$, put $\Gamma_{\lambda,i} = \mathrm{Ind}_{Z^F U^F}^{G^F}(\eta_i \lambda)$. Then we have the following:

$$\Gamma_\lambda = \sum_{i=1}^{c} \Gamma_{\lambda,i} ,$$

$$\langle \Gamma_{\lambda,i}, \Gamma_{\lambda,j} \rangle_{G^F} = \delta_{ij} \cdot \frac{1}{c} \cdot \langle \Gamma_\lambda, \Gamma_\lambda \rangle_{G^F} = \delta_{ij} \left\{ \frac{r(q-1)}{c} + 1 \right\} \qquad (1 \leqq i, j \leqq c) .$$

§2. Schur algebras associated with Γ_λ.

First we quote from [18] some results concerning Schur algebras. We recall that a Schur algebra is a simple direct summand of the group algebra of a finite group over a field of characteristic zero.

Let H be a finite group and N be a normal subgroup of H. Let χ be an irreducible character of H which is induced by an irreducible character ψ of N. Let k be a field of characteristic zero. We assume that $k(\chi) = k$, where $k(\chi)$ is the field generated over k by the values of χ. Set $L = \{ f \in H \mid \psi^f = \psi^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(k(\psi)/k) \}$. Let Nf_i ($i = 0, 1, \cdots, t-1; f_0 = 1$) be all the distinct cosets of N in L, and set $\tau(f_i) = \tau_i$ ($\tau_0 = 1$). Then $L/N \simeq \{ \tau_0, \tau_1, \cdots, \tau_{t-1} \} = \mathrm{Gal}(k(\psi)/k)$ and $k(\psi^L) = k$ ([18], Prop. 3.4). Furthermore let $f_i f_j = n_{ij} f_{\nu(i,j)}$, $n_{ij} \in N$, $\nu(i,j) \in \{0, 1, \cdots, t-1\}$. Suppose that ψ is a linear character of N. Put $\beta(\tau_i, \tau_j) = \psi(n_{ij})$ ($i, j = 0, 1, \cdots, t-1$). Then β is a factor set of $\mathrm{Gal}(k(\psi)/k)$ consisting of roots of unity, and the Schur algebra $A(\psi^L, k)$ over k associated with ψ^L

is isomorphic over k to the cyclotomic algebra $B=(\beta(\tau_i, \tau_j), k(\psi)/k)$ over k:

$$B=\sum_{i=0}^{t-1} k(\psi)u_{\tau_i} \qquad \text{(direct sum)}$$

$$u_{\tau_i}u_{\tau_j}=\beta(\tau_i, \tau_j)u_{\tau_i\tau_j}, \qquad u_{\tau_i}x=x^{\tau_i}u_{\tau_i} \quad (x\in k(\psi))$$

([ibid.], Prop. 3.5).

Suppose that H/N is cyclic. Then L/N is cyclic and we can take $f_i=f^i$ ($i=0, 1, \cdots, t-1; f=f_1$). Then, putting $\tau=\tau(f)$, we have $\tau_i=\tau^i$ ($i=0, 1, \cdots, t-1$), and $f^if^j=f^{i+j}$ if $i+j\leqq t-1$ and $f^if^j=gf^{i+j-t}$ if $i+j\geqq t$ ($g=f^t\in N$). Thus $\beta(\tau_i, \tau_j)=1$ if $i+j\leqq t-1$ and $=\theta=\psi(g)$ if $i+j\geqq t$. Hence

$$B=\sum_{i=0}^{t-1} k(\psi)u^i \qquad (u=u_\tau),$$

$$u^t=\theta u_0 \quad (u_0=u^0=1), \qquad ux=x^\tau u \quad (x\in k(\psi)).$$

Therefore B is the cyclic algebra $(k(\psi)/k, \tau, \theta)$ over k.

Now we wish to investigate the rationality-properties of the characters Γ_λ, $\lambda\in\Lambda$, given in §1. If $p=2$, then U^F/U^F_* is an elementary abelian 2-group, so that any Γ_λ is realizable in $\boldsymbol{Q}$. So from now on, we shall assume that $p\neq 2$.

Let ζ_p be a fixed primitive p-th root of unity in $\boldsymbol{C}$ and put $\Pi=\mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})$. Put $\hat{\boldsymbol{F}}_q=\mathrm{Hom}(\boldsymbol{F}_q^+, \boldsymbol{C}^\times)$, where $\boldsymbol{F}_q^+$ is the additive group of $\boldsymbol{F}_q$. As $\boldsymbol{F}_q^+$ is an elementary abelian p-group, Π acts on $\hat{\boldsymbol{F}}_q$ naturally. Let us fix $\chi\in\hat{\boldsymbol{F}}_q$, $\chi\neq 1$. For $a\in\boldsymbol{F}_q$, we define a character χ_a of $\boldsymbol{F}_q^+$ by $\chi_a(x)=\chi(ax)$ for $x\in\boldsymbol{F}_q^+$. Then we have $\hat{\boldsymbol{F}}_q=\{\chi_a \mid a\in\boldsymbol{F}_q\}$ and it is easy to see that

$$\{\chi^\sigma \mid \sigma\in\Pi\}=\{\chi_a \mid a\in\boldsymbol{F}_p^\times\}.$$

We define an action of B^F on Λ by $\lambda^b(u)=\lambda(bub^{-1})$ for $b\in B^F$, $\lambda\in\Lambda$, $u\in U^F$; if $b=tu$ with $t\in T^F$ and $u\in U^F$, then $\lambda^b=\lambda^t$ for all $\lambda\in\Lambda$; B^F stabilizes Λ_0. Via isomorphism ϕ in §1, Π also acts on Λ and Λ_0. Let us fix $\lambda\in\Lambda_0$. Set:

$$L=\{b\in B^F \mid \lambda^b=\lambda^{\tau(b)} \text{ for some } \tau(b)\in\Pi\}.$$

Then we have $L=MU^F$ with $M=L\cap T^F$. We have:

$$M=\{t\in T^F \mid \text{for some } x\in\boldsymbol{F}_p^\times : \alpha(t)=x \text{ for all } \alpha\in\Delta\}.$$

This shows that the group L is independent of the choice of $\lambda\in\Lambda_0$. The mapping $b\to\tau(b)$ is a homomorphism of L into the cyclic group Π with kernel Z^FU^F (cf. $Z=\bigcap_{\alpha\in\Delta}\mathrm{Ker}\,\alpha$). Let f be an element of M such that $\langle fZ^F\rangle=M/Z^F$, i.e., $M=\langle f, Z^F\rangle$. Put $\sigma=\tau(f)$; $\langle\sigma\rangle=\tau(M)$. Put $h=(M:Z^F)$.

Let $\lambda\in\Lambda$, $\lambda\neq 1$. Let $\eta_1, \cdots, \eta_c$ be as in §1 all the irreducible characters of Z^F. For $i=1, \cdots, c$, put

$$\mu_i=\mathrm{Ind}_{Z^FU^F}^{L}(\eta_i\lambda).$$

Then we see that $\mu_1, \cdots, \mu_c$ are mutually different irreducible characters of L and

$$\lambda^L = \mu_1 + \cdots + \mu_c .$$

We note that we have:

$$\lambda^L(u) = c \cdot \sum_{j=0}^{h-1} \lambda^{\sigma^j}(u) \qquad (u \in U^F),$$

$$\mu_i(zu) = \eta_i(z) \sum_{j=0}^{h-1} \lambda^{\sigma^j}(u) \qquad (z \in Z^F, u \in U^F).$$

Let:

$$k = \mathbf{Q}(\lambda^L) = \mathbf{Q}(\zeta_p)^{\langle\sigma\rangle}, \qquad k_i = \mathbf{Q}(\mu_i) = k(\eta_i) \quad (i = 1, \cdots, c).$$

For $i = 1, \cdots, c$, let A_i be the simple direct summand of the group algebra $k_i[L]$ of L over k_i associated with μ_i; let σ_i be the automorphism of $k_i(\zeta_p)$ over k_i such that $\sigma_i | \mathbf{Q}(\zeta_p) = \sigma$ (cf. $k_i \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$); put $\theta_i = \eta_i(f^h)$ (cf. $f^h \in Z^F$). Then we see that for $i = 1, \cdots, c$, A_i is isomorphic over k_i to the cyclic algebra $B_i = (k_i(\zeta_p)/k_i, \sigma_i, \theta_i)$ over k_i.

§3. Calculation of M for SU_{l+1} and $Spin_{2l}^-$.

Let K, G, F, etc., be as in §1. We assume that G is simply-connected. Let $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ (as to the numbering of the simple roots, we follow that of [2]), and let $\alpha_1^\vee, \cdots, \alpha_l^\vee$ be the corresponding simple coroots. As G is simply-connected, the mapping $h: (x_1, \cdots, x_l) \to \prod_{i=1}^l \alpha_i^\vee(x_i)$ defines an isomorphism of $(K^\times)^l$ with T. For $i = 1, \cdots, l$, we have

$$\alpha_i(h(x_1, \cdots, x_l)) = \prod_{j=1}^{l} x_j^{\langle \alpha_i, \alpha_j^\vee \rangle},$$

where $(\langle \alpha_i, \alpha_j^\vee \rangle)_{1 \le i, j \le l}$ is the Cartan matrix of G. We have

$$F \circ \alpha_i^\vee = q(\rho\alpha_i)^\vee \qquad (i = 1, \cdots, l)$$

(see [15], 11.4.7). It follows that, for $s = h(x_1, \cdots, x_l) \in T$, we have $F(s) = s$ if and only if $x_j = x_i^q$ when $\rho\alpha_i = \alpha_j$. And we see that M consists of those elements $t = h(x_1, \cdots, x_l)$ of T^F such that there is an element $\xi \in \mathbf{F}_p^\times$ such that

$$(0)_\xi \qquad \prod_{j=1}^{l} x_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = \xi \qquad \text{for} \quad i = 1, \cdots, l.$$

Here we treat the case where G is of type $({}^2A_l)$ or of type $({}^2D_l)$. Thus $G^F = SU_{l+1}(\mathbf{F}_q)$ or $G^F = Spin_{2l}^-(\mathbf{F}_q)$.

In the following, for a positive integer n, n_2 is the 2-part of n and $n_{2'}$ is the odd part of n; if $n_2 = 2^e$, we write $\mathrm{ord}_2 n = e$. For integers m, n, $m|n$ means that n is divisible by m; (m, n) is the greatest common divisor of m, n. $2\|n$ means that $\mathrm{ord}_2 n=1$. Let ω be a primitive element of F_{q^2}; we put $\nu=\omega^{(q^2-1)/(p-1)}$ (a primitive element of F_p).

LEMMA 3. *Assume that G is of type $({}^2A_l)$ $(p\neq 2)$. We have $Z^F=\{h(x, x^2, \cdots, x^l) \mid x^{(l+1, q+1)}=1\}\simeq \boldsymbol{Z}/(l+1, q+1)\boldsymbol{Z}$. If $2|l$, then we have $\tau(M)=\Pi$ and we can write as $M=\langle f\rangle\times Z^F$ with $f^{p-1}=1$. Suppose $2\nmid l$. Then the following holds: If q is square (i.e. an even power of p), then we have $\tau(M)=\Pi$ and we can write as $M=\langle f, Z^F\rangle$ with $|\langle f^{p-1}\rangle|=2$. Suppose that q is non-square. Then, if $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(q+1)$, we have $\tau(M)=\Pi$ and we can write as $M=\langle f, Z^F\rangle$ with $|\langle f^{p-1}\rangle|=c_2$; if $\mathrm{ord}_2(l+1)\leqq\mathrm{ord}_2(q+1)$ and $p\equiv 1 \pmod 4$, we have $(\Pi:\tau(M))=2$ and we can write as $M=\langle f, Z^F\rangle$ with $|\langle f^{(p-1)/2}\rangle|=2$; if $\mathrm{ord}_2(l+1)\leqq\mathrm{ord}_2(q+1)$ and $p\equiv -1 \pmod 4$, we have $(\Pi:\tau(M))=2$ and we can write as $M=\langle f\rangle\times Z^F$ with $f^{(p-1)/2}=1$.*

PROOF. Let $s\in T$, $s=h(x_1, \cdots, x_l)$. If $l=2m$, then $s\in T^F$ if and only if s is of the form $h(x_1, \cdots, x_m, x_m^q, \cdots, x_1^q)$ with $x_1, \cdots, x_m\in F_{q^2}^\times$. If $l=2m+1$, then $s\in T^F$ if and only if s is of the form $h(x_1, \cdots, x_m, x_{m+1}, x_m^q, \cdots, x_1^q)$ with $x_1, \cdots, x_m\in F_{q^2}^\times$ and $x_{m+1}\in F_q^\times$. Then Cartan matrix of type (A_l) can be seen in [2], Appendix. In this case, the system of equations $(0)_\xi$ reads as follows:

$$(1)_\xi \qquad \begin{cases} x_1^2x_2^{-1}=\xi\,, \quad x_1^{-1}x_2^2x_3^{-1}=\xi\,, \quad x_2^{-1}x_3^2x_4^{-1}=\xi\,, \\ \cdots\cdots\cdots\cdots \\ x_{l-2}^{-1}x_{l-1}^2x_l^{-1}=\xi\,, \qquad x_{l-1}^{-1}x_l^2=\xi\,. \end{cases}$$

We must further consider the following condition:

$$x_{m+i}=x_{m-i+1}^q \qquad (i=1, \cdots, m)$$

if $l=2m$;

$$(2)_i \qquad x_{m+i+1}=x_{m+1-i}^q \qquad (i=1, \cdots, m)$$

and

$$(3) \qquad x_{m+1}^q=x_{m+1}$$

if $l=2m+1$.

We see that $(1)_\xi \Leftrightarrow (4)_{\xi,i}$ $(i=1, \cdots, l)$, $(5)_\xi$, where

$$(4)_{\xi,i} \qquad x_i=\xi^{-i(i-1)/2}x_1^i \qquad (i=1, \cdots, l)$$

and

$$(5)_\xi \qquad x_1^{l+1}=\xi^{l(l+1)/2}\,.$$

We remark also that, when $l=2m+1$, for x_i given by $(4)_{\xi,i}$ and $(5)_\xi$, the conditions $(2)_i$ and (3) are satisfied if and only if $(6)_\xi$ and $(6)'$ are satisfied, where

$$(6)_\xi \qquad x_1^{q+1}=\xi^l\,,$$

and

$$x_1^{mq} = x_1^{m+1-q}. \tag{6)'$$

First, let us calculate Z^F. Assume that $l=2m+1$. Since an element s of T^F lies in Z^F if and only if $\alpha_i(s)=1$ for $i=1, \cdots, l$, it suffices to consider the conditions $(1)_1$, $(2)_i$ $(i=1, \cdots, m)$, (3). Let $s=h(x_1, \cdots, x_l)$. By the above remarks ($q=$odd) $s \in Z^F$ if and only if $s=h(x_1, x_1^2, \cdots, x_1^l)$ with $x_1^{l+1}=x_1^{q+1}=1$. Hence the assertion in the lemma holds. For the case $l=2m$, the proof is similar.

Next we calculate the group M. First, assume that $2|l$. Put $f=h(x_1, \cdots, x_l)$ where $x_i=\nu^{i(l-i+1)/2}$ $(i=1, \cdots, l)$. Then we see that $f \in T^F$ and $\alpha_i(f)=\nu$ for $i=1, \cdots, l$. Hence we have $\tau(M)=\Pi$ and $M=\langle f \rangle \times Z^F$ with $f^{p-1}=1$. (This case is already treated in [13].) So, in the rest of this proof, we assume that $l=2m+1$ $(m \geqq 0)$.

We show that the condition $(6)_\nu$ implies that $x_1 \notin F_p^\times$. (This means that if $\tau(M)=\Pi$ and if we write as $M=\langle f, Z^F \rangle$, then we cannot have $f^{p-1}=1$, that is, we cannot write as $M=\langle f \rangle \times Z^F$.) In fact, assuming $(6)_\nu$, suppose $x_1 \in F_p^\times$. Then $x_1=\nu^k$ for some integer k. Since l is odd, it is impossible.

Assume that q is square. Let ε be an element of F_q such that $\varepsilon^2=\nu$. Put $f=h(x_1, \cdots, x_l)$ where $x_i=\varepsilon^{i(l-i+1)}$ $(i=1, \cdots, l)$. Then we see that $x_1, \cdots, x_l$ satisfy $(1)_\nu$, $(2)_i$ $(i=1, \cdots, m)$ and (3). Hence we have $\tau(M)=\Pi$ and $M=\langle f, Z^F \rangle$. We have $f^{p-1}=h(-1, 1, -1, 1, \cdots, -1)$. Hence $|\langle f^{p-1} \rangle|=2$.

In the rest of this proof, we assume that q is non-square. Suppose that $x_1, \cdots, x_l$ satisfy $(1)_\nu$, $(2)_i$ $(i=1, \cdots, m)$ and (3). Let $x_1=\omega^k$. Then, by $(6)_\nu$, we have

$$k=\frac{q-1}{p-1}l+t(q-1) \tag{7}$$

for some integer t. Then, by $(5)_\nu$, we have

$$t(l+1)=\frac{l(l+1)}{2}\cdot\frac{q-1}{p-1}+v(q+1)$$

for some integer v. Put $((l+1)/2, q+1)=e$ and write: $q+1=ea$, $(l+1)/2=eb$, $(a, b)=1$. Then:

$$2tb=lb\cdot\frac{q-1}{p-1}+va.$$

Then $b|va$, hence $b|v$ as $(a, b)=1$. So, putting $v=bv'$, we get:

$$2t=l\cdot\frac{q-1}{p-1}+v'a.$$

As q is non-square, $\mathrm{ord}_2(q-1)=\mathrm{ord}_2(p-1)$, so $l\cdot(q-1)/(p-1)$ is odd. Thus v' and a must be odd. Hence $\mathrm{ord}_2(l+1)/2 \geqq \mathrm{ord}_2(q+1)$. This shows that if $\mathrm{ord}_2(l+1) \leqq \mathrm{ord}_2(q+1)$, then no $f \in T^F$ satisfies that $\alpha_i(f)=\nu$ for $i=1, \cdots, l$ and we cannot have $\tau(M)=\Pi$.

Suppose $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(q+1)$. Then $t=\{l(q-1)/(p-1)+v'a\}/2$ (v' odd), hence, by (7), we have:

$$k=\frac{q-1}{p-1}l+\frac{1}{2}\left(l\cdot\frac{q-1}{p-1}+v'a\right)(q-1) \qquad (v' \text{ odd}). \tag{8}$$

Conversely, if we put $x_1=\omega^k$ where k is of the form (8) and $x_i=v^{-i(i-1)/2}x_1^i$ $(i=1,\cdots,l)$, then we see that $x_1,\cdots,x_l$ satisfy $(1)_v$, $(2)_i$ $(i=1,\cdots,m)$ and (3). Therefore we have $\tau(M)=\Pi$. Put $f=h(x_1,\cdots,x_l)$ with the x_i being as above where we take $v'=((l+1)/2,q+1)_{2'}$. Then we have $M=\langle f,Z^F\rangle$ and $f^{p-1}=h(x_1^{p-1},x_1^{2(p-1)},\cdots,x_1^{l(p-1)})$. Let us compute the order of f^{p-1}. Clearly it suffices to compute the order of x_1^{p-1}. We have $x_1^{p-1}=-\omega^{v'a(p-1)(q-1)/2}$. As $\mathrm{ord}_2((l+1)/2)\geqq\mathrm{ord}_2(q+1)$, we have $v'a=((l+1)/2,q+1)_{2'}\cdot(q+1)/((l+1)/2,q+1)=(q+1)_{2'}$. Suppose $p\equiv1 \pmod 4$. Then $q\equiv1 \pmod 4$ and $\mathrm{ord}_2(q+1)=1$. As $2|(p-1)/2$, we have $q^2-1|v'a(p-1)(q-1)/2$. Hence $x_1^{p-1}=-1$. Hence $|\langle f^{p-1}\rangle|=2=c_2$. Suppose $p\equiv-1 \pmod 4$. Then $(p-1)/2$ is odd. Since $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(q+1)$ and $c=(l+1,q+1)$, we have $c_2=(q+1)_2$. Then we see that c_2 is the order of $x_1^{p-1}=-\omega^{(q+1)_{2'}(q-1)(p-1)/2}$.

Assume that $\mathrm{ord}_2(l+1)\leqq\mathrm{ord}_2(q+1)$. We have seen above that $\tau(M)\neq\Pi$. But, by Lemma 2 of [13], we have $(\Pi:\tau(M))=2$. So let us consider the equations $(1)_{v^2}$, $(2)_i$ $(i=1,\cdots,m)$ and the condition (3). Then we have $(6)_{v^2}$: $x_1^{q+1}=v^{2l}$. Put $x_1=\omega^k$. Then $k\equiv2l\cdot(q-1)/(p-1) \pmod{q-1}$, i.e.,

$$k=\frac{q-1}{p-1}2l+t(q-1) \tag{9}$$

for some integer t. By $(5)_{v^2}$, we have

$$t(l+1)=\frac{q-1}{p-1}l(l+1)+s(q+1)$$

for some integer s. By the same argument as before $t=l\cdot(q-1)/(p-1)+s'a'$, where $(l+1,q+1)=e'$, $q+1=e'a'$, $l+1=e'b'$ and $s=b's'$. Hence, by (9), we have $k=l\cdot(q^2-1)/(p-1)+s'a'(q-1)$. Hence $x_1=v^l\omega^{s'a'(q-1)}$ and $x_i=v^{-i(i-1)}x_1^i$ $(i=1,\cdots,1)$ (cf. $(4)_{v^2,i}$). Put $f=h(x_1,\cdots,x_l)$. Then $\tau(f)$ has order $(p-1)/2$ in Π and $f^{(p-1)/2}=h(x_1^{(p-1)/2},(x_1^2)^{(p-1)/2},\cdots,(x_1^l)^{(p-1)/2})$. We have

$$x_1^{(p-1)/2}=-\omega^{s'a'(q-1)(p-1)/2}. \tag{$*$}$$

Now let us consider the problem: Is there an integer s' such that $\omega^{s'a'(q-1)(p-1)/2}=-1$, i.e.,

$$s'a'(q-1)(p-1)/2\equiv(q^2-1)/2 \pmod{q^2-1}?$$

This congruence relation can be rewritten as:

$$s'a'\cdot\frac{p-1}{2}\equiv\frac{q+1}{2} \pmod{q+1}. \tag{10}$$

Suppose $p\equiv 1 \pmod 4$. Then $2|(p-1)/2$ and $2\nmid(q+1)/2$. Hence the answer is negative. Take: $s'=0$. Then $x_1^{(p-1)/2}=-1$ and $|\langle f^{(p-1)/2}\rangle|=2$ (we have $M=\langle f, Z^F\rangle$).

Suppose $p\equiv -1 \pmod 4$. Then $2\nmid(p-1)/2$ and $((p-1)/2, q+1)=1$. Hence there is an integer v such that $v\cdot(p-1)/2\equiv 1 \pmod{q+1}$. By multiplying both hand sides of (10) by v, we get

$$s'a'\equiv\frac{q+1}{2}v \qquad (\mathrm{mod}\ q+1)\,. \tag{11}$$

Here $a'=(q+1)/(l+1, q+1)$. So $\mathrm{ord}_2 a'<\mathrm{ord}_2(q+1)$, hence $a'|(q+1)/2$. Hence, from (11), we get

$$s'\equiv\frac{q+1}{2a'}v \qquad \left(\mathrm{mod}\ \frac{q+1}{a'}\right).$$

Therefore the problem has a positive answer. That is, we have $\omega^{s'a'(q-1)(p-1)/2}=-1$ for some integer s'. By taking such s' in x_1, we have $x_1^{(p-1)/2}=1$ (cf. $(*)$). Thus we can write as $M=\langle f\rangle\times Z^F$ with $f^{(p-1)/2}=1$. This completes the proof of Lemma 3.

LEMMA 4. *Assume that G is of type $({}^2D_l)$ $(l\geqq 3)$. We have $Z^F\simeq \boldsymbol{Z}/4\boldsymbol{Z}$ if $2\nmid l$ and $4|q+1$ and $Z^F=\langle h(1,\cdots,1,-1,-1)\rangle\simeq\boldsymbol{Z}/2\boldsymbol{Z}$ otherwise. If $4|l(l-1)$, then we have $\tau(M)=\Pi$ and we can write as $M=\langle f\rangle\times Z^F$ with $f^{p-1}=1$. If* (a) $2\|l$ *or* (b) $2\|l-1$ *and either* (b_1) *q is square or* (b_2) *q is non-square and $p\equiv 1 \pmod 4$, then we have $\tau(M)=\Pi$ and we can write as $M=\langle f, Z^F\rangle$ with $|\langle f^{p-1}\rangle|=2$. If $2\|l-1$, q is non-square and $p\equiv -1$* (mod 4), *then we have $(\Pi:\tau(M))=2$ and we can write as $M=\langle f\rangle\times Z^F$ with $f^{(p-1)/2}=1$.*

PROOF. Let $s\in T$, $s=h(x_1,\cdots,x_l)$. Then $s\in T^F$ if and only if $x_1,\cdots,x_{l-2}\in \boldsymbol{F}_q^\times$, $x_{l-1}, x_l\in \boldsymbol{F}_{q^2}^\times$ and $x_l=x_{l-1}^q$ (hence $x_{l-1}=x_l^q$). The Cartan matrix of type (D_l) can be seen in [2], Appendix. In this case the system of equations $(0)_\xi$ can be read as follows:

$$(12)_\xi \qquad \begin{cases} x_1^2x_2^{-1}=\xi\,, \quad x_1^{-1}x_2^2x_3^{-1}=\xi\,,\\ \cdots\cdots\cdots \\ x_{l-3}^{-1}x_{l-2}^2x_{l-1}^{-1}x_l^{-1}=\xi\,,\quad x_{l-2}^{-1}x_{l-1}^2=\xi\,,\quad x_{l-2}^{-1}x_l^2=\xi\,. \end{cases}$$

The system of equations $(12)_\xi$ is equivalent to $(13)_{\xi,i}$, $(14)_\xi$, $(15)_\xi$ and $(16)_\xi$, where

$$(13)_{\xi,i} \qquad x_i=\xi^{-i(i-1)/2}x_1^i \qquad (i=1,\cdots,l-2)\,,$$

$$(14)_\xi \qquad x_{l-1}x_l=\xi^{-(l-1)(l-2)/2}x_1^{l-1}\,,$$

$$(15)_\xi \qquad x_{l-1}^2=\xi^{-(l-1)(l-4)/2}x_1^{l-2}\,,$$

$$(16)_\xi \qquad x_1^2=\xi^{2l-2}\,.$$

Let us give several remarks. From $(14)_\xi$, $(15)_\xi$ and $(16)_\xi$, we have $x_l^2=x_{l-1}^2$. Moreover we can rewrite the above system as follows:

$$(17)_\xi \qquad \begin{cases} x_i=\rho^i\xi^{i(l-1)} & (i=1, \cdots, l-2; \rho=\pm 1)\,, \\ x_{l-1}^2=\rho^{l-2}\xi^{l(l-1)/2}\,, \\ x_{l-1}x_l=\rho^{l-1}\xi^{l(l-1)/2}\,. \end{cases}$$

To construct an element of T^F, we must further consider the following condition:

$$(18) \qquad x_l=x_{l-1}^q\,.$$

Using $(17)_\xi$ and (18), we can calculate the centre Z^F: we have $Z^F=\langle h(-1, \cdots, -1, \omega^{(q^2-1)/4}, \omega^{q(q^2-1)/4})\rangle \simeq \boldsymbol{Z}/4\boldsymbol{Z}$ if $2|l-1$ and $4|q+1$, and $Z^F=\langle h(1, \cdots, 1, -1, -1)\rangle \simeq \boldsymbol{Z}/2\boldsymbol{Z}$ otherwise.

Let us calculate the group M. Assume that $4|l(l-1)$. Put $f=h(x_1, \cdots, x_{l-2}, \nu^{l(l-1)/4}, \nu^{l(l-1)/4})$ where $x_i=\nu^{i(2l-i-1)/2}$ $(i=1, \cdots, l-2)$. Then we see that $f\in T^F$ and $\alpha_i(f)=\nu$ for $i=1, \cdots, l$. Hence we have $\tau(M)=\Pi$ and $M=\langle f\rangle \times Z^F$ with $f^{p-1}=1$. (This case is treated in Lemma 4 of [13].)

Assume that $4\nmid l(l-1)$. Let $s=h(x_1, \cdots, x_l)$ be an element of T^F. Suppose that $x_1, \cdots, x_l$ satisfy $(12)_\nu$. We show that $x_{l-1}\notin \boldsymbol{F}_p^\times$. In fact, suppose that $x_{l-1}\in \boldsymbol{F}_p^\times$. Then $x_l=x_{l-1}^q=x_{l-1}\in \boldsymbol{F}_p^\times$. Then both of $x_{l-1}^2=\rho^{l-2}\nu^{l(l-1)/2}$, $x_{l-1}x_l=\rho^{l-1}\nu^{l(l-1)/2}$ must belong to $(\boldsymbol{F}_p^\times)^2$. But it is impossible since $l(l-1)/2$ is odd. This means that if $\tau(M)=\Pi$, then we cannot write as $M=\langle f\rangle \times Z^F$ with $f^{p-1}=1$.

Assume that q is square. Put $f=h(x_1, \cdots, x_{l-2}, \varepsilon^{l(l-1)/2}, \varepsilon^{l(l-1)/2})$ where $x_i=\varepsilon^{i(2l-i-1)}$ for $i=1, \cdots, l-2$. Then we see that $f\in T^F$ and $\alpha_i(f)=\nu$ for $i=1, \cdots, l$. Hence we have $\tau(M)=\Pi$ and $M=\langle f, Z^F\rangle$. As $f^{p-1}=h(1, \cdots, 1, -1, -1)$, we have $|\langle f^{p-1}\rangle|=2$. So, in the rest of this proof, we assume that q is non-square $(2\| l(l-1))$.

Suppose $2\| l$. Let ε be an element of $\boldsymbol{F}_{q^2}$ such that $\varepsilon^2=\nu$. Put $f=h(x_1, \cdots, x_{l-2}, \varepsilon^{l(l-1)/2}, \varepsilon^{(l(l-1)/2)q})$ where $x_1=-\nu^{l-1}$ and $x_i=\nu^{-i(i-1)/2}x_1^i$ for $i=1, \cdots, l-2$. Then we see that $f\in T^F$ and $\alpha_i(f)=\nu$ for $i=1, \cdots, l$. Hence we have $\tau(M)=\Pi$ and $M=\langle f, Z^F\rangle$. As $f^{p-1}=h(1, \cdots, 1, -1, -1)$, we have $|\langle f^{p-1}\rangle|=2$.

Suppose $2\| l-1$. Let $s=h(x_1, \cdots, x_l)$, $s\in T^F$. Suppose that $x_1, \cdots, x_l$ satisfy $(12)_\nu$. Then, by $(16)_\nu$, we have $x_1=\pm\nu^{l-1}$. We first show that $x_1=\nu^{l-1}$ is impossible.

In fact, suppose $x_1=\nu^{l-1}$. Let ε be as above an element of $\boldsymbol{F}_{q^2}$ such that $\varepsilon^2=\nu$. As q is non-square, $\varepsilon\notin \boldsymbol{F}_q$ (this can be easily checked), so $\varepsilon^q\neq\varepsilon$, hence $\varepsilon^q=-\varepsilon$. By $(17)_\nu$ (where $\rho=1$), $x_{l-1}^2=\nu^{l(l-1)/2}=\varepsilon^{l(l-1)}$ and $x_{l-1}x_l=\nu^{l(l-1)/2}=x_{l-1}^2$. Since $x_{l-1}=\eta\varepsilon^{l(l-1)/2}$ $(\eta=\pm 1)$, $x_l=x_{l-1}^q=\eta^q(\varepsilon^{l(l-1)/2})^q=(-1)^{l(l-1)/2}x_{l-1}=-x_{l-1}$, which is a contradiction.

Thus $x_1=-\nu^{l-1}$. Let μ be an element of $\boldsymbol{F}_{q^2}$ such that $\mu^2=-1$. By $(17)_\nu$, $x_{l-1}=\eta\mu\varepsilon^{l(l-1)/2}$ $(\eta=\pm 1)$. Then $x_l=x_{l-1}^q=\eta\mu^q(-1)\varepsilon^{l(l-1)/2}$, and $x_{l-1}x_l=\mu^{1+q}(-1)\nu^{l(l-1)/2}$, which is equal to $\nu^{l(l-1)/2}$ by $(17)_\nu$. Hence, for the existence of solutions, it is necessary and sufficient that $\mu^{1+q}=-1$, i.e. $(1+q)/2$ is odd, i.e. $p\equiv 1 \pmod 4$.

Suppose $p\equiv 1 \pmod 4$. Put $f=h(x_1, \cdots, x_l)$ where $x_1=-\nu^{l-1}$, $x_i=\nu^{-i(i-1)/2}x_1^i$ $(i=2, \cdots, l-2)$, $x_{l-1}=\omega^{(q^2-1)/4+((q^2-1)/2(p-1))(l^2-l)/2}$ and $x_l=x_{l-1}^q$. Then we see that $f\in T^F$ and $\alpha_i(f)=\nu$ for $i=1, \cdots, l$. Hence we have $\tau(M)=\Pi$ and $M=\langle f, Z^F\rangle$. We have

$f^{p-1}=h(1,\cdots,1,-1,-1)$, hence $|\langle f^{p-1}\rangle|=2$.

Finally, suppose $p\equiv -1 \pmod 4$. Then we cannot have $\tau(M)=\Pi$. Put $f=h(x_1,\cdots,x_l)$ where $x_1=v^{2l-2}$, $x_i=v^{-i(i-1)}x_1^i$ $(i=1,\cdots,l-2)$ and $x_{l-1}=x_l=v^{(l^2-l)/2+(p-1)/2}$. Then we see that $f\in T^F$ and $\alpha_i(f)=v^2$ for $i=1,\cdots,l$ and $f^{(p-1)/2}=1$. Hence we have $(\Pi:\tau(M))=2$ and $M=\langle f\rangle\times Z^F$. This completes the proof of Lemma 4.

§4. Schur indices of the μ_i.

In the following, if χ is an irreducible character of a finite group and E is a field of characteristic zero, then $m_E(\chi)$ is the Schur index of χ with respect to E.

In this section we determine local Schur indices of the irreducible characters $\mu_1,\cdots,\mu_c$ of L (see §2) for $G=SU_{l+1}$, $Spin_{2l}^-$.

First we assume that $G=SU_{l+1}$. Let $\lambda\in\Lambda$, $\lambda\neq 1$. By Lemma 3, we have $Z^F=\langle z\rangle$, $z=h(\omega^k,\omega^{2k},\cdots,\omega^{lk})$, $k=(q^2-1)/(l+1,q+1)$. Let ζ_c be a fixed primitive c-th root of unity. We arrange the characters $\eta_1,\cdots,\eta_c$ so that $\eta_i(z)=\zeta_c^i$ $(i=1,\cdots,c)$. Let f be an element of M described in Lemma 3. For $i=1,\cdots,c$, let A_i be the Schur algebra associated with μ_i (see §2). Then

$$A_i=(k_i(\zeta_p)/k_i,\sigma_i,\theta_i),\quad \theta_i=\eta_i(f^h)\qquad (i=1,\cdots,c),$$

where $h=p-1$ or $(p-1)/2$.

Suppose $2|l$. Then, by Lemma 3, we have $\tau(M)=\Pi$ and $M=\langle f\rangle\times Z^F$ with $f^{p-1}=1$ (hence $h=p-1$). Hence $k=\mathbf{Q}$, $k_i=\mathbf{Q}(\eta_i)=\mathbf{Q}(\zeta_c^i)$ and $\theta_i=1$ $(i=1,\cdots,c)$. Hence, for $i=1,\cdots,c$, A_i splits over k_i and $m_{\mathbf{Q}}(\mu_i)=1$. Hence, by a theorem of Schur (see, e.g., [3], p. 479, Exercise 2), $\lambda^L=\mu_1+\cdots+\mu_c$ is realizable in $\mathbf{Q}$. Hence $\Gamma_\lambda=(\lambda^L)^{G^F}$ is realizable in $\mathbf{Q}$.

Suppose that $2\nmid l$ and q is square. Then, by Lemma 3, we have $\tau(M)=\Pi$ and $M=\langle f,Z^F\rangle$ with $f^{p-1}=h(-1,1,-1,1,\cdots,-1)=z^{c/2}$ $(h=p-1)$. Hence we have $k=\mathbf{Q}$ and $k_i=\mathbf{Q}(\eta_i)=\mathbf{Q}(\zeta_c^i)$ $(i=1,\cdots,c)$. And:

$$\theta_i=\eta_i(f^{p-1})=\eta_i(z^{c/2})=\eta_i(z)^{c/2}=\zeta_c^{i\cdot c/2}=(-1)^i$$

$(i=1,\cdots,c)$. Thus:

$$A_i=(\mathbf{Q}(\zeta_c^i)(\zeta_p)/\mathbf{Q}(\zeta_c^i),\sigma_i,(-1)^i)\qquad (i=1,\cdots,c).$$

If i is even, then A_i splits and we have $m_{\mathbf{Q}}(\mu_i)=1$. Suppose that i is odd. If v is a finite place of k_i such that $v\nmid p$, then v is unramified in $k_i(\zeta_p)/k_i$, so that the Hasse invariant of A_i at v is $\equiv 0 \pmod 1$. Hence we have $m_{\mathbf{Q}_r}(\mu_i)=1$ for any prime number $r\neq p$. Let $i=c/2$ (cf. $2\|c$). Then $A_i=A_{c/2}=(\mathbf{Q}(\zeta_p)/\mathbf{Q},\sigma,-1)$. $A_{c/2}$ has the invariants $1/2 \bmod 1$ at ∞, p (see [10]). Hence we have $m_{\mathbf{R}}(\mu_{c/2})=m_{\mathbf{Q}_p}(\mu_{c/2})=2$. [For other odd i, we have $A_i=A_{c/2}\otimes_{\mathbf{Q}}/\mathbf{Q}(\zeta_c^i)$, and, for any place v of k_i over p, the invariant of A_i at v is $\equiv\frac{1}{2}[\mathbf{Q}_p(\zeta_c^i):\mathbf{Q}_p] \pmod 1$, and $[\mathbf{Q}_p(\zeta_c^i):\mathbf{Q}_p]$ is equal to the least positive integer s such that $p^s \equiv 1 \pmod{c/(i, c)}$.] Thus Γ_λ is realizable in $\boldsymbol{Q}_r$ for any prime number $r \neq p$ and, for some i, we have $m_{\boldsymbol{Q}}(\mu_i) = m_{\boldsymbol{R}}(\mu_i) = m_{\boldsymbol{Q}_p}(\mu_i) = 2$.

Suppose that $2 \nmid l$, q is non-square and $\mathrm{ord}_2(l+1) > \mathrm{ord}_2(q+1)$. Then, by Lemma 3, we have $\tau(M) = \Pi$ and $M = \langle f, Z^F \rangle$ with $|\langle f^{p-1} \rangle| = c_2$. So $k = \boldsymbol{Q}$ and $k_i = \boldsymbol{Q}(\eta_i) = \boldsymbol{Q}(\zeta_c^i)$ $(i = 1, \cdots, c)$. And we have $f^{p-1} = z^{c_2' u}$ for some odd integer u. Hence, for $i = 1, \cdots, c$, we have

$$\theta_i = \eta_i(f^{p-1}) = \eta_i(z^{c_2' u}) = \zeta_c^{ic_2' u} = \zeta_{c_2}^i$$

for some primitive c_2th root of unity ζ_{c_2}. Hence

$$A_i = (\boldsymbol{Q}(\zeta_c^i)(\zeta_p)/\boldsymbol{Q}(\zeta_c^i), \sigma_i, \zeta_{c_2}^i) \qquad (i = 1, \cdots, c).$$

Let us fix i. If v is a finite place of k_i such that $v \nmid p$, then v is unramified in $k_i(\zeta_p)/k_i$, so that the invariant of A_i at v is $\equiv 0 \pmod 1$. Hence we have $m_{\boldsymbol{Q}_r}(\mu_i) = 1$ for any prime number $r \neq p$. Let $i = c/2$. Then $A_i = A_{c/2} = (\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q}, \sigma, -1)$. Hence we have $m_{\boldsymbol{Q}}(\mu_{c/2}) = m_{\boldsymbol{R}}(\mu_{c/2}) = m_{\boldsymbol{Q}_p}(\mu_{c/2}) = 2$. [Let i be another. Let v be a place of k_i above p. Let s be the least positive integer such that $p^s \equiv 1 \pmod{c/(i, c)}$. Then we see that A_i splits over $(k_i)_v$ if and only if $\mathrm{ord}_2(p^s - 1) - \mathrm{ord}_2(p-1) - \mathrm{ord}_2 c_2/(i, c_2) \geqq 0$.] Thus Γ_λ is realizable in $\boldsymbol{Q}_r$ for any prime number $r \neq p$ and, for some i, we have $m_{\boldsymbol{Q}}(\mu_i) = m_{\boldsymbol{R}}(\mu_i) = m_{\boldsymbol{Q}_p}(\mu_i) = 2$.

Suppose that $2 \nmid l$, q is non-square, $\mathrm{ord}_2(l+1) \leqq \mathrm{ord}_2(q+1)$ and $p \equiv 1 \pmod 4$. Then, by Lemma 3, we have $(\Pi : \tau(M)) = 2$ and $M = \langle f, Z^F \rangle$ with $f^{(p-1)/2} = h(-1, 1, -1, 1, \cdots, -1) = z^{c/2}$. Hence $h = (p-1)/2$, and we have $k = \boldsymbol{Q}(\sqrt{p})$ and $k_i = \boldsymbol{Q}(\sqrt{p})(\zeta_c^i)$ $(i = 1, \cdots, c)$. And we have $\theta_i = (-1)^i$ $(i = 1, \cdots, c)$. Thus:

$$\begin{aligned} A_i &= (\boldsymbol{Q}(\sqrt{p}, \zeta_c^i)(\zeta_p)/\boldsymbol{Q}(\sqrt{p}, \zeta_c^i), \sigma_i, (-1)^i) \\ &= (\boldsymbol{Q}(\sqrt{p})(\zeta_p)/\boldsymbol{Q}(\sqrt{p}), \sigma', (-1)^i) \otimes_{\boldsymbol{Q}(\sqrt{p})} \boldsymbol{Q}(\sqrt{p}, \zeta_c^i) \qquad (i = 1, \cdots, c), \end{aligned}$$

where σ' is a certain automorphism of $\boldsymbol{Q}(\sqrt{p}, \zeta_p)$ over $\boldsymbol{Q}(\sqrt{p})$. If i is even, then A_i splits over k_i and we have $m_{\boldsymbol{Q}(\sqrt{p})}(\mu_i) = 1$. Suppose that i is odd. Put $B = (\boldsymbol{Q}(\sqrt{p})(\zeta_p)/\boldsymbol{Q}(\sqrt{p}), \sigma', -1)$. Then, by [10], B has non-zero invariants ($\equiv 1/2 \bmod 1$) only at two real places of $\boldsymbol{Q}(\sqrt{p})$. Thus we have $m_{\boldsymbol{Q}_r}(\mu_i) = 1$ for any prime number r. For $i = c/2$ (which is odd since $2 \| q+1$), we have $A_i = B$. So we have $m_{\boldsymbol{Q}(\sqrt{p})}(\mu_{c/2}) = m_{\boldsymbol{R}}(\mu_{c/2}) = 2$. Thus Γ_λ is realizable in $\boldsymbol{Q}(\sqrt{p})_v$ for any finite place v of $\boldsymbol{Q}(\sqrt{p})$.

Finally, suppose that $2 \nmid l$, q is non-square, $\mathrm{ord}_2(l+1) \leqq \mathrm{ord}_2(q+1)$ and $p \equiv -1 \pmod 4$. Then, by Lemma 3, we have $(\Pi : \tau(M)) = 2$ and $M = \langle f \rangle \times Z^F$ with $f^{(p-1)/2} = 1$. Hence $k = \boldsymbol{Q}(\sqrt{-p})$ and $k_i = \boldsymbol{Q}(\sqrt{-p})(\zeta_c^i)$ $(i = 1, \cdots, c)$. As $f^{(p-1)/2} = 1$, we have $\theta_i = 1$ $(i = 1, \cdots, c)$. Thus each A_i splits over k_i and we have $m_{\boldsymbol{Q}(\sqrt{-p})}(\mu_i) = 1$ $(i = 1, \cdots, c)$. Hence Γ_λ is realizable in $\boldsymbol{Q}(\sqrt{-p})$.

Thus we have

PROPOSITION 2. *Let $G = SU_{l+1}$. Let $\lambda \in \Lambda$, $\lambda \neq 1$. Then: If $2 \mid l$, Γ_λ is realizable in $\boldsymbol{Q}$. Suppose $2 \nmid l$. If q is square, or, q is non-square and* $\mathrm{ord}_2(l+1) > \mathrm{ord}_2(q+1)$, *then Γ_λ is* *realizable in* $\boldsymbol{Q}_r$ *for any prime number* $r \neq p$ *and, for some* i, *we have* $m_{\boldsymbol{Q}}(\mu_i)=m_{\boldsymbol{R}}(\mu_i)=m_{\boldsymbol{Q}_p}(\mu_i)=2$. *If* q *is non-square*, $\mathrm{ord}_2(l+1)\leqq \mathrm{ord}_2(q+1)$ *and* $p\equiv 1 \pmod 4$, *then* Γ_λ *is realizable in* $\boldsymbol{Q}(\sqrt{p})_v$ *for any finite place* v *of* $\boldsymbol{Q}(\sqrt{p})$ *and, for some* i, *we have* $m_{\boldsymbol{Q}(\sqrt{p})}(\mu_i)=m_{\boldsymbol{R}}(\mu_i)=2$. *If* q *is non-square*, $\mathrm{ord}_2(l+1)\leqq \mathrm{ord}_2(q+1)$ *and* $p\equiv -1 \pmod 4$, *then* Γ_λ *is realizable in* $\boldsymbol{Q}(\sqrt{-p})$.

Next we assume that $G=Spin_{2l}^{-}$ ($l\geqq 3$). Let $\lambda\in\Lambda$, $\lambda\neq 1$. Suppose $4|l(l-1)$. Then, by Lemma 4, we have $\tau(M)=\Pi$ and $M=\langle f\rangle\times Z^F$ with $f^{p-1}=1$. Hence, by the argument in the case that $G=SU_{l+1}$, we see that Γ_λ is realizable in $\boldsymbol{Q}$.

Suppose that $2\|l$ or that $2\|l-1$ and q is square or that $2\|l-1$, q is non-square and $p\equiv 1 \pmod 4$. Put $z=h(1,\cdots,1,-1,-1)$. Then, by Lemma 4, $Z^F=\langle z\rangle\simeq \boldsymbol{Z}/2\boldsymbol{Z}$ and we have $\tau(M)=\Pi$ and $M=\langle f, Z^F\rangle$ with $f^{p-1}=z$. We arrange η_1, η_2 so that $\eta_i(z)=(-1)^i$ ($i=1,2$). Then $k=k_i=\boldsymbol{Q}$ ($i=1,2$) and $\theta_i=(-1)^i$ ($i=1,2$). Thus:

$$A_i=(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q}, \sigma, (-1)^i) \qquad (i=1,2)\,.$$

Hence Γ_λ is realizable in $\boldsymbol{Q}_r$ for any prime number $r\neq p$ and we have $m_{\boldsymbol{Q}}(\mu_1)=m_{\boldsymbol{R}}(\mu_1)=m_{\boldsymbol{Q}_p}(\mu_1)=2$.

Suppose that $2\|l-1$, q is non-square and $p\equiv -1 \pmod 4$. Then, by Lemma 4, we have $(\Pi:\tau(M))=2$ and $M=\langle f\rangle\times Z^F$ with $f^{(p-1)/2}=1$. Hence $k=\boldsymbol{Q}(\sqrt{-p})$ and $\theta_i=1$ ($i=1,2,3,4$). Hence, by the argument in the case that $G=SU_{l+1}$, we see that Γ_λ is realizable in $\boldsymbol{Q}(\sqrt{-p})$.

Thus we get:

PROPOSITION 3. *Assume that* $G=Spin_{2l}^{-}$ ($l\geqq 3$). *Let* $\lambda\in\Lambda$, $\lambda\neq 1$. *Then*: *If* $4|l(l-1)$, *then* Γ_λ *is realizable in* $\boldsymbol{Q}$. *Suppose that* $2\|l(l-1)$. *If* $2\|l$, *or*, $2\|l-1$ *and* q *is square, or*, $2\|l-1$ *and* q *is non-square and* $p\equiv 1 \pmod 4$, *then* Γ_λ *is realizable in* $\boldsymbol{Q}_r$ *for any prime number* $r\neq p$ *and, for some* i, *we have* $m_{\boldsymbol{Q}}(\mu_i)=m_{\boldsymbol{R}}(\mu_i)=m_{\boldsymbol{Q}_p}(\mu_i)=2$. *If* $2\|l-1$, q *is non-square and* $p\equiv -1 \pmod 4$, *then* Γ_λ *is realizable in* $\boldsymbol{Q}(\sqrt{-p})$.

§5. Main results.

LEMMA 5. *Let* G *be as in* §1. *Let* E *be a field of characteristic* 0. *Assume that, for any* $\lambda\in\Lambda$, Γ_λ *is realizable in* E. *Then, if* χ *is an irreducible character of* G^F *such that* $\langle\chi,\Gamma_\lambda\rangle_{G^F}=1$ *for some* $\lambda\in\Lambda$ *or (when* p *is good for* G) $p\nmid\chi(1)$, *then* $m_E(\chi)=1$.

PROOF. If $\langle\chi,\Gamma_\lambda\rangle_{G^F}=1$ for some λ, then the assertion follows from the theorem of Schur. Assume that p is a good prime for G and $p\nmid\chi(1)$. Then the assertion can be proved by a method similar to the proof of Corollary 4 to Proposition 1 of [13].

A large part of the following two theorems is contained in [13].

THEOREM 1. *Let* $G=SU_{l+1}$ *or* $Spin_{2l}^{-}$ *(in either case* $p\neq 2$). *Let* χ *be an irreducible character of* G^F *such that* $\langle\chi,\Gamma_\lambda\rangle_{G^F}=1$ *for some* $\lambda\in\Lambda$ *or* $p\nmid\chi(1)$. *Then, in any one of the* *following cases, we have* $m_{\boldsymbol{Q}}(\chi)=1$: SU_{l+1} $(2|l)$; $Spin_{2l}^{-}$ $(4|l(l+1))$.

PROOF. By Propositions 2, 3, we see that in any one of the cases above any Γ_λ is realizable in $\boldsymbol{Q}$. Hence the assertion follows from Lemma 5.

THEOREM 2. *Let* $G=SU_{l+1}$ *or* $Spin_{2l}^{-}$ *(in either case* $p\neq 2$*). Let* χ *be an irreducible character of* G^F *such that* $\langle\chi, \Gamma_\lambda\rangle_{G^F}=1$ *for some* $\lambda\in\Lambda$ *or* $p\nmid\chi(1)$. *Then in any one of the following cases we have* $m_{\boldsymbol{Q}_r}(\chi)=1$ *for any prime number* $r\neq p$:

SU_{l+1}: *q is square; q is non-square and* $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(q+1)$;

$Spin_{2l}^{-}$: $2\|l$; $2\|l-1$ *and q is square*; $2\|l-1$, *q is non-square and* $p\equiv 1 \pmod 4$.

PROOF. By Propositons 2, 3, we see that in any one of the cases above, for any prime number $r\neq p$, and Γ_λ is realizable in $\boldsymbol{Q}_r$. Hence the assertion follows from Lemma 5.

The following theorem is announced in [14].

THEOREM 3. *Assume that q is an even power of* $p\neq 2$. *Let* χ *be an irreducible character of* $SU_{l+1}(\boldsymbol{F}_q)$. *Then, for any prime number* $r\neq p$, *we have* $m_{\boldsymbol{Q}_r}(\chi)=1$.

REMARK. Let q be an arbitrary power of $p\neq 2$. Then it is known that the Schur index (over $\boldsymbol{Q}$) of any irreducible character of $SU_{l+1}(\boldsymbol{F}_q)$ is at most two (Gow [6]). One can prove that, for $l+1\neq 2$, 4, $SU_{l+1}(\boldsymbol{F}_q)$ has a rational-valued unipotent character χ such that $m_{\boldsymbol{Q}}(\chi)=m_{\boldsymbol{R}}(\chi)=m_{\boldsymbol{Q}_p}(\chi)=2$ and $m_{\boldsymbol{Q}_r}(\chi)=1$ for any prime number $r\neq p$. (Also see Theorem 4 below.)

LEMMA 6. *Let the situations be as in Theorem 3. Then, for any unipotent element u of* $SU_{l+1}(\boldsymbol{F}_q)$, $\chi(u)$ *is a rational integer and, for any prime number* $r\neq p$, $m_{\boldsymbol{Q}_r}(\chi)|\chi(u)$.

PROOF. Let $G=SL_{l+1}(K)$ and $G_1=GL_{l+1}(K)$, and let F be the endomorphism of G_1 defined by $F((x_{ij}))={}^t(x_{ij}^q)^{-1}$ (${}^tA$ is the transpose of a matrix A). Then $G^F=SU_{l+1}(\boldsymbol{F}_q)$ and $G_1^F=U_{l+1}(\boldsymbol{F}_q)$.

Let u by any unipotent element of G^F. Then $u\in G_1^F$ $(\subset GL_{l+1}(\boldsymbol{F}_{q^2}))$. Let u_μ be a Jordan canonical form of u in G_1, where $\mu=(\mu_1, \cdots, \mu_r)$ is a certain partition of $l+1$:

$$u_\mu=\begin{pmatrix} \underbrace{\begin{matrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{matrix}}_{\mu_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\begin{matrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{matrix}}_{\mu_r} \end{pmatrix}.$$

Then u and u_μ are conjugate in $GL_{l+1}(\boldsymbol{F}_{q^2})$. Let:

$$L_\mu = \left\{ \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_r \end{pmatrix} \middle| \, A_i \in GL_{\mu_i}(K),\, i=1, \cdots, r \right\}.$$

Then L_μ is an F-stable reductive subgroup of G_1 and is isomorphic to $GL_{\mu_1}(K) \times \cdots \times GL_{\mu_r}(K)$. Clearly u_μ is a regular unipotent element of L_μ. Let u_0 be a regular unipotent elment of L_μ contained in L_μ^F ($\subset GL_{l+1}(F_{q^2})$). Then u_μ and u_0 are conjugate in $L_\mu^{F^2} = GL_{\mu_1}(F_{q^2}) \times \cdots \times GL_{\mu_r}(F_{q^2})$, hence conjugate in $GL_{l+1}(F_{q^2})$. Hence u and u_0 are conjugate in $GL_{l+1}(F_{q^2})$. As u, $u_0 \in G_1^F = U_{l+1}(F_q)$, by a result of Ennola ([1], E-11, I, 3.5), u and u_0 are conjugate in G_1^F. Hence there is an element $g \in G_1^F$ such that $u = gu_0g^{-1}$. We note that $u_0 \in G^F$.

Let χ be an irreducible character of G^F. Let χ^g be the character of G^F defined by $\chi^g(x) = \chi(gxg^{-1})$ for $x \in G^F$. Then $\chi(u) = \chi(gu_0g^{-1}) = \chi^g(u_0)$. As u_0 is unipotent, u_0 belongs to $(L'_\mu)^F = SU_{\mu_1}(F_q) \times \cdots \times SU_{\mu_r}(F_q)$ (L_μ denotes the derived group of L_μ). So we can write as $u_0 = (u_1, \cdots, u_r)$ with $u_i \in SU_{\mu_i}(F_q)$ $(i=1, \cdots, r)$. Clearly, for $i=1, \cdots, r$, u_i is a regular unipotent element of SU_{μ_i} ($= SU_{\mu_i}(K)$ with Frobenius map F).

For $i=1, \cdots, r$, let U_i be the unipotent radical of the Borel subgroup B_i of SU_{μ_i} containing u_i; as $F(u_i) = u_i$ and such B_i is unique, B_i is F-stable; thus U_i is also F-stable and $u_i \in U_i^F$. Put $H = U_1^F \times \cdots \times U_r^F$; we consider H as a subgroup of G^F. Let Λ_μ and N_μ be respectively the set of linear characters of H and the set of non-linear irreducible characters of H. Then we have

$$\chi^g | H = \sum_{\lambda \in \Lambda_\mu} a_\lambda \lambda + \sum_{\rho \in N_\mu} b_\rho \rho ,$$

where $a_\lambda = \langle \chi^g | H, \lambda \rangle_H$ for $\lambda \in \Lambda_\mu$ and $b_\rho = \langle \chi^g | H, \rho \rangle_H$ for $\rho \in N_\rho$. As $U_1 \times \cdots \times U_r$ is a maximal unipotent subgroup of $L'_\mu = SU_{\mu_1} \times \cdots \times SU_{\mu_r}$ and u_0 is a regular unipotent element of L'_μ in $H = U_1^F \times \cdots \times U_r^F$, by a result of Lehrer [11], we have $\rho(u_0) = 0$ for all ρ in N_μ. Hence we have:

$$\chi^g(u_0) = \sum_{\lambda \in \Lambda_\mu} a_\lambda \lambda(u_0) .$$

Let $\lambda \in \Lambda_\mu$. Then we can write as $\lambda = \lambda_1 \cdot \cdots \cdot \lambda_r$, where, for $i=1, \cdots, r$, λ_i is a linear character of U_i^F. Hence, by Theorem (43.2) of [3], p. 316, we have:

$$\lambda^{(L'_\mu)^F} = \lambda_1^{SU_{\mu_1}(F_q)} \# \cdots \# \lambda_r^{SU_{\mu_r}(F_q)} ,$$

where # denotes the outer tensor product of characters. Let t be any prime number $\neq p$. As q is square, by Proposition 2 (or Proposition 1, (ii) of [13]), each $\lambda_i^{SU_{\mu_i}(F_q)}$ is realizable in Q_t. Hence $\lambda^{(L'_\mu)^F}$ is realizable in Q_t, and so is λ^{G^F}. Hence, by the theorem of Schur, $m_{Q_t}(\chi^g) | a_\lambda$. As $m_{Q_t}(\chi) = m_{Q_t}(\chi^g)$, $m_{Q_t}(\chi) | a_\lambda$.

For $i=1, \cdots, r$, let T_i be an F-stable maximal torus of B_i and let f_i be an element of T_i^F described in Lemma 3, i.e., $\lambda_i^{f_i}=\lambda_i^\sigma$ for any linear character λ_i of U_i^F, where σ is a certain generator of $\Pi=\mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})$. Let $f=(f_1,\cdots,f_r)$ (an element of $(L'_\mu)^F$). As $f\in G^F$, $(\chi^g)^f=\chi^g$ and f interchanges the characters in Λ_μ. Thus we have:

$$\chi^g(u_0)=(\chi^g)^f(u_0)=\sum_{\lambda\in\Lambda_\mu} a_\lambda\lambda^f(u_0)=\sum_{\lambda\in\Lambda_\mu} a_\lambda\lambda_1^{f_1}(u_1)\cdot\cdots\cdot\lambda_r^{f_r}(u_r)$$

$$=\sum_{\lambda\in\Lambda_\mu} a_\lambda\lambda_1^\sigma(u_1)\cdot\cdots\cdot\lambda_r^\sigma(u_r)=(\chi^g(u_0))^\sigma\,.$$

As σ is a generator of Π, this shows that $\chi^g(u_0)$ lies in $\boldsymbol{Q}$. Hence $\chi(u)=\chi^g(u_0)$ is a rational integer. Put $m=m_{\boldsymbol{Q}_t}(\chi)$. Then we have an expression:

$$\chi^g(u_0)/m=\sum_{\lambda\in\Lambda_\mu}(a_\lambda/m)\lambda(u_0)\,.$$

As we have seen above, m divides each a_λ, hence the right hand side of this expression is an algebraic integer. As $\chi^g(u_0)/m$ is a rational number, we conclude that $\chi^g(u_0)/m$ is a rational integer. Hence $m|\chi^g(u_0)$. Hence $m|\chi(u)$. This completes the proof of Lemma 6.

PROOF OF THEOREM 3. Let G, G_1 and F be as in the proof of Lemma 6. Let χ be an irreducible character of G^F. Let $\chi^{(1)},\cdots,\chi^{(s)}$ be the G_1^F-conjugates of χ. Then, by Clifford theory, there is an irreducible character χ_1 of G_1^F such that

$$\chi_1|G^F=\chi^{(1)}+\cdots+\chi^{(s)}\,.$$

Then, by Theorem C of [12] and by the Ennola conjecture ([18], [9]), there is a unipotent element u of G_1^F such that $\chi_1(u)=\pm p$-power. As u is unipotent, $u\in G^F$. Let r be any prime number $\neq p$, and put $m=m_{\boldsymbol{Q}_r}(\chi)=m_{\boldsymbol{Q}_r}(\chi^{(i)})$ $(i=1,\cdots,s)$. By Lemma 6, for $i=1,\cdots,s$, $\chi^{(i)}(u)$ is a rational integer and $m|\chi^{(i)}(u)$. Thus we have an expression:

$$\chi_1(u)/m=(\chi^{(1)}(u)/m)+\cdots+(\chi^{(s)}(u)/m)\in\boldsymbol{Z}\,.$$

Hence m divides a power of p. But, by the result of Gow [6], we have $m\leqq 2$. Hence $m=1$ (cf. $p\neq 2$). This completes the proof of Theorem 3.

THEOREM 4. *Let $G=SU_{l+1}$ or $Spin_{2l}^-$ (in either case $p\neq 2$). Then in any one of the following cases G^F has an irreducible character χ such that $m_{\boldsymbol{Q}}(\chi)=2$:*

SU_{l+1}: $2\nmid l$ and q is square; $2\nmid l$, $q\equiv 1$ (mod 4), *q is non-square and* $\mathrm{ord}_2(l+1)>\mathrm{ord}_2(q+1)$; *$2\nmid l$, q is non-square,* $\mathrm{ord}_2(l+1)\leqq\mathrm{ord}_2(q+1)$ *and $p\equiv 1$* (mod 4);

$Spin_{2l}^-$: $2\|l$ and $q\equiv 1$ (mod 4); *$2\|l-1$ and q is square; $2\|l-1$, q is non-square and $p\equiv 1$* (mod 4).

PROOF. The following proof was inspired by [6]. Let $\lambda\in\Lambda_0$. By Propositions 2, 3, Lemmas 3, 4 and Proposition 1, we find that, in any one of the cases in the theorem, there is an irreducible character μ_i such that $m_{k_i}(\mu_i)=2$ and $\langle\Gamma_{\lambda,i},\Gamma_{\lambda,i}\rangle_{G^F}$ is odd. By [13], the Schur index (over $\boldsymbol{Q}$) of any irreducible component of $\Gamma_{\lambda,i}$ is at most two. We note that the character $\Gamma_{\lambda,i}$ takes its values in k_i. Suppose that all the irreducible components of $\Gamma_{\lambda,i}$ have the index 1 over k_i. Let A be the set of irreducible components of $\Gamma_{\lambda,i}$. Let $\bar{k}_i$ be an algebraic closure of k_i. Then $\mathrm{Gal}(\bar{k}_i/k_i)$ acts on A, and we have a decomposition $A=A_1\cup\cdots\cup A_t$, where $A_1,\cdots,A_t$ are the orbits under the action of $\mathrm{Gal}(\bar{k}_i/k_i)$. For $j=1,\cdots,t$, put $\phi_j=\sum_{\chi\in A_j}\chi$. Then, by the theorem of Schur, $\phi_1,\cdots,\phi_t$ are realizable in k_i. Hence $\Gamma_{\lambda,i}=\phi_i+\cdots+\phi_t$ are realizable in k_i. Hence, by the theorem of Schur, $2=m_{k_i}(\mu_i)$ must divide $\langle\mu_i,\Gamma_{\lambda,i}|L\rangle_L=\langle\Gamma_{\lambda,i},\Gamma_{\lambda,i}\rangle_{G^F}$, which is odd. This is a contradiction. Therefore $\Gamma_{\lambda,i}$ must contain some irreducible character χ such that $m_{k_i}(\chi)=2$. We note that if $\Gamma_{\lambda,i}$ is realizable in $(k_i)_v$, then we have $m_{(k_i)_v}(\chi)=1$. Such v can be easily determined by Propositions 2, 3.

Since $\boldsymbol{Q}(\Gamma_\lambda)\neq\boldsymbol{Q}$ generally, our method is not sufficient to determine the Schur indices over $\boldsymbol{Q}$. We hope to find other general methods to determine $m_{\boldsymbol{Q}}(\chi)$.

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Present Address:
Iwamizawa College, Hokkaido University of Education,
Midorigaoka, Iwamizawa, Hokkaido, 068 Japan.