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Extrapolation Theorem on Some Quasi-Banach Spaces

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Abstract. In [5], the author has proved extended sharp extrapolation theorem on L^p spaces with $\sum$-method ([1]), which asserted $\sum_{1<p<q}((p-1)^{-\alpha}L^p)^{p/q}=(L\log^\alpha L+L^q)^{1/q}$. In the present paper, on that result, we shall consider the case $q\approx 1$.

1. Introduction and result.

Let (Ω, μ) be a σ-finite measure space. In extrapolation theory on L^p-spaces, we treat the operator which satisfies the following assumptions.

ASSUMPTION. Let $1<q<\infty$ and fix it.

(1) T is a sub-additive operator on $L^p(\Omega, \mu)$ for $1<\forall p<q$, i.e. $|T(f+g)|\le|Tf|+|Tg|$ a.e. for any $f, g\in L^p(\Omega, \mu)$.

(2) For any $f\in L^p(\Omega, \mu)$,

$$\left[\int_\Omega |Tf(x)|^p d\mu(x)\right]^{1/p}\le\frac{A}{(p-1)^\alpha}\left[\int_\Omega |f(x)|^p d\mu(x)\right]^{1/p}. \tag{1.1}$$

Here, positive constants A and α are independent of p and f.

In real analysis, we can find many operators satisfying such conditions (§3). In [5], for such operator, the author has proved the following extrapolation theorem:

THEOREM A. *Let T satisfy the assumption above. Then, for any $f\in L(\log L)^\alpha+L^q(\Omega, \mu)$,*

$$\begin{aligned}&\int_{|Tf|\le 1}|Tf(x)|^q d\mu(x)+\int_{|Tf|>1}|Tf(x)|d\mu(x)\\ &\le\frac{C_A}{(q-1)^\alpha}\left[\int_{|f|\le 1}|f(x)|^q d\mu(x)+\int_{|f|>1}|f(x)|(1+\log|f(x)|)^\alpha d\mu(x)\right].\end{aligned} \tag{1.2}$$

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As a corollary of this result, we can get S. Yano's classical extrapolation theorem:

THEOREM B ([9]). *Let (Ω, μ) be a finite measure space. Then, for the operator T satisfying the assumption above and the function $f \in L\log^{\alpha} L$,*

$$\int_{\Omega} |Tf(x)|d\mu(x) \le C_{q,\alpha,A}\left[\int_{\Omega} |f(x)|(1+\log^{+}|f(x)|)^{\alpha}d\mu(x)+\mu(\Omega)\right]. \tag{1.3}$$

The original proof of Theorem B depends upon the property

$$L^1(\Omega) \supset L^p(\Omega) \supset L^q(\Omega) \qquad \text{for any} \quad 1<p<q .$$

After the proof of Theorem B, $\sum_{0<\theta<1} A_\theta$, the $\sum$-extrapolation space of strong compatible family of (quasi-)Banach spaces $\{A_\theta\}_{0<\theta<1}$, was defined as

$$\sum_{0<\theta<1} A_\theta = \text{the closure of the linear hull of} \bigcup_{0<\theta<1} A_\theta$$

with the (quasi-)norm

$$\|a\|_{\Sigma} = \inf\{\textstyle\sum \|a_n\|_{\theta_n} : a=\sum a_n,\ a_n \in A_{\theta_n}\} .$$

It was proved that this space is the "widest" space (see [1, Proposition 2.3]).

For $\alpha \ge 0$, the author proved $\sum_{1<p<q}((p-1)^{-\alpha}L^p(\Omega))^{p/q}$ is the space of all measurable function f satisfying

$$\left[\int_{|f|\le 1} |f(x)|^q d\mu(x) + \int_{|f|>1} |f(x)|(1+\log|f(x)|)^{\alpha}d\mu(x)\right]^{1/q} < \infty$$

and Theorem A was proved from it. Here, each $((p-1)^{-\alpha}L^p(\Omega))^{p/q}$ is the space of all p-th integrable functions on (Ω, μ) with quasi-norm

$$((p-1)^{-\alpha}\|f\|_{L^p(\Omega,\mu)})^{p/q} = \left[\frac{1}{(p-1)^{p\alpha}}\int_{\Omega} |f(x)|^p d\mu(x)\right]^{1/q} .$$

However, the original purpose of these studies is the approach to $L^1(\Omega)$. As is known, we can never get (1.2) for $q=1$. For example, we shall consider the Hilbert transform H. Let $\chi(x)=1$ for $0\le x\le 1$, and $\chi(x)=0$ elsewhere. Then it is easy to show $H\chi \in L^p(\boldsymbol{R})$ for $p>1$ but $\notin L^1(\boldsymbol{R})$. So, instead of the case $q=1$, we shall investigate the following function classes.

DEFINITION (cf. [3]). Let $\alpha, \beta \ge 0$. $f \in \mathscr{L}^{*}_{\beta,\alpha}$ if and only if f is a measurable function on Ω such that

$$\int_{|f|\le 1} \frac{|f(x)|}{(1-\log|f(x)|)^{\beta}}\, d\mu(x) + \int_{|f|>1} |f(x)|(1+\log|f(x)|)^{\alpha}d\mu(x) < \infty .$$

Once, the author tried to get some estimation on these classes ([3, Theorem 2]). In this paper, we shall prove the following result, which is the sharpened one.

THEOREM. *Let (Ω, μ) be a σ-finite measure space and let T be an operator satisfying the assumption above. If $f \in \mathscr{L}^*_{\varepsilon,\alpha}$, then $Tf \in \mathscr{L}^*_{\alpha+\varepsilon,0}$ for arbitrary $\varepsilon > 0$. Moreover,*

$$(1.4)\quad \int_{|Tf|\leq 1} \frac{|Tf(x)|}{(1-\log|Tf(x)|)^{\alpha+\varepsilon}} d\mu(x) + \int_{|Tf|>1} |Tf(x)| d\mu(x)$$

$$\leq C_{q,\alpha,\varepsilon,A}\left[\int_{|f|\leq 1} \frac{|f(x)|}{(1-\log|f(x)|)^{\varepsilon}} d\mu(x) + \int_{|f|>1} |f(x)|(1+\log|f(x)|)^{\alpha} d\mu(x)\right].$$

Here, the positive constant $C_{q,\alpha,\varepsilon,A}$ depends only on q, α, ε and A.

Moreover, we can treat the spaces $\mathscr{L}^*_{\beta,\alpha}$ as Orlicz spaces. Put

$$(1.5)\quad \Phi_{\beta,\alpha}(t) = \begin{cases} \dfrac{|t|}{(\beta+1)(1-\log|t|)^{\beta}} & (0 \leq |t| \leq 1) \\ \dfrac{t(1+\log|t|)^{\alpha}}{\alpha+1} + \dfrac{\alpha-\beta}{(\alpha+1)(\beta+1)} & |t| > 1) \end{cases}$$

and

$$\Phi_{\beta,\alpha}(L) = \left\{ f : \int_{\Omega} \Phi_{\beta,\alpha}(f(x)) d\mu(x) < \infty \right\}$$

for all $\alpha, \beta \geq 0$. Then, it is easy to show $\Phi_{\beta,\alpha}(L) = \mathscr{L}^*_{\beta,\alpha}$ and each $\Phi_{\beta,\alpha}(L)$ is an Orlicz space with the Luxemberg norm

$$(1.6)\quad \|f\|_{\Phi_{\beta,\alpha}(L)} = \inf\left\{\lambda > 0 : \int_{\Omega} \Phi_{\beta,\alpha}\left(\frac{f(x)}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

Then we can get

COROLLARY. *Let (Ω, μ) be a σ-finite measure space and let T be an operator satisfying the assumption above. If $f \in \Phi_{\varepsilon,\alpha}(L)$, then $Tf \in \Phi_{\alpha+\varepsilon,0}(L)$ for arbitrary $\varepsilon > 0$. Moreover,*

$$(1.7)\quad \|Tf\|_{\Phi_{\alpha+\varepsilon,0}(L)} \leq C \|f\|_{\Phi_{\varepsilon,\alpha}(L)}.$$

Here the positive constant C is independent of f.

This corollary can be proved similarly to [4, Theorem 1 and 2] and we here omit it.

2. Proof of the theorem.

In this section, we shall prove our theorem. First, we shall prove the following lemma.

LEMMA. *Let $f \in \mathscr{L}^*_{\beta,\alpha}$ and $\beta > 0$. Then,*

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\beta}}\int_{|f|\le 1}|f(x)|^{1+(q-1)/n}d\mu(x)\overset{q,\beta}{\approx}\int_{|f|\le 1}\frac{|f(x)|}{(1-\log|f(x)|)^{\beta}}d\mu(x)\,. \tag{2.1}$$

Here, "$X\overset{q,\beta}{\approx}Y$" means that there exist two positive constants c_1, c_2 which depend only on q and β such that $c_1Y\le X\le c_2Y$.

PROOF. It suffices to prove

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\beta}}a^{(q-1)/n}\overset{q,\beta}{\approx}\frac{1}{(1-\log a)^{\beta}} \tag{2.2}$$

for any $0<a\le 1$. First, we consider the case $0<a<e^{-1}$. For $n=1, 2, 3, \cdots$,

$$\frac{1}{2^{1+\beta}}\frac{a^{(q-1)/n}}{n^{1+\beta}}\le\int_n^{n+1}\frac{a^{(q-1)/t}}{t^{1+\beta}}dt\le 2^{1+\beta}\frac{a^{(q-1)/(n+1)}}{(n+1)^{1+\beta}}\,.$$

So,

$$\sum_{n=1}^{\infty}\frac{a^{(q-1)/n}}{n^{1+\beta}}\overset{\beta}{\approx}\int_1^{\infty}\frac{a^{(q-1)/t}}{t^{1+\beta}}dt=(q-1)^{-\beta}\left(\int_0^{-(q-1)\log a}x^{\beta-1}e^{-x}dx\right)\frac{1}{(-\log a)^{\beta}}\,.$$

For simplicity, we denote

$$\Gamma(\beta;\,t)=\int_0^t x^{\beta-1}e^{-x}dx$$

for any $0<t\le\infty$. As is known, $\Gamma(\beta;\,\infty)=\Gamma(\beta)$. Then, we have

$$\frac{(q-1)^{-\beta}\Gamma(\beta;\,q-1)}{(1-\log a)^{\beta}}\le\frac{(q-1)^{-\beta}\Gamma(\beta;\,-(q-1)\log a)}{(-\log a)^{\beta}}\le\frac{(q-1)^{-\beta}2^{\beta}\Gamma(\beta)}{(1-\log a)^{\beta}}\,. \tag{2.3}$$

If $e^{-1}<a\le 1$, we have

$$e^{1-q}\sum_{n=1}^{\infty}\frac{1}{n^{1+\beta}}\le\sum_{n=1}^{\infty}\frac{1}{n^{1+\beta}}a^{(q-1)/n}\le\sum_{n=1}^{\infty}\frac{1}{n^{1+\beta}}\approx 1+\frac{1}{\beta}\,, \tag{2.4}$$

$$\frac{1}{2}<\frac{1}{(1-\log a)^{\beta}}\le 1\,.$$

Now, we conclude (2.2).

Now, we prove our theorem. From Theorem A, we have

$$\begin{aligned}&\int_{|Tf|\le 1}|Tf(x)|^{p}d\mu(x)+\int_{|Tf|>1}|Tf(x)|d\mu(x)\\&\le C\left(\frac{1}{p-1}\right)^{\alpha}\left[\int_{|f|\le 1}|f(x)|^{p}d\mu(x)+\int_{|f|>1}|f(x)|(1+\log|f(x)|)^{\alpha}d\mu(x)\right],\end{aligned} \tag{2.5}$$

for any $1<p\leq q$. Here the constant C depends only on A. Put $p=1+(q-1)/n$, $n\in \boldsymbol{N}$. Multiplying both sides of (2.5) by $n^{-(\alpha+1+\varepsilon)}$ and summing them up with respect to n, we get

$$\text{(2.6)}\quad \sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1+\varepsilon}}\int_{|Tf|\leq 1}|Tf(x)|^{1+(q-1)/n}d\mu(x)+\int_{|Tf|>1}|Tf(x)|d\mu(x)$$

$$\leq C_{q,\alpha,\varepsilon,A}\left[\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\int_{|f|\leq 1}|f(x)|^{1+(q-1)/n}d\mu(x)+\int_{|f|>1}|f(x)|(1+\log|f(x)|)^{\alpha}d\mu(x)\right],$$

where the positive constant C depends upon only q, α, β and A. By using the lemma above,

$$\text{the first term of RHS. of (2.6)} \overset{q,\varepsilon}{\approx} \int_{|f|\leq 1}\frac{|f(x)|}{(1-\log|f(x)|)^{\varepsilon}}\,d\mu(x)\,,$$

$$\text{the first term of LHS. of (2.6)} \overset{q,\alpha,\varepsilon}{\approx} \int_{|Tf|\leq 1}\frac{|Tf(x)|}{(1-\log|Tf(x)|)^{\alpha+\varepsilon}}\,d\mu(x)\,.$$

Now, we have completed the proof.

REMARK 1. We can never get this theorem for $\varepsilon=0$. For example, $H\chi$, defined in §1, is not a member of $\mathscr{L}^{*}_{1,0}(\boldsymbol{R})$.

REMARK 2. Assume $q\geq 2$. Multiplying the both sides of (2.5) by $n^{-1-\alpha}(\log(1+n))^{-(1+\delta)}$, $\delta>0$, we can prove a better estimate

$$\int_{|Tf|\leq 1}\frac{|Tf(x)|}{(1-\log|Tf(x)|)^{\alpha}(1+\operatorname{LogLog}|Tf(x)|)^{1+\delta}}\,d\mu(x)+\int_{|Tf|>1}|Tf(x)|d\mu(x)$$

$$\leq C_{q,\alpha,\delta,A}\left[\int_{|f|\leq 1}\frac{|f(x)|}{(1+\operatorname{LogLog}|f(x)|)^{1+\delta}}\,d\mu(x)+\int_{|f|>1}|f(x)|(1+\log|f(x)|)^{\alpha}d\mu(x)\right].$$

Here, $\operatorname{LogLog}y=\log(1-\log y)$ for any $0<y\leq 1$.

By using other suitable sequence x_n, $\{x_n\}\in l^1$, we can get similar results.

REMARK 3. T. Miyamoto ([2]) has proved that "If the operator T is of weak-type (1, 1) and (q, q) for some $q>1$, then T satisfies (1.4) for $\alpha=1$". But our theorem includes his result because the Marcinkiewicz interpolation theorem says "If the operator T is of weak-type (1, 1) and (q, q) for some $q>1$, then T satisfies (1.1) for $\alpha=1$".

3. Applications.

In this section, we state some applications of our theorem.

EXAMPLE 1 (Maximal function). Let $\mathscr{R}_n$ denote the collection of all rectangles in $\boldsymbol{R}^n$ whose sides are parallel to the coordinate axes for $n\geq 2$ and let $\mathscr{R}_1$ denote the collection of a finite open interval in $\boldsymbol{R}^1$. For any $f\in L^1_{loc}(\boldsymbol{R}^n)$, we consider the maximal function

$$\mathcal{M}f(x)=\sup_{\substack{R\ni\mathcal{R}_n\\ R\ni x}}\frac{1}{|R|}\left|\int_R f(x-y)dy\right|.$$

Here, $|R|$ denotes the Lebesgue measure of the set R. Then, as is known, the operator $\mathcal{M}$ is of weak-type (1, 1) for $n=1$. But for $n\geq 2$, $\mathcal{M}$ is not of weak-type (1, 1) (see [7, X §2.3]). Of course, even if $n\geq 2$, $\mathcal{M}$ is of type (p, p) for $p>1$. Moreover,

$$\|\mathcal{M}f\|_{L^p(\boldsymbol{R}^n)}\leq\frac{A^n}{(p-1)^n}\|f\|_{L^p(\boldsymbol{R}^n)}$$

(see Remark 3 and [7, II §5.20]). Therefore, instead of the L^1-boundedness, we have

$$\|\mathcal{M}f\|_{\Phi_{n+\varepsilon,0}(L)}\leq C\|f\|_{\Phi_{\varepsilon,n}(L)}$$

for any $\varepsilon>0$ (see (1.6)). Here, the positive constant C is independent of f.

EXAMPLE 2 (Differential operators). Suppose f is of class $C^2(\boldsymbol{R}^n)$ and has compact support. Let $\Delta f=\sum_{j=1}^n \partial^2 f/\partial x_j^2$. Then we have

$$\left\|\frac{\partial^2 f}{\partial x_j\partial x_k}\right\|_{\Phi_{2+\varepsilon,0}(L)}\leq C\|\Delta f\|_{\Phi_{\varepsilon,2}(L)}$$

for any $\varepsilon>0$. Hence, the positive constant C is independent of f.

In fact, by using the Riesz transform of the function f,

$$R_jf=\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\lim_{\delta\to 0}\int_{|y|>\delta}\frac{y_j}{|y|^{n+1}}f(x-y)dy\,,\qquad j=1, 2, 3, \cdots, n\,,$$

we have

$$\frac{\partial^2 f}{\partial x_j\partial x_k}=-R_jR_k\Delta f$$

(see [6, III §1.3]). As is known, each R_j is of weak-type (1, 1) and of strong type (2, 2). So, as remarked above, $T=R_jR_k$ satisfies our assumption for $\alpha=2$. Hence we can apply our theorem.

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