TOKYO J. MATH.
VOL. 18, No. 2, 1995

Lower Bounds for the Class Number and the Caliber of Certain Real Quadratic Fields

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(Communicated by T. Nagano)

Introduction.

We give some canonical cycles of reduced ideals for the real quadratic fields $K=\mathbf{Q}(\sqrt{m})$ with $m=4q^2+1$, $m=q^2+4$ (q odd), $m=q^2+1$ (q odd) and $m=q^2\pm 2$ (q odd). Lower bounds of the class number $h(K)$ and the caliber Cal(K) (number of reduced ideals) are given. Some of those lower bounds for the class number are obtained, by other methods, in [2].

Let m be a square free integer, K the quadratic field $\mathbf{Q}(\sqrt{m})$, $\theta=(1+\sqrt{m})/2$ if $m\equiv 1$ (4) and $\sqrt{m}$ otherwise, θ^τ the conjugate of θ, $F(X)=(X-\theta)(X-\theta^\tau)$ the fundamental polynom of K and $D(K)=(\theta-\theta^\tau)^2$ the discriminant of K. Every reduced ideal $\mathfrak{a}$ of K is presented in the form $[a, \theta-c]$ where a is the norm of $\mathfrak{a}$, c an integer such that $0<\theta-c<a$ and $F(c)=-ab$. $\mathfrak{a}=[a, \theta-c]$ is reduced when $(a+b)^2\leq D(K)$ and a cycle of reduced ideals starts from $\mathfrak{a}$ to the reduced ideal $\mathfrak{a}_1=((\theta^\tau-c)/a)\mathfrak{a}=[b, \theta-c_1]$, this operation is repeated until we obtain $\mathfrak{a}$ another time (see [1]). The class number of K is equal to the number of cycles and the caliber is the sum of the numbers counting reduced ideals in every cycle. In what follows $\tau(x)$ denotes the number of distinct positive divisors of the integer x.

I. $\boldsymbol{m=4q^2+1}$, $F(X)=X^2-X-q^2$. Let d be a proper divisor of q, and put $q=\lambda d$. From

$$F(1)=-q^2, \qquad F(q)=-q=-d\lambda\,,$$

$$F(q+1-\lambda)=-\lambda(2q+1-d-\lambda)\,, \qquad F(q+1-d)=-d(2q+1-d-\lambda)\,,$$

we construct the cycles:

-$[1, \theta-q]\rightarrow[q, \theta-1]\rightarrow[q, \theta-q]$,

-$[d, \theta-q]\rightarrow[\lambda, \theta-(q+1-\lambda)]\rightarrow[2q+1-\lambda-d, \theta-(q+1-d)]$.

Received June 3, 1994

As the reduced ideals $[d, \theta-q]$ associated to every proper divisor of q are in distinct cycles of length three we have the following results:

THEOREM 1. *For* $K=Q(\sqrt{m})$ *with* $m=4q^2+1$,

i) $h(K)\geq\tau(q)-1$,

ii) $\mathrm{Cal}(K)\geq 3(\tau(q)-1)$.

II. $m=q^2+4$ $(q=2k+1)$, $F(X)=X^2-X-k^2-k-1$. From

$$F(k+1)=-1\,,\qquad F(k)=-q=-d\lambda\,,$$

$$F(k+2-\lambda)=-\lambda(q+2-d-\lambda)\,,\qquad F(k+2-d)=-d(q+2-d-\lambda)\,,$$

we construct the following cycles:

-$[1, \theta-(k+1)]$,

-$[d, \theta-k]\to[\lambda, \theta-(k+2-\lambda)]\to[q+2-d-\lambda, \theta-(k+2-d)]$.

THEOREM 2. *For* $K=Q(\sqrt{m})$ *with* $m=q^2+4$ $(q=2k+1)$,

i) $h(K)\geq\tau(q)-1$,

ii) $\mathrm{Cal}(K)\geq 3\tau(q)-5$.

III. $m=q^2+1$ $(q=2k+1)$, $F(X)=X^2-q^2-1$. From

$$F(1)=-q^2\,,\qquad F(q)=-1\,,$$

$$F(q-1)=-2q=-2d\lambda\ (d \text{ an odd proper divisor of } q)\,,$$

$$F(q-2\lambda+1)=-2\lambda(2q+2-2\lambda-d)\,,\qquad F(q-2d+1)=-2d(2q+2-2d-\lambda)\,,$$

$$F(q+1-d)=-d(2q+2-2\lambda-d)\,,\qquad F(q+1-\lambda)=-\lambda(2q+2-2d-\lambda)\,,$$

we have the following cycles of reduced ideals of K:

-$[1, \theta-q]$,

-$[2, \theta-(q-1)]\to[q, \theta-1]\to[q, \theta-(q-1)]$,

-$[d, \theta-(q-1)]\to[2\lambda, \theta-(q-2\lambda+1)]$
$\to[2q+2-2\lambda-d, \theta-(q+1-d)]$,

-$[2d, \theta-(q-1)]\to[\lambda, \theta-(q+1-\lambda)]$
$\to[2q+2-2d-\lambda, \theta-(q-2d+1)]$.

The cycles of the reduced ideals $[d, \theta-(q-1)]$ and $[2d, \theta-(q-1)]$ are in distinct cycles of length three then:

THEOREM 3. *For* $K=\mathbf{Q}(\sqrt{m})$ *with* $m=q^2+1$ $(q=2k+1)$,
i) $h(K)\geq 2\tau(q)-2$,
ii) $\mathrm{Cal}(K)\geq 6\tau(q)-8$.

IV. $\boldsymbol{m=q^2-2}$ (q odd and $q\not\equiv 0(3)$). From the following values of the fundamental polynom $F(X)=X^2-q^2+2$ of K:

$$F(q-1)=-(2q-3)=-d\lambda\ (d\text{ a proper odd divisor of }2q-3)\,,$$

$$F(q-2)=-2(2q-3)=-2d\lambda\,,\qquad F(q-1-d)=-2d\left(q+\frac{\lambda-d}{2}-1\right),$$

$$F(q-1-\lambda)=-2\lambda\left(q+\frac{d-\lambda}{2}-1\right),$$

$$F\left(\frac{\lambda+d}{2}\right)=-\left(q-1+\frac{\lambda-d}{2}\right)\left(q-1+\frac{d-\lambda}{2}\right),$$

we construct the following cycles of length four and six:

$$\text{-}[1,\theta-(q-1)]\rightarrow[2q-3,\theta-(q-2)]\rightarrow[2,\theta-(q-2)]\rightarrow[2q-3,\theta-(q-1)]\,,$$

$$\text{-}[d,\theta-(q-1)]\rightarrow[\lambda,\theta-(q-2)]\rightarrow[2d,\theta-(q-1-d)]\rightarrow\left[q-1+\frac{\lambda-d}{2},\theta-\frac{\lambda+d}{2}\right]$$

$$\rightarrow\left[q-1+\frac{d-\lambda}{2},\theta-(q-1-\lambda)\right]\rightarrow[2\lambda,\theta-(q-2)]\,,$$

and if d is a proper divisor of $2q+3$, with $2q+3=\lambda d$, we have from

$$F(q+1-d)=-d(2q+2-\lambda-d)=-2d\left(q+1-\frac{\lambda+d}{2}\right),$$

$$F(q+1-\lambda)=-\lambda(2q+2-\lambda-d)=-2\lambda\left(q+1-\frac{\lambda+d}{2}\right),$$

$$F(q+2-d)=-d(2q+4-d-2\lambda)\,,\qquad F(q+2-\lambda)=-\lambda(2q+4-\lambda-2d)\,,$$

$$F(q+2-2d)=-2d(2q+4-\lambda-2d)\,,\qquad F(q+2-2\lambda)=-2\lambda(2q+4-d-2\lambda)\,,$$

the following cycle of length eight:

$$\text{-}[d,\theta-(q+1-d)]\rightarrow[2q+2-\lambda-d,\theta-(q+1-\lambda)]\rightarrow[\lambda,\theta-(q+2-\lambda)]$$

$$\rightarrow[2q+4-\lambda-2d,\theta-(q+2-2d)]\rightarrow[2d,\theta-(q+1-d)]\rightarrow\left[q+1-\frac{\lambda+d}{2},\theta-(q+1-\lambda)\right]$$

$$\rightarrow[2\lambda,\theta-(q+2-2\lambda)]\rightarrow[2q+4-d-2\lambda,\theta-(q+2-d)]\,.$$

When $q \not\equiv 0$ (3), the integers $2q-3$ and $2q+3$ are coprime and so we have:

THEOREM 4. *For $K=\mathbf{Q}(\sqrt{m})$ with $m=q^2-2$, q odd, and $q \not\equiv 0$ (3),*
i) $h(K) \geq \tau(2q+3)+\tau(2q-3)-3$,
ii) $\mathrm{Cal}(K) \geq 6\tau(2q-3)+8\tau(2q+3)-24$.

V. $\boldsymbol{m=q^2+2}$ (q odd, and $q \equiv 0$ (3)), $F(X)=X^2-q^2-2$. From

$$F(q)=-2\,, \qquad F(q-1)=-(2q+1)=-d\lambda \ (d \text{ a proper divisor of } 2q+1)\,,$$

$$F(q+2-\lambda)=-\lambda(2q+4-\lambda-2d)\,, \qquad F(q+2-2d)=-2d(2q+4-\lambda-2d)\,,$$

$$F(q-d-1)=-2d\left(q-1+\frac{\lambda-d}{2}\right),$$

$$F\left(\frac{\lambda+d}{2}\right)=-\left(q-1+\frac{\lambda-d}{2}\right)\left(q-1+\frac{d-\lambda}{2}\right),$$

$$F(q-1-\lambda)=-2\lambda\left(q-1+\frac{d-\lambda}{2}\right), \qquad F(q+2-2\lambda)=-2\lambda(2q+4-d-2\lambda)\,,$$

$$F(q+2-d)=-d(2q+4-d-2\lambda)\,,$$

we can give the following cycles of length two and eight:

-$[1, \theta-q] \rightarrow [2, \theta-q]$,

-$[d, \theta-(q-1)] \rightarrow [\lambda, \theta-(q+2-\lambda)] \rightarrow [2q+4-\lambda-2d, \theta-(q+2-2d)] \rightarrow [2d, \theta-(q-1-d)]$

$$\rightarrow\left[q-1+\frac{\lambda-d}{2}, \theta-\left(\frac{\lambda+d}{2}\right)\right]\rightarrow\left[q-1+\frac{d-\lambda}{2}, \theta-(q-1-\lambda)\right]$$

$$\rightarrow [2\lambda, \theta-(q+2-2\lambda)] \rightarrow [2q+4-d-2\lambda, \theta-(q+2-d)]\,.$$

We can also construct from

$$F(q-2)=-2(2q-1)=-2d\lambda \ (d \text{ a proper divisor of } 2q-1)\,,$$

$$F(q+1-\lambda)=-2\lambda\left(q+1-\frac{\lambda+d}{2}\right)=-\lambda(2q+2-\lambda-d)\,,$$

$$F(q+1-d)=-2d\left(q+1-\frac{\lambda+d}{2}\right)=-d(2q+2-\lambda-d)\,,$$

the following cycle of length six:

-$[d, \theta-(q-2)] \rightarrow [2\lambda, \theta-(q+1-\lambda)] \rightarrow \left[q+1-\dfrac{\lambda+d}{2}, \theta-(q+1-d)\right]$

$$\rightarrow [2d, \theta-(q-2)] \rightarrow [\lambda, \theta-(q+1-\lambda)] \rightarrow [2q+2-\lambda-d, \theta-(q+1-d)]\,.$$

THEOREM 5. *For* $K=\boldsymbol{Q}(\sqrt{m})$ *with* $m=q^2+2$, *q odd, and* $q\equiv 0$ (3),

i) $h(K)\geq\tau(2q+1)+\tau(2q-1)-3$,

ii) $\mathrm{Cal}(K)\geq 6\tau(2q-1)+8\tau(2q+1)-26$.

References

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