TOKYO J. MATH.
VOL. 18, No. 2, 1995

A Normal Form of First Order Partial Differential Equations with Singular Solution

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(Communicated by T. Nagano)

Abstract. We give a local normal form of first order partial differential equations with singular solution up to contact diffeomorphism.

In [3] we establish the notion of *first order differential equations with singular solution* by using the method which is originated in Kossowski [4]. In this note we give a local normal form of such equations up to contact diffeomorphism. The method using here depends heavily on ([3], [4]), however we have never seen a normal form theorem for such a class of equations. We now review notions and results in [3]. *A first order differential equations* (or briefly, *an equation*) is a relation $F=0$, where $F: (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)\to(\boldsymbol{R}, 0)$ is a submersion germ on the 1-jet space of functions of n-variables. Let θ be the canonical contact form on $J^1(\boldsymbol{R}^n, \boldsymbol{R})$ which is given by $\theta=dy-\sum_{i=1}^n p_i dx_i$, where (x, y, p) are canonical coordinates of $J^1(\boldsymbol{R}^n, \boldsymbol{R})$. Throughout the remainder of this note, we shall consider $J^1(\boldsymbol{R}^n, \boldsymbol{R})$ as a contact manifold whose contact structure is given by the canonical 1-form. The notion of a solution of an equation is given by the philosophy of Lie. *A geometric solution* (*or, a Legendrian solution*) *of* $F=0$ is defined to be an immersion $i: (L, q_0)\to(J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$ from an n-dimensional manifold such that $i^*\theta=0$ and $i(L)\subset F^{-1}(0)$ (i.e. a Legendrian submanifold which is contained in $F^{-1}(0)$). The following notion is quite important to consider the notion of singular solutions. We say that z_0 is *a contact singular point* (*or, characteristic point*) if $\theta(T_{z_0}F^{-1}(0))=0$. We denote the set of contact singular points by $\Sigma_c(F)$. We call it *a contact singular set of F.* The notion of singular solutions (in the classical sense) has been appeared accompanied by the notion of complete solutions in classical treatises. *A complete solution* of $F=0$ is defined to be a foliation whose leaves are geometric solutions of $F=0$. We defined the notion of singular solutions (in the strict sense) as follows. A geometric solution $i: (L, q_0)\to(J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$ of $F=0$ is called *a singular solution* (in the strict sense) if it satisfies the following condition:

(*) There exists a representative $\tilde{i}: U\to F^{-1}(0)$ of i such that $\tilde{i}(V)$ is not contained

Received June 20, 1994

in a leaf of any complete solutions of $F=0$ for any open subset $V\subset U$.

In [3] we have shown the following results.

LEMMA. *An equation* $F\colon (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)\to(\boldsymbol{R}, 0)$ *has a singular solution (in the strict sense) if and only if* $\Sigma_c(F)$ *is an n-dimensional submanifold. Moreover,* $\Sigma_c(F)$ *is a singular solution of* $F=0$.

We call the equation which satisfies the condition in Lemma *an equation with singular solution.*

We remark that z_0 is a contact singular point of $F=0$ if and only if $F(z_0)=\partial F/\partial p_i(z_0)=(\partial F/\partial x_i+p_i(\partial F/\partial z_i))(z_0)=0$. So we can easily check that $\Sigma_c(F)$ is a submanifold or not. Our main result is as follows:

THEOREM. *Let* $F\colon (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)\to(\boldsymbol{R}, 0)$ *be an equation with singular solution. Then there is a contact diffeomorphism germ* $f\colon (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)\to(J^1(\boldsymbol{R}^n, \boldsymbol{R}), 0)$ *such that* $f(F^{-1}(0))=\{y=0\}$.

For the proof we quote the following very important result.

KOSTANT-STERNBERG'S THEOREM. ([2]) *Let* (P, ω) *be a symplectic manifold, L a Lagrangian submanifold and* α *a smooth* 1*-form on a neighbourhood of L in P with* $\alpha|L=0$ *and* $d\alpha=\omega$. *Then there exist a tubular neighbourhood V of L in P, a neighbourhood U of zero section L in* T^*L *and a unique "local" vector bundle isomorphism* $K\colon (V, L)\to(U, L)$ *such that K is the identity on L and* $K^*\theta_L=\alpha$. *Here,* θ_L *is the canonical* 1*-form on* T^*L.

Let $F\colon (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)\to(\boldsymbol{R}, 0)$ be an equation such that z_0 is a contact singular point. If $F_y=0$ at z_0, then $F_{x_i}=F_{p_i}=0$ at z_0 for any $i=1, \cdots, n$. This contradicts the fact that F is a submersion germ. Then we have $F_y\neq 0$. By the implicit function theorem, there exists a function germ $h\colon (T^*\boldsymbol{R}^n, (x_0, p_0))\to(\boldsymbol{R}, y_0)$ such that $F^{-1}(0)=\{(x, y, p)\,|\,y=h(x, p)\}$, where $T^*\boldsymbol{R}^n$ is the cotangent bundle of $\boldsymbol{R}^n$ and $z_0=(x_0, y_0, p_0)$. Here, we consider that $J^1(\boldsymbol{R}^n, \boldsymbol{R})\cong T^*\boldsymbol{R}^n\times\boldsymbol{R}$. In the terminology of Kossowski [4] an equation of the above form is called *a graphlike equation.* The following method is originated by Kossowski. We now define a map germ

$$graph(h)\colon (T^*\boldsymbol{R}^n, (x_0, p_0))\to(J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$$

by

$$graph(h)(x, p)=(x, h(x, p), p)\,.$$

We define a 1-form on $T^*\boldsymbol{R}^n$ by $\theta_h=graph(h)^*\theta=dh-\sum_{i=1}^n p_i dx_i$. Then we have the following one to one correspondence:

$$\begin{array}{c}\{L \mid L \text{ is a solution of } y-h(x, p)=0\}\\ {\scriptstyle graph(h)}\Big\uparrow\Big\downarrow{\scriptstyle \Pi_*}\\ \{L \mid i\colon L\subset T^*\boldsymbol{R}^n \text{ is a maximal integral submanifold of } \theta_h=0\}\,,\end{array}$$

where $\Pi(x, y, p)=(x, p)$ and $\Pi_*(L)=\Pi(L)$. Thus a solution of a graphlike equation $y-h(x, p)=0$ may be regarded as a maximal isotropic submanifold of $(T^*\boldsymbol{R}^n, \theta_h)$. Since $-d\theta_h=\sum_{i=1}^n dp_i \wedge dx_i$ is the canonical symplectic two form, a solution of $y-h(x, p)=0$ is a Lagrangian submanifold of $(T^*\boldsymbol{R}^n, \omega)$, where $\omega=-d\theta_h$. For the definition and properties of Lagrangian submanifolds, see [1]. Under the above preparations, we can prove the normal form theorem.

PROOF OF THEOREM. We have $F^{-1}(0)=\{y-h(x, p)=0\}$ and $G_h^{-1}(\Sigma_c(F))=L_h$ is a Lagrangian submanifold of $T^*\boldsymbol{R}^n$ on which θ_h vanishes, where $\theta_h=dh-\sum_{i=1}^n p_i dx_i$. Kostant-Sternberg's theorem asserts that there exist a tubular neighbourhood V of L_h in $T^*\boldsymbol{R}^n$ and a unique (local) vector bundle isomorphism K: $V\to T^*L_h$ such that K is identity on L_h and $K^*\theta_{L_h}=-\theta_h$. We denote local coordinates of L_h as $(x'_1, \cdots, x'_n)$ and the corresponding canonical coordinates of $T^*L'_h$ is denoted by $(x'_1, \cdots, x'_n, p'_1, \cdots, p'_n)$. We define a diffeomorphism germ Φ: $V\times \boldsymbol{R}\to T^*L_h\times \boldsymbol{R}$ by $\Phi(x, p, y)=(K(x, p), y-h(x, p))$. On the other hand, we have the canonical contact structure on $T^*L_h\times \boldsymbol{R}$ given by the contact form $dy'-\sum_{i=1}^n p'_i dx'_i$, where $(x'_1, \cdots, x'_n, p'_1, \cdots, p'_n, y')$ is the canonical coordinate on $T^*L_h\times \boldsymbol{R}$ induced by the previous arguments. It follows that $\Phi^*(dy'-\sum_{i=1}^n p'_i dx'_i)=dy-dh+\theta_h=dy-\sum_{i=1}^n p_i dx_i$. Since $V\times \boldsymbol{R}$ may be considered as an open set of $J^1(\boldsymbol{R}^n, \boldsymbol{R})$, Φ is a local contact diffeomorphism. By definition, we have $\Phi(\{y=h(x, p)\})=\{y'=0\}$ and $\Phi(L_h)=\{p'_1=\cdots=p'_n=0\}$. This completes the proof.

We have some examples of first order differential equations with singular solution in [3], however we only give a typical example here.

EXAMPLE. Consider the following equation around the origin:

$$y-p^m=0 \qquad (n=1, m\geq 2).$$

We can calculate that $\Sigma_\pi(F)=\Sigma_c(F)=\{y=p=0\}$. We consider the following diffeomorphism germ at the origin:

$$\begin{cases} X=x-\dfrac{m}{m-1}p^{m-1} \\ Y=y-p^m \\ P=p. \end{cases}$$

Then it is easy to show that this local diffeomorphism is a contact diffeomorphism and it sends $\{y-p^m=0\}$ to $\{Y=0\}$.

Finally, we remark that the normal form theorem can be easily generalized to the case for overdetermined systems of first order equations.

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