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Ringed Spaces of Valuation Rings over Hilbert Rings

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Introduction.

Given a field K and a subring A of K, we consider the local ringed space $\mathrm{Zar}(K|A)$ consisting of all valuation rings of K which contain A (see [4] or [5]). If A is a Hilbert ring, in other words, if any prime ideals of A are intersections of maximal ideals (see [1], p. 373), then the ringed space $X=\mathrm{Zar}(K|A)$ satisfies the condition

(1) $$\beta_X: t(X_{\mathrm{cl}}, \mathscr{F}_X|_{X_{\mathrm{cl}}}) \rightrightarrows (X, \mathscr{F}_X)\,.$$

Here X_{cl} is the set of closed points of X and $\mathscr{F}_X$ is the structure sheaf on X. For the morphism β_X of ringed spaces, see (17). Given a topological space W, we denote by tW the set of irreducible closed subsets of W. If $(W, \mathscr{F}_W)$ is a ringed space, then tW also has a structure of ringed spaces donoted by $t(W, \mathscr{F}_W)$. The correspondence $(W, \mathscr{F}_W)\mapsto t(W, \mathscr{F}_W)$ gives rise to a covariant functor from the category of ringed spaces to itself. Moreover, if W is a T_1-space, then the ringed space $(X, \mathscr{F}_X)=t(W, \mathscr{F}_W)$ satisfies the condition (1), and the morphism $f: X\to Y$ of ringed spaces obtained by t from a morphism of T_1-ringed spaces satisfies the condition

(2) $$f(X_{\mathrm{cl}})\subset Y_{\mathrm{cl}}\,.$$

In this case, t gives an equivalence of the categories (see section 1). Therefore, we shall consider the following problem.

PROBLEM 1. Characterize the ringed spaces $(X, \mathscr{F}_X)$ satisfying the condition (1).

EXAMPLES. (i) Let X be an affine scheme $\mathrm{Spec}\,A$. Then X satisfies the condition (1) if and only if A is a Hilbert ring.

(ii) Any integral scheme X of finite type over a field satisfies the condition (1).

For a local ringed space $(W, \mathcal{O}_W)$, we introduce a morphism $\pi_W: W\to \mathrm{Spec}\,\mathcal{O}_W(W)$ defined by $\pi_W(x)=\rho_{W,x}^{-1}(m(\mathcal{O}_{W,x}))$. Here $\rho_{W,x}: \mathcal{O}_W(W)\to\mathcal{O}_{W,x}$ are the canonical mappings and $m(R)$ denotes the unique maximal ideal of a local ring R. The next problem is closely related to Problem 1.

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PROBLEM 2. Characterize the local ringed space $(W, \mathcal{O}_W)$ satisfying the condition:

(3) $\overline{\pi_W(\pi_W^{-1}(F))}=F$, for any closed subsets F of $\operatorname{Spec}\mathcal{O}_W(W)$.

Relating to these problems, the following three theorems are obtained.

THEOREM 1. *Let A be an integral domain, $X=\operatorname{Spec}A$ and $W=X_{\mathrm{cl}}$. Then the next three conditions are equivalent:*
(a) *X satisfies the condition* (1).
(b) *W is irreducible and satisfies the condition* (3).
(c) *A is a Hilbert ring.*

For a field K and a subring A of K, let $\operatorname{Loc}(K|A)$ denote the set of local subrings of K which contain A. Then the set $\operatorname{Loc}(K|A)$ has a structure of local ringed spaces (see [6]).

THEOREM 2. *Let $X=\operatorname{Loc}(K|A)$ and $W=X_{\mathrm{cl}}$. Then*

(i) *X satisfies the condition* (1) *if and only if A is a Hilbert Prüfer ring with quotient field K.*

(ii) *W is irreducible and satisfies the condition* (3) *if and only if A is Hilbert.*

THEOREM 3. *Let $X=\operatorname{Zar}(K|A)$ and $W=X_{\mathrm{cl}}$. Then the next three conditions are equivalent:*
(a) *X satisfies the condition* (1).
(b) *W is irreducible and satisfies the condition* (3).
(c) *A is a Hilbert ring.*

COROLLARY. *Suppose that A is a Hilbert ring and i: $W\to X$ is the inclusion mapping. Then*

(i) $\mathcal{O}_W=i^{-1}\mathcal{O}_X$ *and* $\mathcal{O}_X=i_*\mathcal{O}_W$.

(ii) *Let Ω_X^m (resp. Ω_W^m) be the sheaf of regular differential forms on X (resp. W) for any multi-index m. Then $\Omega_W^m=i^{-1}\Omega_X^m$, $\Omega_X^m=i_*\Omega_W^m$ and hence $\Omega_X^m(X)=\Omega_W^m(W)$* (see also [6], Theorem 2).

Given an integral Hilbert ring A, we introduce the following three categories.

$\mathscr{C}_0(A)$: the category of fields K which contain A and A-ring homomorphisms.

$\mathscr{C}_1(A)$: the category of local ringed spaces $\operatorname{Zar}(K|A)$ and dominant morphisms over $\operatorname{Spec}A$ satisfying the condition (2).

$\mathscr{C}_2(A)$: the category of local ringed spaces $\operatorname{Zar}(K|A)_{\mathrm{cl}}$ and dominant morphisms over $\operatorname{Spec}A$.

We can give an explicit characterization for objects of both the categories $\mathscr{C}_1(A)$ and $\mathscr{C}_2(A)$ among local ringed spaces (see Theorem 1 in [5] and Lemma 15).

From Theorem 1 in [5], the category $\mathscr{C}_0(A)$ is anti-equivalent to $\mathscr{C}_1(A)$. Moreover, the next result is obtained as an application of Theorem 3.

THEOREM 4. *Let A be an integral Hilbert ring. Then*

(i) *the categories $\mathscr{C}_1(A)$ and $\mathscr{C}_2(A)$ are equivalent. Therefore the categories $\mathscr{C}_0(A)$ and $\mathscr{C}_2(A)$ are anti-equivalent.*

(ii) *If A is an algebraically closed field k and K is a field finitely generated over k, then*

$$\mathrm{Zar}(K|k)_{\mathrm{cl}} \simeq \mathrm{proj.lim}\, V,$$

where V runs over all complete algebraic varieties over k with rational function field K.

In the following we shall prove Theorems 1, 2, 3 and 4.

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§1. Here we collect some properties of the functor t omitting proofs. First we consider in topological spaces. By (Top) we denote the category of topological spaces and continuous mappings.

For a topological space X, let tX denote the totality of irreducible closed subsets of X. There exists a unique topology on tX with the family of closed subsets $\{tE \mid E$ is closed in $X\}$. Then the mapping: {closed subsets of X}$\to${closed subsets of tX} defined by

(4) $$E \mapsto tE, \qquad \text{for closed subsets } E \text{ of } X$$

is an inclusion-preserving bijection, and

(5) $$E \text{ is irreducible if and only if } tE \text{ is irreducible},$$

(6) $$tE = \overline{\{E\}} : \text{the closure in } tX, \qquad \text{for any} \quad E \in tX.$$

Thus the mapping: $tX \to t(tX)$ defined by

(7) $$E \mapsto \overline{\{E\}}, \qquad \text{for} \quad E \in tX$$

is also an inclusion-preserving bijection.

For a continuous mapping $f\colon X \to Y$, a mapping $tf\colon tX \to tY$ is defined by

(8) $$(tf)(E) = \overline{f(E)} : \text{the closure in } Y, \qquad \text{for} \quad E \in tX.$$

Then

(9) $$(tf)^{-1}(tF) = t(f^{-1}(F)), \qquad \text{for any closed subsets } F \text{ of } Y.$$

Therefore tf is continuous, and hence t: (Top)$\to$(Top) is a covariant functor.

For a topological space X, a mapping $\alpha_X\colon X \to tX$ is defined by

(10) $$\alpha_X(x) = \overline{\{x\}} : \text{the closure in } X, \qquad \text{for} \quad x \in X.$$

Then, for any closed subsets E of X,

(11) $$\alpha_X^{-1}(tE)=E\,, \qquad \overline{\alpha_X(E)}=tE\,,$$

and hence α_X is continuous and dominant. Therefore α: id$\to t$ is a natural transformation, where id is the identity functor of the category (Top). Moreover the mapping: {closed subsets of tX}$\to${closed subsets of X} defined by

(12) $$F\mapsto\alpha_X^{-1}(F)\,, \qquad \text{for closed subsets } F \text{ of } tX$$

is an inclusion-preserving bijection and is the inverse of the mapping defined by (4).

LEMMA 1. *Let X be a topological space. Then*
(i) *X satisfies $T_0\Leftrightarrow\alpha_X$ is injective*
$\Leftrightarrow\alpha_X$ is an into-homeomorphism.
(ii) *$\alpha_{tX}: tX\to t(tX)$ is an inclusion-preserving homeomorphism.*
(iii) *X satisfies T_1 if and only if X satisfies T_0 and* $\operatorname{Im}\alpha_X=(tX)_{\mathrm{cl}}$.

We introduce the following three conditions for a continuous mapping $f: X\to Y$.

(13) $tf: tX\to tY$ is a homeomorphism.

(14) $f: X\to f(X)$ is a closed mapping and $\overline{F\cap f(X)}=F$ for any closed subsets F of Y.

(15) $f: X\to Y$ is an into-homeomorphism and $\overline{F\cap f(X)}=F$ for any closed subsets F of Y.

LEMMA 2. *Let $f: X\to Y$ be a continuous mapping. Then*
(i) *f is dominant if and only if tf is dominant.*
(ii) $(15)\Rightarrow(13)\Rightarrow(14)$.

Next we consider the functor t in ringed spaces. By (R. Spaces), we denote the category of ringed spaces.

For a ringed space $(X,\mathscr{F}_X)$, we put $t(X,\mathscr{F}_X)=(tX,\alpha_{X*}\mathscr{F}_X)$. We also write $\mathscr{F}_{tX}=\alpha_{X*}\mathscr{F}_X$.

For a morphism $(f,f^\#): (X,\mathscr{F}_X)\to(Y,\mathscr{F}_Y)$ of ringed spaces, we put $t(f,f^\#)=(tf,\alpha_{Y*}f^\#)$. We also write $(tf)^\#=\alpha_{Y*}f^\#$. Accordingly t becomes a functor: (R. Spaces)$\to$(R. Spaces).

Letting $\alpha_X^\#$ be the natural identity of $\alpha_{X*}\mathscr{F}_X$ for any ringed space $(X,\mathscr{F}_X)$, we obtain a morphism $\alpha_{(X,\mathscr{F}_X)}=(\alpha_X,\alpha_X^\#): (X,\mathscr{F}_X)\to t(X,\mathscr{F}_X)$ of ringed spaces. Thus α: id$\to t$ is a natural transformation, where id is the identity functor of (R. Spaces). Note that $\mathscr{F}_{tX,Y}=\mathscr{F}_{X,Y}$ for any $Y\in tX$, and hence $(\alpha_X^\#)_x: \mathscr{F}_{tX,\alpha_X(x)}\to\mathscr{F}_{X,x}$ is the identity mapping for any $x\in X$. Moreover,

(16) any irreducible closed subset of X has a unique generic point in X
$\Leftrightarrow\alpha_X$ is bijective

$\Leftrightarrow \alpha_{(X,\mathscr{F}_X)}$ is an isomorphism of ringed spaces.

LEMMA 3. *Let X be a topological space. Then the category of sheaves on X and the category of sheaves on tX are equivalent by the functors* α_{X*} *and* α_X^{-1}.

LEMMA 4. *Let* $(f, f^{\#}): (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y)$ *be a morphism of ringed spaces. Then*

(i) $t(f, f^{\#})$ *is an isomorphism of ringed spaces if and only if tf is a homeomorphism and* $f^{\#}: \mathscr{F}_Y \to f_*\mathscr{F}_X$ *is an isomorphism of sheaves on Y.*

(ii) *If* $\mathscr{F}_X = f^{-1}\mathscr{F}_Y$, *then* $\mathscr{F}_{tX} = (tf)^{-1}\mathscr{F}_{tY}$.

(iii) *If tf is a homeomorphism and* $\mathscr{F}_X = f^{-1}\mathscr{F}_Y$, *then* $\mathscr{F}_Y = f_*\mathscr{F}_X$, *and hence* $t(f, f^{\#})$ *is an isomorphism of ringed spaces.*

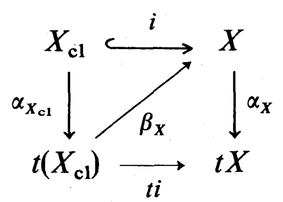
COROLLARY. *Let* $(X, \mathscr{F}_X)$ *be a ringed space,* $W \subset X$, $\mathscr{F}_W = \mathscr{F}_X|_W$ *and let* $(i, i^{\#}): (W, \mathscr{F}_W) \to (X, \mathscr{F}_X)$ *be the inclusion morphism of ringed spaces. Then* $t(i, i^{\#})$ *is an isomorphism of ringed spaces if and only if* $\overline{E \cap W} = E$ *for any closed subsets E of X.*

Suppose that α_X is an isomorphism of ringed spaces for a ringed space $(X, \mathscr{F}_X)$. Then we can define a morphism β_X of ringed spaces by

(17) $$\beta_X = \alpha_X^{-1} \circ ti: \quad t(X_{\mathrm{cl}}) \to X.$$

Here $i: X_{\mathrm{cl}} \to X$ is the inclusion morphism of ringed spaces. Thus the following diagram commutes:

(18)



Therefore $(X, \mathscr{F}_X)$ satisfies the condition (1) if and only if any irreducible closed subset of X has a unique generic point in X and $\overline{E \cap X_{\mathrm{cl}}} = E$ for any closed subsets E of X.

Let us introduce the following two categories.

$\mathscr{C}_1$: the category of ringed spaces satisfying the condition (1) and morphisms of ringed spaces satisfying the condition (2).

$\mathscr{C}_2$: the full subcategory of ringed spaces consisting of objects which satisfy the separable condition T_1.

Then the functor $t: \mathscr{C}_2 \to \mathscr{C}_1$ gives an equivalence of categories, and $X \mapsto X_{\mathrm{cl}}$ is the inverse functor of t.

§2. In this section we study the relationship between the functor t and intersection sheaves.

Let K be a field, A a subring of K, X an irreducible topological space and

$s: X \to \mathrm{Loc}(K|A)$ a continuous mapping. For any non empty open subsets V of X, define $\mathcal{O}_X(V)$ to be $\bigcap_{x\in V} s(x)$. Thus we obtain an integral local ringed space $(X, \mathcal{O}_X)$. Then $\mathcal{O}_X$ is said to be an intersection sheaf of X with respect to the mapping s (see [6]).

LEMMA 5. *Let K, A, X and s be as above. For any irreducible subset Y of X, we put $\xi_Y = \bigcup_{x\in Y} s(x) \subset K$. Then*

(i) *$\xi_Y \in \mathrm{Loc}(K|A)$ and $\overline{s(Y)} = \overline{\{\xi_Y\}}$ in $\mathrm{Loc}(K|A)$. Therefore s is dominant if and only if $\xi_X = K$.*

(ii) *If Y is dense in X, then $\xi_Y = \xi_X$.*

(iii) *Let $\mathcal{O}_X$ be the intersection sheaf of X with respect to s. Then $\mathcal{O}_{X,Y} \simeq \xi_Y$. Thus $\mathrm{Rat}\,X \simeq \xi_X$. In what follows, we identify the above two rings. Then $\mathrm{dom}(\alpha) = s^{-1}(\mathrm{Loc}(K|A[\alpha]))$ for any $\alpha \in \mathrm{Rat}\,X \subset K$.*

(iv) *$(X, \mathcal{O}_X)$ satisfies the condition (8) in [5] if and only if $\mathrm{Rat}\,X$ is a field.*

PROOF. For $\alpha \in K$, we put $Y(\alpha) = Y \cap s^{-1}(\mathrm{Loc}(K|A[\alpha]))$. Then $\alpha \in \xi_Y$ if and only if $Y(\alpha) \neq \emptyset$.

(i) For any $\alpha, \beta \in \xi_Y$, there exists $x \in Y$ such that $\alpha, \beta \in s(x)$. Thus ξ_Y is a subring of K. Note that $\xi_Y^\times = \bigcup_{x\in Y} s(x)^\times$. Since $\xi_Y - \xi_Y^\times$ is an ideal of ξ_Y, we obtain $\xi_Y \in \mathrm{Loc}(K|A)$. Moreover,

$$(19) \qquad \bigcap_{x\in Y} m(s(x)) \subset m(\xi_Y)\,.$$

It is clear that $s(Y) \subset \overline{\{\xi_Y\}}$. If we put $V = \mathrm{Loc}(K|A[\alpha_1, \cdots, \alpha_r])$ for any $\alpha_1, \cdots, \alpha_r \in K$, then $Y \cap s^{-1}(V) = Y(\alpha_1) \cap \cdots \cap Y(\alpha_r)$. If $\xi_Y \in V$, then $Y(\alpha_i) \neq \emptyset$ $(i = 1, \cdots, r)$. Since Y is irreducible, we obtain $Y \cap s^{-1}(V) \neq \emptyset$. Thus $s(Y) \cap V \neq \emptyset$ and hence $\xi_Y \in \overline{s(Y)}$. Therefore $\overline{s(Y)} = \overline{\{\xi_Y\}}$.

(ii) Since $s(X) = s(\bar{Y}) \subset \overline{s(Y)} \subset \overline{s(X)}$, we have $\overline{s(Y)} = \overline{s(X)}$. By (i), we see that $\overline{\{\xi_Y\}} = \overline{s(Y)} = \overline{s(X)} = \overline{\{\xi_X\}}$. Thus $\xi_Y = \xi_X$.

(iii) The mapping $\xi_Y \to \mathcal{O}_{X,Y}$ defined by $\alpha \mapsto \langle X(\alpha), \alpha \rangle_Y$ is an isomorphism of rings.

(iv) The "only if" part is verified from Lemma 7 in [5]. For "if" part, it suffices to prove that $\bigcap_{x\in V} \pi_V(x) = 0$ for any non empty open sets V of X, by Lemma 3 in [5]. By (19), (ii) and (iii), $\bigcap_{x\in V} \pi_V(x) = \bigcap_{x\in V} m(s(x)) \subset m(\xi_V) = m(\xi_X) = m(\mathrm{Rat}\,X) = 0$. Q.E.D.

LEMMA 6. *Let K be a field, A a subring of K and X a topological space. Then*

(i) *the mapping: $C(tX, \mathrm{Loc}(K|A)) \to C(X, \mathrm{Loc}(K|A))$ defined by $r \mapsto r \circ \alpha_X$ is a bijection. Here $C(X, Y)$ is the set of continuous mappings from X to Y.*

(ii) *Assume that X is irreducible. Let $s = r \circ \alpha_X$ and let $\mathcal{O}_X$ (resp. $\mathcal{O}_{tX}$) be the intersection sheaf of X (resp. tX) with respect to s (resp. r). Then $\mathcal{O}_X = \alpha_X^{-1}\mathcal{O}_{tX}$, $\mathcal{O}_{tX} = \alpha_{X*}\mathcal{O}_X$ and $\mathrm{Rat}(X, \mathcal{O}_X) = \mathrm{Rat}(tX, \mathcal{O}_{tX})$.*

PROOF. (i) Since $\mathrm{Loc}(K|A)$ is a T_0-space, the mapping in question is injective. For any continuous mapping $s: X \to \mathrm{Loc}(K|A)$, we put $r(Y) = \xi_Y$ for $Y \in tX$. Then $r: tX \to \mathrm{Loc}(K|A)$ is continuous and $s = r \circ \alpha_X$.

(ii) is induced from Lemma 2 in [6], Lemmas 3 and 5.

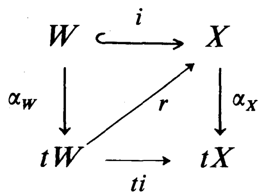
COROLLARY. (i) *If* $(X, \mathcal{O}_X)$ *is a local ringed space and* $\mathcal{O}_X$ *is an intersection sheaf, then* $t(X, \mathcal{O}_X)$ *is also a local ringed space and* $\mathcal{O}_{tX}$ *is an intersection sheaf.*

(ii) *If* $(X, \mathcal{O}_X)$ *is an integral local ringed space satisfying the condition* (8) *in* [5], *then* $t(X, \mathcal{O}_X)$ *is an integral local ringed space satisfying the condition* (8) *in* [5].

LEMMA 7. *Let* W *be a subset of* $\mathrm{Loc}(K|A)$. *Take a continuous mapping* $r: tW \to \mathrm{Loc}(K|A)$ *such that* $r \circ \alpha_W$ *is the inclusion mapping from* W *to* $\mathrm{Loc}(K|A)$. *If we put* $X = \mathrm{Im}\, r$, *then*

(i) $W \subset X$ *and* $r: tW \to X$ *is a homeomorphism. Moreover the mapping*: $X \to tW$ *defined by* $R \mapsto \overline{\{R\}} \cap W$ *is the inverse mapping of* r.

(ii) *Assume that* W *is irreducible. Let* $\mathcal{O}_W$ (*resp.* $\mathcal{O}_X$) *be the intersection sheaf of* W (*resp.* X) *with respect to the inclusion mapping. Then* $r: t(W, \mathcal{O}_W) \to (X, \mathcal{O}_X)$ *is an isomorphism of local ringed spaces, and the following diagram commutes.*



The proof is complete from Lemmas 5 and 6.

REMARK. If $W = X_{\mathrm{cl}}$, then $r = \beta_X$ by (18).

§3. Using some elementary properties of Hilbert rings, we shall prove Theorems 1, 2 and 3.

For Hilbert rings, the next two lemmas are well-known (see [1]).

LEMMA 8. *The following four conditions for a ring* A *are equivalent*:

(a) *A is a Hilbert ring.*

(b) $\overline{F \cap \text{m-Spec}\, A} = F$, *for any closed subsets* F *of* $\mathrm{Spec}\, A$.

(c) *If* $\varphi: A \to B$ *is a ring homomorphism of finite type and* $m \in \text{m-Spec}\, B$, *then* $\varphi^{-1}(m) \in \text{m-Spec}\, A$.

(d) *For any* $f \in A$, *let* $\varphi: A \to A_f$ *denote the canonical mapping. Then* $\varphi^{-1}(m) \in \text{m-Spec}\, A$ *for any* $m \in \text{m-Spec}\, A_f$.

LEMMA 9. *Let* A *be a ring and* B *a ring integral over* A. *Then* A *is Hilbert if and only if* B *is Hilbert.*

COROLLARY. *Suppose that* A *and* B *are subrings of a field* K *which satisfy* $\mathrm{Zar}(K|A) = \mathrm{Zar}(K|B)$. *Then* A *is Hilbert if and only if* B *is Hilbert.*

LEMMA 10. *Let* $(W, \mathcal{O}_W)$ *be a local ringed space such that* $\pi_W(W) = \text{m-Spec}\, \mathcal{O}_W(W)$.

Then $(W, \mathcal{O}_W)$ *satisfies the condition* (3) *if and only if* $\mathcal{O}_W(W)$ *is a Hilbert ring.*

The proof is obvious from Lemma 8.

LEMMA 11. *Let* A *be an integral domain,* $W=\text{m-Spec}\,A$ *and* $\mathcal{O}_W=\tilde{A}|_W$. *If* W *is irreducible, then* $\mathcal{O}_W(W)=A$ *and* $\pi_W(W)=\text{m-Spec}\,\mathcal{O}_W(W)$.

PROOF. Since A is integral, the structure sheaf $\tilde{A}$ on $\text{Spec}\,A$ is the intersection sheaf with respect to the mapping: $\text{Spec}\,A \to \text{Loc}(QA|A)$ defined by $P \mapsto A_P$. Thus $\mathcal{O}_W$ is also an intersection sheaf and hence $\mathcal{O}_W(W)=\bigcap_{m\in W} A_m=A$. Since $\pi_W: W \to \text{Spec}\,A$ is the inclusion mapping, we obtain $\pi_W(W)=\text{m-Spec}\,A$. Q.E.D.

Now the proof of Theorem 1 is complete from Lemmas 8, 10 and 11.

LEMMA 12. *Let* K *be a field,* A *a subring of* K *and* $W=\text{Loc}(K|A)_{\text{cl}}$. *Then* $W=\{A_m \mid m\in \text{m-Spec}\,A\}$.

The proof is easy.

COROLLARY. *If* W *is irreducible, then* $\mathcal{O}_W(W)=A$ *and* $\pi_W(W)=\text{m-Spec}\,\mathcal{O}_W(W)$. *Here* $\mathcal{O}_W$ *is the intersection sheaf of* W *with respect to the inclusion mapping.*

PROOF OF THEOREM 2. (i) First, we show the "only if" part. Note that $X=\text{Im}\,r$ by Lemma 7. For any $P\in \text{Spec}\,A$, there exists $Y\in t(\text{m-Spec}\,A)$ such that $A_P=\bigcup_{m\in Y} A_m$. Then $P=\bigcap_{m\in Y} m$. Thus A is Hilbert. For any $P\in\text{Spec}\,A$, there exists $R\in \text{Zar}(K|A)$ such that R dominates A_P. Since $\text{Loc}(K|A)=\{A_P \mid P\in \text{Spec}\,A\}$, we can take $Q\in\text{Spec}\,A$ such that $R=A_Q$. Then $P=Q$ and hence $A_P=R\in\text{Zar}(K|A)$. Thus A is a Prüfer ring with quotient field K. Then we check the "if" part. Since $\text{Loc}(K|A)=\text{Zar}(K|A)\simeq\text{Spec}\,A$ and $\beta_{\text{Spec}\,A}: t(\text{m-Spec}\,A)\to\text{Spec}\,A$ is an isomorphism of local ringed spaces, $\beta_X=r: tW\to X$ is also an isomorphism of local ringed spaces.

(ii) is derived from Lemmas 8, 10 and the corollary to Lemma 12. Q.E.D.

LEMMA 13. *Let* K *be a field,* A *a subring of* K, $X=\text{Zar}(K|A)$ *and* $W=X_{\text{cl}}$. *Suppose that* $\mathcal{O}_X$ *is the intersection sheaf of* X *with respect to the inclusion mapping and* $\mathcal{O}_W=\mathcal{O}_X|_W$.

(i) *If* W *is irreducible, then* $\mathcal{O}_W(W)=\mathcal{O}_X(X)$ *and* $\pi_W(W)=\text{m-Spec}\,\mathcal{O}_W(W)$.

(ii) *The following three conditions are equivalent.*

(a) $\bar{W}=X$.

(b) W *is irreducible and* $K=\text{Rat}\,W$.

(c) *For any intermediate ring* B *between* A *and* K *such that* B *is of finite type over* A, *there exists* $m\in\text{m-Spec}\,B$ *such that* $A\cap m\in\text{m-Spec}\,A$.

(iii) *The following two conditions are equivalent.*

(a′) $r: tW\to X$ *is an isomorphism of local ringed spaces.*

(c′) *If a ring* B *is an intermediate ring between* A *and* K *such that* B *is of finite type over* A *and* $m\in\text{m-Spec}\,B$, *then* $A\cap m\in\text{m-Spec}\,A$.

PROOF. (i) is induced from Lemma 7 and Proposition 8 in [4].

(ii) The equivalence between (a) and (b) is verified from Lemma 5. (a) $\Rightarrow$ (c): let $V = \mathrm{Zar}(K|B)$ for any B. Then $V \cap W \neq \varnothing$. Take $R \in V \cap W$ and let $m = B \cap m(R)$. Since $A/A \cap m \subset B/m \subset R/m(R)$ are integral extensions by Lemma 7 in [4], we obtain $m \in \text{m-Spec} B$ and $A \cap m \in \text{m-Spec} A$. (c) $\Rightarrow$ (a): it suffices to prove that $\mathrm{Zar}(K|B) \cap W \neq \varnothing$ for any B. There exists $m \in \text{m-Spec} B$ such that $A \cap m \in \text{m-Spec} A$. By a weak form of Hilbert's zero-point theorem, B/m is integral over $A/A \cap m$. On the other hand, since the mapping $\Phi_{K|B}$: $\mathrm{Zar}(K|B)_{\mathrm{cl}} \to \text{m-Spec} B$ is onto, there exists $R \in \mathrm{Zar}(K|B)_{\mathrm{cl}}$ such that $m = B \cap m(R)$. Then $R/m(R)$ is integral over $A/A \cap m$. By Lemma 7 in [4], we have $R \in W$. Therefore $\mathrm{Zar}(K|B) \cap W \neq \varnothing$.

(iii) (a′) $\Rightarrow$ (c′): given B and m, there exists $R_0 \in \mathrm{Zar}(K|B)$ such that $m = B \cap m(R_0)$, since $\Phi_{K|B}$ is surjective. We let $E = \overline{\{R_0\}}^{-}$. By the corollary to Lemma 4, we obtain $\overline{E \cap W} = E \ni R_0$, and hence $\mathrm{Zar}(K|B) \cap E \cap W \neq \varnothing$. If $R \in \mathrm{Zar}(K|B) \cap E \cap W$, then $m = B \cap m(R)$ and $R/m(R)$ is integral over $A/A \cap m$. Thus $A \cap m \in \text{m-Spec} A$. (c′) $\Rightarrow$ (a′): it suffices to prove $r(tW) = X$ by Lemma 7. The inclusion $r(tW) \subset X$ is easy. Conversely, for any $R_0 \in X$, we put $Y = \overline{\{R_0\}} \cap W$. Let $V = \mathrm{Zar}(K|B)$ for any intermediate ring B between A and K such that B is of finite type over A. If $R_0 \in V$, then $B \subset R_0$. By Proposition 8 in [4], there exists $R \in \mathrm{Zar}(K|B)_{\mathrm{cl}}$ such that $R \subset R_0$. By Lemma 7 in [4], $R/m(R)$ is integral over $B/B \cap m(R)$, and so $B \cap m(R) \in \text{m-Spec} B$. By the assumption (c′), we have $A \cap m(R) \in \text{m-Spec} A$. By a weak form of Hilbert's zero-point theorem, $B/B \cap m(R)$ is integral over $A/A \cap m(R)$. Therefore $R \in W$ and hence $V \cap Y \neq \varnothing$. This implies $R_0 \in \bar{Y}$ and $\overline{\{R_0\}} = \bar{Y}$. Since $Y \in tW$ and $R_0 = r(Y)$, we obtain $X = r(tW)$. Q.E.D.

PROOF OF THEOREM 3. The equivalence between (a) and (c) is verified from Lemmas 8 and 13. The equivalence between (b) and (c) is induced from Lemmas 9, 10 and 13.

§4. Here we characterize the local ringed spaces $\mathrm{Zar}(K|A)_{\mathrm{cl}}$ explicitly, and prove Theorem 4.

For an integral domain A and a local ringed space $(W, \mathcal{O}_W)$, we introduce the following six conditions:

(20) W satisfies the separable condition T_0.

(21) $(W, \mathcal{O}_W)$ is an integral local ringed space satisfying the condition (8) in [5].

(22) $(W, \mathcal{O}_W)$ is a local ringed space over $\mathrm{Spec} A$ and the structure morphism is dominant.

REMARK. By (21) and (22), $\mathrm{Rat}\, W$ is a field and $A \hookrightarrow \mathcal{O}_W(W) \hookrightarrow \mathcal{O}_{W,x} \hookrightarrow \mathrm{Rat}\, W$ for any $x \in W$.

(23) The topology of W is generated by $\{\mathrm{dom}(\alpha) \mid \alpha \in \mathrm{Rat}\, W\}$.

(24) For any $x \in W$, the stalk $\mathcal{O}_{W,x}$ is a valuation ring of $\mathrm{Rat}\, W$

and $\mathcal{O}_{W,x}/m(\mathcal{O}_{W,x})$ is an integral extension over $A/A \cap m(\mathcal{O}_{W,x})$.

(25) If R is a valuation ring of Rat W which contains A, then there exists $x \in W$ such that $\mathcal{O}_{W,x} \subset R$.

LEMMA 14. (i) *Let K be a field, A a subring of K and $W = \mathrm{Zar}(K|A)_{\mathrm{cl}}$. If $\bar{W} = \mathrm{Zar}(K|A)$, then $(W, \mathcal{O}_W)$ satisfies the conditions* (20), (21), (22), (23), (24), (25) *and $K = \mathrm{Rat}\, W$.*

(ii) *Conversely, suppose that an integral domain A and a local ringed space $(W, \mathcal{O}_W)$ satisfy the conditions* (20), (21), (22), (23), (24) *and* (25). *If we put $K = \mathrm{Rat}\, W$, then K is a field containing A that satisfies $\overline{\mathrm{Zar}(K|A)_{\mathrm{cl}}} = \mathrm{Zar}(K|A)$ and $(W, \mathcal{O}_W) \simeq \mathrm{Zar}(K|A)_{\mathrm{cl}}$.*

PROOF. (i) is induced from Lemma 7, Proposition 8 in [4] and Lemma 5.

(ii) By Lemmas 6, 7 in [5], Lemma 3 in [6] and (21), W is irreducible, $K = \mathrm{Rat}\, W$ is a field and $\mathcal{O}_W$ is the intersection sheaf of W with respect to Ψ_W. By (22), A is a subring of K. Note that (20), (23) induce that Ψ_W is an into-homeomorphism, and (24), (25) imply that $\Psi_W(W) = \mathrm{Zar}(K|A)_{\mathrm{cl}}$. Thus $W \simeq \mathrm{Zar}(K|A)_{\mathrm{cl}}$. By Lemma 5, we obtain $\overline{\mathrm{Zar}(K|A)_{\mathrm{cl}}} = \overline{\{K\}} = \mathrm{Zar}(K|A)$. Q.E.D.

Here we consider the following two categories for an integral ring A.

$\mathscr{C}'_1(A)$: the category of local ringed spaces $(X, \mathcal{O}_X)$ satisfying the conditions (29), (30), (32), (33), (35) and (36) in [5] and morphisms $f: X \to Y$ of local ringed spaces over Spec A satisfying the condition (2).

$\mathscr{C}'_2(A)$: the full subcategory of local ringed spaces over Spec A consisting of local ringed spaces $(W, \mathcal{O}_W)$ which satisfy the conditions (20), (21), (22), (23), (24) and (25).

By Theorem 1 in [5], the objects of $\mathscr{C}_1(A)$ coincide with those of $\mathscr{C}'_1(A)$. Moreover,

LEMMA 15. *Let A be an integral Hilbert ring. Then*

(i) *the functor $t: \mathscr{C}'_2(A) \to \mathscr{C}'_1(A)$ gives an equivalence of categories.*

(ii) *A local ringed space $(W, \mathcal{O}_W)$ is an object of $\mathscr{C}_2(A)$ if and only if $(W, \mathcal{O}_W)$ satisfies the conditions* (20), (21), (22), (23), (24) *and* (25). *Therefore the category $\mathscr{C}_i(A)$ is a subcategory of $\mathscr{C}'_i(A)$ obtained by assuming morphisms to be dominant* ($i = 1, 2$).

PROOF. (i) is induced from Theorem 1 in [5], Lemmas 1, 14 and Theorem 3. (ii) is obvious from Theorem 1 in [5] and Lemma 14.

REMARK. Let $\mathscr{C}'_0(A)$ be the category of projective fields over A, in which morphisms are places that fix all elements of A. Then $\mathscr{C}'_0(A)$ and $\mathscr{C}'_1(A)$ are anti-equivalent (see [5], Lemma 11).

PROOF OF THEOREM 4. (i) is verified from Theorem 1 in [5], Lemmas 2 and 15. In order to show (ii), we first notice by Theorem 2 in [5];

(26) $$\mathrm{Zar}(K|k)\simeq \mathrm{proj.lim}\, X\,,$$

where X runs over all integral schemes proper over Spec$k$ with rational function field K. By Examples, (ii) and Theorem 3, all objects and morphisms in (26) belong to the category $\mathscr{C}_1$, and X_{cl} become complete algebraic varieties V. Since $\mathscr{C}_1$ and $\mathscr{C}_2$ are equivalent, we obtain

$$\mathrm{Zar}(K|k)_{\mathrm{cl}}\simeq \mathrm{proj.lim}\, V\,. \qquad \text{Q.E.D.}$$

References

[1] R. GILMER, *Multiplicative Ideal Theory*, Marcel Dekker (1972).
[2] A. GROTHENDIECK and J. DIEUDONNÉ, *Eléments de Géométrie Algébrique*, I.H.E.S. (1960–1967).
[3] R. HARTSHORNE, *Algebraic Geometry*, Springer-Verlag (1977).
[4] K. SEKIGUCHI, Prüfer domain and affine scheme, Tokyo J. Math. **13** (1990), 259–275.
[5] K. SEKIGUCHI, Ringed spaces of valuation rings and projective limits of schemes, Tokyo J. Math. **16** (1993), 191–203.
[6] K. SEKIGUCHI, Differential forms on ringed spaces of valuation rings, Tokyo J. Math. **18** (1995), 133–145.

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