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Harmonic Analysis on Homogeneous Vector Bundles on Hyperbolic Spaces

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Introduction.

Harmonic analysis on hyperbolic spaces $X=U(p,q;\boldsymbol{F})/(U(1;\boldsymbol{F})\times U(p-1,q;\boldsymbol{F}))$ $(\boldsymbol{F}=\boldsymbol{R},\boldsymbol{C},\boldsymbol{H})$ has been studied by many people. Faraut [1], Limic, Niederle, and Raczka [12], Molčhanov [14], Rossmann [15], and Strichartz [23] proved the Plancherel formula on hyperbolic spaces. One method of proof is to use the explicit expression of K-finite eigenfunctions of the Laplacian. Schlichtkrull [18], Sekiguchi [19], and Shitikov [22] studied the Poisson transformation for hyperbolic spaces. Schlichtkrull and Shitikov used the explicit expressions of K-finite eigenfunctions of the Laplacian, and of K-finite functions in degenerate principal series representations.

In this paper we generalize these results for a homogeneous vector bundle on X associated with an irreducible representation δ of $U(1;\boldsymbol{F})$. The basic tools are K-finite functions.

The first result (Theorem 5.2) is the Plancherel formula on the associated vector bundle. We decompose every f in a dense subspace of $L^2(X,\delta)$, the space of L^2-sections of the associated vector bundle, in terms of eigenfunctions of the Laplacian. We describe the Plancherel measure explicitly in terms of the $\boldsymbol{c}$-function ((4.10), (4.11)) for the associated degenerate series representation.

The second result of this paper is the determination of the closed $U(p,q;\boldsymbol{F})$-invariant subspaces of the eigenspaces of the Laplacian on the associated vector bundle (Theorem 6.2), and of the image and the kernel of the Poisson transformation (Theorem 6.4). The result is also new in the Riemannian case, $p=1$.

The degenerate principal series representations, which correspond to the boundaries of hyperbolic spaces, have been studied by Molčhanov [13], Klimyk and Gruber [4, 5, 6, 7], Vilenkin and Klimyk [25], and Howe and Tan [3], by using the explicit K-decompositions. The Poisson transformation gives an intertwining operator from a degenerate principal series representation to an eigenspace of the Laplacian. We use their results to construct K-finite eigenfunctions of the Laplacian and invariant subspaces

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of the eigenspace representation.

The methods and results of this paper are certainly not all new. It has been pointed out by Kobayashi [8, 10], Vilenkin and Klimyk [25], and Howe and Tan [3, concluding remark] that by the isomorphism

$$U(dp, dq; \boldsymbol{R})/U(dp-1, dq; \boldsymbol{R}) \simeq U(p, q; \boldsymbol{F})/U(p-1, q; \boldsymbol{F}),$$

where $d=\dim_{\boldsymbol{R}}\boldsymbol{F}$, harmonic analysis on the associated vector bundle follows from harmonic analysis on real hyperboloid $U(dp, dq; \boldsymbol{R})/U(dp-1, dq; \boldsymbol{R})$, and from the decomposition of the degenerate series representations associated with the real hyperboloid under the projective action of $U(1, \boldsymbol{F})$. From this point of view, the key result is Lemma 4.1, which enables one to reduce the Poisson and Fourier transformations, and intertwining operator for $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$ to those for $\boldsymbol{F}=\boldsymbol{R}$. Vilenkin and Klimyk [25] proved the Plancherel theorem for the associated vector bundle in this line; their description of the Plancherel measure is formally different from the one given here.

The results for $\boldsymbol{F}=\boldsymbol{C}$ were announced in [20, 21].

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1. Notation.

Let $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ denote the fields of real, complex and quaternion numbers, respectively, and let $\boldsymbol{F}$ be one of these fields. Let $\boldsymbol{Z}$ denote the ring of integers and $\boldsymbol{N}$ the set of nonnegative integers.

Let [,] be the Hermitian form on $\boldsymbol{F}^{p+q}$ given by

$$[x, y]=\bar{y}_1x_1+\cdots+\bar{y}_px_p-\bar{y}_{p+1}x_{p+1}-\cdots-\bar{y}_{p+q}x_{p+q} \tag{1.1}$$

for $x=(x_1, x_2, \cdots, x_{p+q})$ and $y=(y_1, y_2, \cdots, y_{p+q})$ in $\boldsymbol{F}^{p+q}$ and put $|x|=\sqrt{[x, x]}$. Let $G=U(p, q; \boldsymbol{F})$ be the group of all $(p+q)\times(p+q)$ matrices with coefficients in $\boldsymbol{F}$ preserving the Hermitian form [,]. In the standard notation, $G=O(p, q)$, $U(p, q)$, and $Sp(p, q)$ for $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}$, and $\boldsymbol{H}$ respectively. Let $\sim$ denote the equivalence relation of $\boldsymbol{F}^{p+q}$ defined by

$$x\sim y \Leftrightarrow y=xu \text{ for some } u\in U(1; \boldsymbol{F}). \tag{1.2}$$

Assume p and q are positive integers. The group $G=U(p, q; \boldsymbol{F})$ acts on the projective space $P_{p+q-1}(\boldsymbol{F})$ and the stabilizer of the vector $x^0=(1, 0, \cdots, 0)$ is the group $H=U(1; \boldsymbol{F})\times U(p-1, q; \boldsymbol{F})$. The homogeneous space $X=X(p, q; \boldsymbol{F})=G/H$ is the projective image of the space

$$Z=Z(p, q; \boldsymbol{F})=\{x\in \boldsymbol{F}^{p+q}; [x, x]=1\}. \tag{1.3}$$

The space X is a pseudo-Riemannian symmetric space.

Let K be the maximal compact subgroup $U(p; \boldsymbol{F})\times U(q; \boldsymbol{F})$ of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the

Lie algebras of G and K respectively, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Iwasawa decomposition. Let $E_{i,j}$ denote the $(p+q)\times(p+q)$ matrices with (i,j)-th entry 1 and all other entries being 0. Let $L\in\mathfrak{p}$ be the matrix $E_{p+q,1}+E_{1,p+q}$ and $a_t=\exp tL$ for $t\in\boldsymbol{R}$. Let $\mathfrak{n}$ be the sum of eigenspaces of $\operatorname{ad}L$ in $\mathfrak{g}$ with positive eigenvalues. Let $A=\{a_t;\, t\in\boldsymbol{R}\}$, $N=\exp\mathfrak{n}$, and $M=Z_H(L)$ be subgroups of G. The group M consists of the matrices $\operatorname{diag}(u, h, u)$, where $u\in U(1;\boldsymbol{F})$ and $h\in U(p-1, q-1;\boldsymbol{F})$. Then $P=MAN$ is a maximal parabolic subgroup of G. Let $\rho=\frac{1}{2}d(p+q)-1$, where d is the dimension of $\boldsymbol{F}$ over $\boldsymbol{R}$ ($d=1$, 2 or 4).

Let

$$(1.4)\qquad \begin{aligned}\Sigma&=S(\boldsymbol{F}^p)\times S(\boldsymbol{F}^q)\\&=\{z\in\boldsymbol{F}^{p+q}\,;\,|z_1|^2+\cdots+|z_p|^2=|z_{p+1}|^2+\cdots+|z_{p+q}|^2=1\}\\&\simeq U(p;\boldsymbol{F})/U(p-1;\boldsymbol{F})\times U(q;\boldsymbol{F})/U(q-1;\boldsymbol{F})\,.\end{aligned}$$

The boundary of X is the homogeneous space $B=G/P\simeq K/K\cap M$, which can be identified with $\Sigma/\sim$.

Let dk, dm, du, and $d\sigma$ denote the invariant measures on K, M, $U(1;\boldsymbol{F})$, and Σ with total measure 1 respectively.

We use the following convention throughout this paper: If $\boldsymbol{F}=\boldsymbol{R}$, let $l=0$ or 1; if $\boldsymbol{F}=\boldsymbol{C}$, let $l\in\boldsymbol{Z}$; and if $\boldsymbol{F}=\boldsymbol{H}$, let $l\in\boldsymbol{N}$. We define the irreducible representation δ_l of $U(1;\boldsymbol{F})$ as follows: If $\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$, let $\delta_l(u)=u^l$ for $u\in U(1;\boldsymbol{F})$; and if $\boldsymbol{F}=\boldsymbol{H}$, let δ_l be the $(l+1)$-dimensional irreducible representation of $U(1;\boldsymbol{H})=Sp(1)\simeq SU(2)$. Let χ_l denote the character of δ_l.

2. The most degenerate series representations of $U(p, q; \boldsymbol{F})$.

We denote the irreducible representation of M that is given by $m\mapsto\delta_l(u)$ for $m=\operatorname{diag}(u, h, u)$ ($u\in U(1;\boldsymbol{F})$, $h\in U(p-1;\boldsymbol{F})\times U(q-1;\boldsymbol{F})$) by δ_l and its character by χ_l. Let $\sigma_{\lambda,l}$ be the irreducible representation of P given by

$$(2.1)\qquad ma_tn\mapsto\delta_l(m)e^{(\rho-\lambda)t}\,,\qquad m\in M\,,\quad t\in\boldsymbol{R}\,,\quad n\in N\,.$$

The representation $\pi_{\lambda,l}$ of $U(p, q;\boldsymbol{F})$ induced by the character $\sigma_{\lambda,l}$ is realized in the space

$$(2.2)\qquad \mathscr{E}_{\lambda,l}(G/P)=\Big\{f\in C^\infty(G)\,;\,f(g)=(\dim\delta_l)^{-1}\int_M f(gm)\chi_l(m)dm\quad\text{and}\quad f(ga_tn)=e^{(\lambda-\rho)t}f(g)\ \text{for all}\ g\in G,\ t\in\boldsymbol{R},\ \text{and}\ n\in N\Big\}\,.$$

As a K-module, $\mathscr{E}_{\lambda,l}(G/P)$ can be identified with

$$(2.3)\qquad \mathscr{E}_l(\Sigma)=\Big\{f\in C^\infty(\Sigma)\,;\,f(\sigma)=(\dim\delta_l)^{-1}\int_{U(1,\boldsymbol{F})}f(\sigma u)\chi_l(u)du\ \text{for all}\ \sigma\in\Sigma\Big\}\,.$$

Let Δ_{dr} be the Laplacian on $S(F^r)\simeq S^{dr-1}$. For $j\in N$, let $\mathscr{H}^j(R^{dr})$ be the eigenspace

(2.4) $$\mathscr{H}^j(R^{dr})=\{f\in C^\infty(S^{dr-1});\ \Delta_{dr}f=-j(j+dr-2)f\}\ .$$

The representation of $O(dr)$ on $\mathscr{H}^j(R^{dr})$ is irreducible. We have the following decomposition of $C^\infty(\Sigma)$ into irreducible representations of $O(dp)\times O(dq)$:

(2.5) $$C^\infty(\Sigma)=\sum_{m,n\geq 0}\mathscr{H}^m(R^{dp})\otimes\mathscr{H}^n(R^{dq})\ .$$

We define

(2.6) $$\mathscr{E}_l^{m,n}(\Sigma)=\mathscr{E}_l(\Sigma)\cap\mathscr{H}^m(R^{dp})\otimes\mathscr{H}^n(R^{dq})$$

and

(2.7) $$E_l=\{(m,n)\in N^2\ ;\ \mathscr{E}_l^{m,n}(\Sigma)\neq\{0\}\}\ .$$

The set E_l is given as follows (cf. [3, Diagram 2.17, 4.16, 5.12, 5.14]):

(2.8) $F=R$.

(a) $q=1$, $E_l=\{(m,n);\ m+n\equiv l \bmod 2,\ n=0 \text{ or } 1\}$.

(b) $p=1$, $E_l=\{(m,n);\ m+n\equiv l \bmod 2,\ m=0 \text{ or } 1\}$.

(c) $p,q>1$, $E_l=\{(m,n);\ m+n\equiv l \bmod 2\}$.

(2.9) $F=C, H$.

(a) $q=1$, $E_l=\{(m,n);\ m+n\equiv l \bmod 2,\ m-n\geq -|l|,\ m+n\geq |l|\}$.

(b) $p=1$, $E_l=\{(m,n);\ m+n\equiv l \bmod 2,\ n-m\geq -|l|,\ m+n\geq |l|\}$.

(c) $p,q>1$, $E_l=\{(m,n);\ m+n\equiv l \bmod 2,\ m+n\geq |l|\}$.

The space $\mathscr{H}^m(R^{dp})\otimes\mathscr{H}^n(R^{dq})$ is invariant under K, but in general not irreducible. We refer [3] for the decomposition of $\mathscr{H}^m(R^{dp})\otimes\mathscr{H}^n(R^{dq})$ into irreducible representations of K.

3. Eigenfunctions of the Laplacian.

In this section we describe the radial part of the Laplacian after Faraut [1] and the K-decomposition of the eigenspaces after Schlichtkrull [18]. The K-finite eigenfunctions on the vector bundle can be written by the Gaussian hypergeometric functions. It is convenient for us to use notation of the Jacobi functions that is introduced by Flensted-Jensen and Koornwinder (cf. [2, Appendix], [11]) instead of that for the hypergeometric functions. We summarize the definition and some results of the Jacobi functions in the appendix.

We put

(3.1) $$\mathscr{E}_l(X)=\left\{f\in C^\infty(Z)\ ;\ f(x)=(\dim\delta_l)^{-1}\int_{U(1,F)}f(xu)\chi_l(u)du \text{ for all } x\in Z\right\}.$$

The space $\mathscr{E}_l(X)$ can be identified with the space of the C^∞-sections of the homogeneous vector bundle on X associated with the representation $\delta_l \otimes 1$ of $H = U(1; \boldsymbol{F}) \times U(p-1; q; \boldsymbol{F})$. Hereafter we denote the representation $\delta_l \otimes 1$ of H by δ_l.

Let Δ denote the Laplace-Beltrami operator on $Z(p, q; \boldsymbol{F}) \simeq Z(dp, dq; \boldsymbol{R})$. The operator Δ maps $\mathscr{E}_l(X)$ into itself, and the algebra of G-invariant differential operators on $\mathscr{E}_l(X)$ is generated by Δ. For $\lambda \in \boldsymbol{C}$ we define

$$\mathscr{E}_{\lambda,l}(X) = \{f \in \mathscr{E}_l(X) ; \Delta f = (\lambda^2 - \rho^2)f\} . \tag{3.2}$$

The map $\Sigma \times (0, \infty) \to Z$ given by

$$(\sigma, t) \mapsto (\sigma_1 \cosh t, \cdots, \sigma_p \cosh t, \sigma_{p+1} \sinh t, \cdots, \sigma_{p+q} \sinh t) \tag{3.3}$$

is a diffeomorphism onto $Z \backslash \{0\}$. We shall frequently use the spherical coordinates $(\sigma, t) \in \Sigma \times (0, \infty)$ on $Z \backslash \{0\}$.

In the spherical coordinates, the Laplacian is given by

$$\Delta = \frac{1}{A(t)} \left[\frac{d}{dt} A(t) \frac{d}{dt} \right] - \frac{1}{\cosh^2 t} \Delta_{dp} + \frac{1}{\sinh^2 t} \Delta_{dq} , \tag{3.4}$$

where $A(t) = (2 \sinh t)^{dq/2-1} (2 \cosh t)^{dp/2-1}$ (cf. [1, VII (2)]). The normalized invariant measure dx on Z is given by

$$\int_Z f(x) dx = \int_\Sigma \int_0^\infty f(\sigma, t) A(t) dt d\sigma .$$

Let $\mathscr{E}_{\lambda,l}^{m,n}(X)$ denote the space of the functions in $\mathscr{E}_{\lambda,l}(X)$ that are K-finite of type $(m, n) \in E_l$. For $f = h(\sigma)F(t) \in \mathscr{E}_{\lambda,l}^{m,n}(X)$, the differential equation $\Delta f = (\lambda^2 - \rho^2)f$ is given by

$$\frac{1}{A(t)} \left[\frac{d}{dt} A(t) \frac{dF}{dt} \right] + \left[\frac{m(m+dp-2)}{\cosh^2 t} - \frac{n(n+dq-2)}{\sinh^2 t} \right] F = (\lambda^2 - \rho^2) F . \tag{3.5}$$

If we put $\varphi = (\sinh t)^{-n} (\cosh t)^{-m} F$, then equation (3.5) can be written as the Jacobi differential equation

$$\left(\frac{d^2}{dt^2} + ((dq/2 + n - 1) \sinh t + (dp/2 + m - 1) \cosh t) \frac{d}{dt} \right) \varphi = (\lambda^2 - (\rho + m + n)^2) \varphi \tag{3.6}$$

and the unique solution of (3.6) that is regular at 0 and $\varphi(0) = 1$ is given by the Jacobi function $\varphi_{-i\lambda}^{(dq/2+n-1, dp/2+m-1)}(t)$. For $h \in \mathscr{E}_l^{m,n}(\Sigma)$ we define a C^∞-function $\Phi_{\lambda,l}(h)$ on $Z \backslash \{0\}$ by

$$\Phi_{\lambda,l}(h)(\sigma, t) = h(\sigma) \frac{(\cosh t)^m (\sinh t)^n}{\Gamma(n + dq/2)} \varphi_{i\lambda}^{(dq/2+n-1, dp/2+m-1)}(t) . \tag{3.7}$$

THEOREM 3.1. *Let $\lambda \in \boldsymbol{C}$, $l \in \boldsymbol{Z}$, and $(m, n) \in E_l$. For $h \in \mathscr{E}_l^{m,n}(\Sigma)$ the function $\Phi_{\lambda,l}(h)$ extends to an element of $\mathscr{E}_{\lambda,l}^{m,n}(X)$, which is given by* (3.7) *for $\sigma \in \Sigma$, $t \in [0, \infty)$. Moreover the map $\Phi_{\lambda,l} \colon \mathscr{E}_l^{m,n}(\Sigma) \to \mathscr{E}_{\lambda,l}^{m,n}(X)$ is a K-isomorphism.*

PROOF. The proof is the same as the proof of [18, Theorem 4.2]. □

4. The Poisson transformations.

Let $P_{\lambda,l}(g)$ be the distribution on G defined by

$$(4.1)\quad P_{\lambda,l}(g)=\begin{cases}\chi_l(m)e^{-(\lambda+\rho)t} & \text{if } g^{-1}\in(\{1\}\times U(p-1,q;\boldsymbol{F}))ma_tN\,,\quad m\in M\,,\ t\in\boldsymbol{R}\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Notice that $P_{\lambda,l}$ is a section of the homogeneous vector bundle over X associated with the representation δ_l of H and it can be considered as a function on $Z=Z(p,q;\boldsymbol{F})$. In the coordinate of Z, $P_{\lambda,l}$ is given by

$$(4.2)\qquad P_{\lambda,l}(x)=|[\sigma^0,x]|^{-\lambda-\rho}\chi_l\left(\frac{[\sigma^0,x]}{|[\sigma^0,x]|}\right),\qquad x\in Z\,,$$

where $\sigma^0=(1,0,\cdots,0,1)$.

For $\varphi\in\mathscr{E}_{\lambda,l}(G/P)\simeq\mathscr{E}_l(\Sigma)$, we define the Poisson integral of φ by

$$(4.3)\qquad \mathscr{P}_{\lambda,l}\varphi(x)=\begin{cases}\displaystyle\int_K P_{\lambda,l}(k^{-1}x)\varphi(k)dk\,, & p=1\\ \displaystyle\frac{1}{\Gamma(\frac{1}{2}(-\lambda-\rho+d+|l|))}\int_K P_{\lambda,l}(k^{-1}x)\varphi(k)dk\,, & p>1\end{cases}$$

for $x\in Z$. The integral converges at least for $\mathrm{Re}(\lambda+\rho)<0$ and admits a meromorphic continuation in λ.

The following lemma is a generalization of [1, Lemma 5.4].

LEMMA 4.1. *For* $\mathrm{Re}\,\lambda>0$, *we have*

$$(4.4)\quad |x|^{\lambda}\chi_l\left(\frac{x}{|x|}\right)=\frac{1}{a(\lambda,l)}\int_{U(1;\boldsymbol{F})}|\mathrm{Re}(xu)|^{\lambda}(\mathrm{sgn}\,\mathrm{Re}(xu))^l\,\overline{\chi_l(u)}du\,,\qquad \textit{for}\quad x\in\boldsymbol{F}^{\times}\,,$$

where

$$(4.5)\qquad a(\lambda,l)=\dim\delta_l\frac{\Gamma(d/2)}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(\frac{1}{2}(\lambda+2))}{\Gamma(\frac{1}{2}(\lambda+d+l))\Gamma(\frac{1}{2}(\lambda+2-l))}\,.$$

PROOF. By reason of homogeneity, there exists constant $a(\lambda,l)$ such that (4.4) is satisfied.

If $\boldsymbol{F}=\boldsymbol{R}$, then $U(1;\boldsymbol{F})=\{1,-1\}$ and it is clear that $a(\lambda,l)=1$.

If $\boldsymbol{F}=\boldsymbol{C}$, then we have

$$a(\lambda, l)=\int_{U(1)} |\operatorname{Re}(u)|^{\lambda}(\operatorname{sgn}\operatorname{Re}(u))^{l}u^{-l}du$$
$$=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(\cos\theta)^{\lambda}e^{-il\theta}d\theta$$
$$=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(\frac{1}{2}(\lambda+2))}{\Gamma(\frac{1}{2}(\lambda+2+l))\Gamma(\frac{1}{2}(\lambda+2-l))}.$$

The last equality follows from the formula in [17, P290].

Let $\boldsymbol{F}=\boldsymbol{H}$. By the identification $U(1;\boldsymbol{H})=Sp(1)\simeq SU(2)$, we have

$$a(\lambda, l)=\int_{Sp(1)} |\operatorname{Re}(u)|^{\lambda}(\operatorname{sgn}\operatorname{Re}(u))^{l}\overline{\chi_l(u)}du$$
$$=\int_{SU(2)} |\operatorname{tr}(u)/2|^{\lambda}(\operatorname{sgn}\operatorname{tr}(u))^{l}\overline{\chi_l(u)}du,$$

where we donote the character of the $(l+1)$-dimensional irreducible representation of $SU(2)$ by χ_l. Since the integrand is constant on classes of conjugate elements, we have by an integral formula and the character formula for $SU(2)$ (cf. [24, Chapter II, Proposition 2.5, Proposition 3.4]),

$$a(\lambda, l)=\frac{2}{\pi}\int_{0}^{\pi} |\cos t|^{\lambda}(\operatorname{sgn}\cos t)^{l}\frac{\sin(l+1)t}{\sin t}\sin^{2}t\,dt$$
$$=\frac{4}{\pi}\int_{0}^{\pi/2}(\cos t)^{\lambda}\sin(l+1)t\sin t\,dt.$$

It can be shown by an addition formula for trigonometric functions and integration by parts that

$$a(\lambda, l)=\frac{\lambda}{l}a(\lambda-1, l-1)+\frac{\lambda+2-l}{l}a(\lambda+1, l-1).$$

Since

$$a(\lambda, 0)=\frac{\pi}{2}B\left(\frac{1}{2}(\lambda+1), \frac{3}{2}\right),$$

where $B(p, q)$ is the beta-function, we can show by induction on l that

$$a(\lambda, l)=\frac{l+1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}(\lambda+1))\Gamma(\frac{1}{2}(\lambda+2))}{\Gamma(\frac{1}{2}(\lambda-l+2))\Gamma(\frac{1}{2}(\lambda+l+4))}. \qquad \square$$

The following proposition enables us to reduce the Poisson integrals for $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$ to those for $\boldsymbol{F}=\boldsymbol{R}$.

PROPOSITION 4.2. *If $p>1$, then*

$$\mathscr{P}_{\lambda,l}\varphi(x)=\frac{\sqrt{\pi}}{\dim\delta_l\Gamma(d/2)}\frac{\Gamma(\frac{1}{2}(-\lambda-\rho+2-|l|))}{\Gamma(\frac{1}{2}(-\lambda-\rho+1))\Gamma(\frac{1}{2}(-\lambda-\rho+2))}$$
$$\times\int_{\Sigma}|\mathrm{Re}([\sigma,x])|^{-\lambda-\rho}(\mathrm{sgn}\,\mathrm{Re}([\sigma,x]))^l\varphi(\sigma)d\sigma \tag{4.6}$$

for $\varphi\in\mathscr{E}_l(\Sigma)$. If $p=1$, then $\mathscr{P}_{\lambda,l}\varphi$ is given by the right-hand side of (4.6) *multiplied by* $\Gamma(\frac{1}{2}(-\lambda-\rho+d+|l|))$.

PROOF. By Lemma 4.1, we have

$$\int_{\Sigma}\int_{U(1;F)}|\mathrm{Re}([\sigma,x]u)|^{-\lambda-\rho}(\mathrm{sgn}\,\mathrm{Re}([\sigma,x]u))^l\overline{\chi_l(u)}\varphi(\sigma)du\,d\sigma$$
$$=\frac{1}{a(-\lambda-\rho,l)}\int_{\Sigma}|\mathrm{Re}([\sigma,x])|^{-\lambda-\rho}(\mathrm{sgn}\,\mathrm{Re}([\sigma,x]))^l\varphi(\sigma)d\sigma\,,$$

hence the proposition follows. □

It follows from Proposition 4.2 and [1, IV(3)] that $\mathscr{P}_{\lambda,l}\varphi$ is in $\mathscr{E}_{\lambda,l}(X)$. We call the G-equivariant map

$$\mathscr{P}_{\lambda,l}:\mathscr{E}_{\lambda,l}(G/P)\to\mathscr{E}_{\lambda,l}(X)$$

the *Poisson transformation.*

The following lemma is a generalization of [18, Lemma 7.2]. The proof can be done in the same way, so we omit it.

LEMMA 4.3. *For $(m,n)\in E_l$ and $\varphi\in\mathscr{E}_l^{m,n}(\Sigma)$ we have*

$$\mathscr{P}_{\lambda,l}\varphi=\beta(\lambda,m,n)\Phi_{\lambda,l}(\varphi)\,, \tag{4.7}$$

where $\beta(\lambda,m,n)$ is given by

$$\beta(\lambda,m,n)=\frac{(-1)^{\frac{1}{2}(m-n-|l|)}}{\dim\delta_l}\frac{\Gamma(dp/2)\Gamma(dq/2)}{\Gamma(d/2)}$$
$$\times\frac{\Gamma(\frac{1}{2}(\lambda+\rho+n+m))}{\Gamma(\frac{1}{2}(\lambda+\rho+|l|))\Gamma(-\frac{1}{2}(\lambda+\rho-dp-m+n))} \tag{4.8}$$

for $p>1$, and the right-hand side of (4.8) *multiplied by* $\Gamma(\frac{1}{2}(-\lambda-\rho+d+|l|))$ *for $p=1$.*

THEOREM 4.4. *Let φ be a K-finite function in $\mathscr{E}_{\lambda,l}(G/P)$. If* $\mathrm{Re}\lambda>0$, *then we have*

$$\lim_{t\to\infty}e^{(\rho-\lambda)t}\mathscr{P}_{\lambda,l}\varphi(a_t)=c_l(\lambda)\varphi(e)\,, \tag{4.9}$$

where the constant $c_l(\lambda)$ is given by

(4.10)

$$c_l(\lambda)=\frac{1}{\dim\delta_l}\frac{2^{\rho-\lambda}}{\pi}\frac{\Gamma(dp/2)\Gamma(dq/2)}{\Gamma(d/2)}\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}(\lambda+\rho+|l|))}\sin\frac{\pi}{2}(\lambda+\rho-dp+2-|l|).$$

for $p>1$ *and*

$$c_l(\lambda)=\frac{1}{\dim\delta_l}2^{\rho-\lambda}\Gamma(dq/2)\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}(\lambda+\rho+|l|))\Gamma(\frac{1}{2}(\lambda+\rho-d+2-|l|))} \tag{4.11}$$

for $p=1$.

PROOF. This theorem follows from (4.7), (4.8), (A.5), and (A.6). □

REMARK 4.5. If $p>1$, then the integral in (4.6) depends moromorphically on λ with simple poles for $\lambda+\rho>0$ and $\lambda+\rho\equiv l+1$ mod 2, and if $p=1$ it is entire in λ (cf. [15, Lemma 5]). Therefore $\mathscr{P}_{\lambda,l}\varphi$ depends holomorphically on λ.

In the case of functions on the hyperbolic spaces, Schlichtkrull [18, Theorem 7.4, Corollary 7.5] showed that the eigenspace representation and the corresponding degenerate series representation are equivalent in the Grothendieck group. We have an analogous result in the case of the vector bundles on the hyperbolic spaces.

Let $\mathscr{I}_{\lambda,l}\subset\mathscr{E}_{\lambda,l}(X)$ denote the closure of the image of $\mathscr{P}_{\lambda,l}$. For $\varphi\in\mathscr{E}_l(\Sigma)$ we define

$$\mathscr{P}'_{\lambda,l}\varphi(x)=\frac{d}{dv}\bigg|_{v=\lambda}\mathscr{P}_{v,l}\varphi(x).$$

If $\mathscr{P}_{\lambda,l}\varphi=0$, then $\mathscr{P}'_{\lambda,l}\varphi\in\mathscr{E}_{\lambda,l}(X)$ (cf. [18, P212]). Let

$$\tilde{\mathscr{P}}'_{\lambda,l}:\ \ker\mathscr{P}_{\lambda,l}\to\mathscr{E}_{\lambda,l}(X)/\mathscr{I}_{\lambda,l}$$

denote the composition of $\mathscr{P}'_{\lambda,l}$ and the projection from $\mathscr{E}_{\lambda,l}(X)$ to $\mathscr{E}_{\lambda,l}(X)/\mathscr{I}_{\lambda,l}$.

THEOREM 4.6. *Let $\lambda\in\mathbf{C}$ and $l\in\mathbf{Z}$. The map $\tilde{\mathscr{P}}'_{\lambda,l}$ is G-equivariant. If* $\mathrm{Re}\,\lambda\geq 0$, *then it is an isomorphism from* $\ker\mathscr{P}_{\lambda,l}$ *onto* $\mathscr{E}_{\lambda,l}(X)/\mathscr{I}_{\lambda,l}$.

PROOF. By Theorem 3.1 and Lemma 4.3 we can prove the theorem in the same way as the proof of [18, Theorem 7.4], so we omit it. □

COROLLARY 4.7. *Every irreducible subquotient of $\mathscr{E}_{\lambda,l}(X)$ is infinitesimally equivalent with an irreducible subquotient of $\mathscr{E}_{\lambda,l}(G/P)$, and vice versa.*

5. The Plancherel formula.

If $p>1$, we define the *Fourier transformation* $\mathscr{F}_{\lambda,l}:\mathscr{E}_l(X)\to\mathscr{E}_{\lambda,l}(G/P)$ by

$$\mathscr{F}_{\lambda,l}f(g)=\frac{1}{\Gamma(\frac{1}{2}(\lambda-\rho+d+|l|))}\int_Z f(x)\overline{P_{-\lambda,l}(g^{-1}x)}dx\,,\qquad g\in G. \tag{5.1}$$

By (4.2), it can be seen easily that

$$\mathcal{F}_{\lambda,l}f(\sigma)=\frac{1}{\Gamma(\frac{1}{2}(\lambda-\rho+d+|l|))}\int_X |[x,\sigma]|^{\lambda-\rho}\chi_l\left(\frac{[x,\sigma]}{|[x,\sigma]|}\right)f(x)dx\,,\qquad \sigma\in\Sigma\,. \tag{5.2}$$

If $p=1$, we define the Fourier transformation by the integral (5.1) without the normalizing factor $1/\Gamma(\frac{1}{2}(\lambda-\rho+d+|l|))$.

The Fourier transformation is G-equivariant. The Fourier transformation for K-finite functions can be written by the Jacobi transform ([11], see Appendix):

PROPOSITION 5.1. *Let $(m,n)\in E_l$. Let $f\in\mathcal{E}_l(X)$ be a function that is of the form $f=(\sinh t)^n(\cosh t)^m F(t)h(\sigma)$, where $F(t)$ is an even function on $\mathbf{R}$ and $h\in\mathcal{E}_l^{m,n}(\Sigma)$. Then the Fourier transform of f is given by*

$$\mathcal{F}_{\lambda,l}f(\sigma)=2^{-2(m+n)}\frac{\beta(-\lambda,m,n)}{\Gamma(n+dq/2)}F^{\wedge}_{\alpha,\beta}(i\lambda)h(\sigma)\,, \tag{5.3}$$

where $\alpha=n+dq/2-1$ and $\beta=m+dp/2-1$ and $F^{\wedge}_{\alpha,\beta}$ is the Jacobi transform of F.

PROOF. ([1, VII (2)]) It suffices to show (5.3) for h that is contained in an irreducible component of $\mathcal{E}_l^{m,n}(\Sigma)$. Since each K-type in $\mathcal{E}_l(\Sigma)$ is multiplicity one, it follows from K-equivariance of $\mathcal{F}_{\lambda,l}$ that there exists constant c such that $\mathcal{F}_{\lambda,l}f(\sigma)=ch(\sigma)$. We have

$$\begin{aligned}\int_X f(x)\overline{P_{-\lambda,l}(x)}dx&=\int_K\int_0^\infty F(t)h(k\cdot\sigma^0)\overline{P_{-\lambda,l}(ka_t)}(\sinh t)^n(\cosh t)^m A(t)dt\,dk\\&=2^{-2(m+n)}\int_0^\infty\left(\int_K h(k\cdot\sigma^0)\overline{P_{-\lambda,l}(ka_t)}dk\right)F(t)\Delta_{\alpha,\beta}(t)dt\,.\end{aligned}$$

We refer (A.2) for the definition of the function $\Delta_{\alpha,\beta}(t)$. If we put $k\cdot\sigma^0=\sigma=(\sigma_1,\cdots,\sigma_{p+q})$, we have

$$\begin{aligned}&\int_K h(k\cdot\sigma^0)\overline{P_{-\lambda,l}(ka_t)}dk\\&=\int_K h(k\cdot\sigma^0)|[ka_t\cdot x^0,\sigma^0]|^{\lambda-\rho}\chi_l\left(\frac{[ka_t\cdot x^0,\sigma^0]}{|[ka_t\cdot x^0,\sigma^0]|}\right)dk\\&=\int_\Sigma h(\sigma)|\sigma_1\cosh t-\sigma_{p+q}\sinh t|^{\lambda-\rho}\chi_l\left(\frac{\sigma_1\cosh t-\sigma_{p+q}\sinh t}{|\sigma_1\cosh t-\sigma_{p+q}\sinh t|}\right)d\sigma\\&=\Gamma(\tfrac{1}{2}(\lambda-\rho+d+|l|))(\mathcal{P}_{-\lambda,l}h)(a_t)\,.\end{aligned}$$

Now the proposition follows from Lemma 4.3. □

THEOREM 5.2. *Let $f \in \mathscr{E}_l(X)$ be K-finite and compact supported.*
(1) $p=1$.

$$(5.4)\qquad f(x)=\frac{1}{2\pi}\int_0^\infty (\mathscr{P}_{i\lambda,l}\circ\mathscr{F}_{i\lambda,l}f)(x)|c_l(i\lambda)|^{-2}d\lambda$$
$$-i\sum_{\substack{0<s<|l|-\rho+d-2\\ s\equiv\rho+l}}(\mathscr{P}_{s,l}\circ\mathscr{F}_{s,l}f)(x)\mathrm{Res}\left[\frac{1}{c_l(\lambda)c_l(-\lambda)};\lambda=s\right].$$

(2) $\boldsymbol{F}=\boldsymbol{R}, p>1$ *and q odd.*

$$(5.5)\qquad f(x)=\frac{1}{2\pi}\int_0^\infty (\mathscr{P}_{i\lambda,l}\circ\mathscr{F}_{i\lambda,l}f)(x)|c_l(i\lambda)|^{-2}d\lambda$$
$$-i\sum_{\substack{s>0\\ s\equiv\rho+l+1 \bmod 2}}(\mathscr{P}_{s,l}\circ\mathscr{F}_{s,l}f)(x)\mathrm{Res}\left[\frac{1}{c_l(\lambda)c_l(-\lambda)};\lambda=s\right].$$

(3) $p>1$. $\boldsymbol{F}=\boldsymbol{R}$, *$q$ even, or* $\boldsymbol{F}=\boldsymbol{C}$ *or* $\boldsymbol{H}$.

$$(5.6)\qquad f(x)=\frac{1}{2\pi}\int_0^\infty (\mathscr{P}_{i\lambda,l}\circ\mathscr{F}_{i\lambda,l}f)(x)|c_l(i\lambda)|^{-2}d\lambda$$
$$-i\sum_{\substack{0<s<\rho+|l|\\ s\equiv\rho+l \bmod 2}}(\mathscr{P}_{s,l}\circ\mathscr{F}_{s,l}f)(x)\mathrm{Res}\left[\frac{1}{c_l(\lambda)c_l(-\lambda)};\lambda=s\right]$$
$$-i\sum_{\substack{s\geq\rho+|l|\\ s\equiv\rho+l \bmod 2}}\left(\mathscr{P}_{s,l}\circ\left.\frac{d}{d\lambda}\right|_{\lambda=s}\mathscr{F}_{\lambda,l}f\right)(x)c_{-2}\left[\frac{1}{c_l(\lambda)c_l(-\lambda)};\lambda=s\right].$$

Here Res *denotes the residue and $c_{-2}[\cdot\,;\lambda=s]$ denotes the coefficient of $(\lambda-s)^{-2}$ in the Laurent development at s.*

PROOF. This theorem follows from (2.8), (2.9), Lemma 4.3, Proposition 5.1, and the inversion formula for the Jacobi transform (A.9). □

REMARK 5.3. If $p=1$, then the set of discrete spectra in the theorem above is empty for $\boldsymbol{F}=\boldsymbol{R}$, and not empty for $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$ and $|l|$ sufficiently large. For $\boldsymbol{F}=\boldsymbol{C}$ the formula (5.4) was given by Flensted-Jensen [2].

REMARK 5.4. We describe relations between Knapp-Stein's intertwining operator and the Fourier and the Poisson transformations. If $p>1$, we define

$$(5.7)\qquad A_{\lambda,l}\varphi(\sigma')=\frac{1}{\Gamma(\frac{1}{2}(\lambda-\rho+d+|l|))}\int_\Sigma |[\sigma,\sigma']|^{\lambda-\rho}\chi_l\left(\frac{[\sigma,\sigma']}{|[\sigma,\sigma']|}\right)\varphi(\sigma)d\sigma,$$

for $\varphi\in\mathscr{E}_l(\Sigma)$, and if $p=1$ we define the operator $A_{\lambda,l}$ by the integral (5.7) without the normalizing factor $1/\Gamma(\frac{1}{2}(\lambda-\rho+d+|l|))$. The operator $A_{\lambda,l}$ gives an intertwining operator from $\mathscr{E}_{-\lambda,l}(G/P)$ to $\mathscr{E}_{\lambda,l}(G/P)$. If $\mathrm{Re}\lambda>0$, then by [1, V (3)] we have

$$A_{\lambda,l}\varphi(k\cdot\sigma^0)=\lim_{t\to\infty} e^{(\lambda-\rho)t}(ka_t)\mathscr{P}_{-\lambda,l}\varphi(ka_t)\,. \tag{5.8}$$

It follows easily from Lemma 4.3, Theorem 4.4, and (A.6) that

$$A_{\lambda,l}\circ\mathscr{F}_{-\lambda,l}=c_l(\lambda)\mathscr{F}_{\lambda,l}\,, \tag{5.9}$$

$$A_{\lambda,l}\circ A_{-\lambda,l}=c_l(\lambda)c_l(-\lambda)\mathrm{id}\,. \tag{5.10}$$

6. Invariant subspaces of the eigenspace representations.

In this section we describe the closed G-invariant subspaces of the eigenspace representations and the images of the Poisson transformations on the vector bundles on the hyperbolic spaces. Schlichtkrull [18] and Shitikov [22] studied the closed invariant subspaces of the eigenspace representations and the images of the Poisson transformations on the hyperbolic spaces. We use the similar method with that of [22].

For $(m,n)\in N^2$ let $A_\lambda^{\pm\pm}(m,n)$ be the following four linear functions:

$$\begin{aligned} A_\lambda^{++}(m,n)&=\lambda-\rho-m-n\,,\\ A_\lambda^{+-}(m,n)&=\lambda-\rho-m+n+dq-2\,,\\ A_\lambda^{-+}(m,n)&=\lambda-\rho+m-n+dp-2\,,\\ A_\lambda^{--}(m,n)&=\lambda-\rho+m+n+dp+dq-4\,. \end{aligned} \tag{6.1}$$

Among the functions $A_\lambda^{\pm\pm}$ we have the relations

$$A_\lambda^{++}+A_\lambda^{--}=A_\lambda^{+-}+A_\lambda^{-+}\,, \tag{6.2}$$

$$A_\lambda^{++}+A_{-\lambda}^{--}=A_\lambda^{+-}+A_{-\lambda}^{-+}=-2\,. \tag{6.3}$$

We call the line $A_\lambda^{**}(m,n)=0$ $(**=\pm\pm)$ a *barrier* if it intersects E_l. The invariant subspaces of the degenerate series representations and those of the eigenspace representations can be described by the barriers.

If the line $A_\lambda^{**}(m,n)=0$ is a barrier, let $V_{\lambda,l}^{**}$ be the closure of the space $\{h\in\mathscr{E}_l^{m,n}(\Sigma);$ $(m,n)\in E_l$ with $A_\lambda^{**}(m,n)\geq 0\}$ and $\mathscr{E}_{\lambda,l}^{**}=\mathscr{E}_{\lambda,l}^{**}(X)$ the closure of the space $\{\Phi_{\lambda,l}(h);$ $h\in\mathscr{E}_l^{m,n}(\Sigma)$ for $(m,n)\in E_l$ with $A_\lambda^{**}(m,n)\geq 0\}$.

THEOREM 6.1 (Molčhanov [13], Klimyk-Gruber [6, 7], Vilenkin-Klimyk [25], and Howe-Tan [3]). *The spaces $V_{\lambda,l}^{\pm\pm}$ are closed G-invariant subspaces of $\mathscr{E}_{\lambda,l}(G/P)$. If $F\neq C$ or $p,q>1$, then any closed G-invariant subspaces of $\mathscr{E}_{\lambda,l}(G/P)$ can be obtained from these spaces by means of the operations of intersection and arithmetic sum. In particular, if $\lambda-\rho$ is not an integer, then $\mathscr{E}_{\lambda,l}(G/P)$ is irreducible.*

THEOREM 6.2. *Let $\mathrm{Re}\lambda\geq 0$. The spaces $\mathscr{E}_{\lambda,l}^{\pm+}$, $\mathscr{E}_{-\lambda,l}^{\pm+}$ are closed G-invariant subspaces of $\mathscr{E}_{\lambda,l}(X)$. If $F\neq C$ or $p,q>1$, then any closed G-invariant subspaces of $\mathscr{E}_{\lambda,l}(X)$ can be* *obtained from these spaces by means of the operations of intersection and arithmetic sum. In particular, if* $\lambda-\rho$ *is not an integer, then* $\mathscr{E}_{\lambda,l}(X)$ *is irreducible.*

The eigenspace $\mathscr{E}_{\lambda,l}(X)$ *has a unique irreducible nonzero subrepresentation except when* $p>1$, dq *is even, and* $\lambda\in|l|+\rho+2N$, *where both* $\mathscr{E}^{-+}_{-\lambda,l}$ *and* $\mathscr{E}^{++}_{\lambda,l}$ *are irreducible subspaces.*

PROOF. For $(m, n)\in E_l$, the action of $\mathfrak{p}$ on $\mathscr{E}_l(\Sigma)$ possibly can take us to any of the four points $(m\pm 1, n\pm 1)$. Whether we can get to one of the points $(m\pm 1, n\pm 1)$ from (m, n) depends on whether the transition coefficient $A^{\pm\pm}_{\lambda}(m, n)$ is nonzero (cf. [3, Lemma 2.3, Lemma 4.1, Theorem 4.2, Corollary 4.3, Lemma 5.3, Lemma 5.4, Theorem 5.5]).

We determine the transition coefficients for the action of $\mathfrak{p}$ on the K-finite eigenfunctions, by using this result, and the G-equivariance of the maps $\mathscr{P}_{\lambda,l}$ and $\tilde{\mathscr{P}}'_{\lambda,l}$, and Theorem 4.5. The transition coefficients for $\mathscr{E}_{\lambda,l}(X)$ are given by

$$A^{\pm +}(m, n)_{\lambda}\cdot\frac{\beta(\lambda, m\pm 1, n+1)}{\beta(\lambda, m, n)}=\mp\frac{1}{2}A^{\pm +}_{\lambda}(m, n)A^{\pm\pm}_{\mp\lambda}(m, n)\,,$$

$$A^{\pm -}(m, n)_{\lambda}\cdot\frac{\beta(\lambda, m\pm 1, n-1)}{\beta(\lambda, m, n)}=\pm 2\,,$$

hence the theorem follows. □

REMARK 6.3. We can prove Theorem 6.2 by generalizing the method of Schlichtkrull [18]. Each K-types has multiplicity one in $\mathscr{E}_l(\Sigma)$ and contains a unique one-dimensional space of vectors which transform by δ_l under the action of M from the left. We can compute the action of L on these vectors as in [18, Section 4, 5]. We have Theorem 6.3 by determining which coefficients vanish. Previously we proved Theorem 6.2 for $\boldsymbol{F}=\boldsymbol{C}$ in this line (cf. [21]).

THEOREM 6.4. *Let* $\lambda\in\boldsymbol{C}$.

(1) *Assume* $p>1$.

(i) $\mathscr{I}_{\lambda,l}=\mathscr{E}^{++}_{-\lambda,l}$ *and* $\ker\mathscr{P}_{\lambda,l}=V^{--}_{\lambda,l}$ *for* $\lambda+\rho+|l|\in -2N$.

(ii) $\mathscr{I}_{\lambda,l}=\mathscr{E}^{-+}_{-\lambda,l}$ *and* $\ker\mathscr{P}_{\lambda,l}=V^{+-}_{\lambda,l}$ *for* (a): $\lambda+\rho\in l+1+2\boldsymbol{Z}$ *and* dp *is odd, or* (b): $\lambda+\rho\in l+2\boldsymbol{Z}$, $\lambda+\rho+|l|>0$, *and* dp *is even.*

(iii) $\mathscr{I}_{\lambda,l}=\mathscr{E}_{\lambda,l}(X)$ *and* $\ker\mathscr{P}_{\lambda,l}=\{0\}$ *in all other cases than* (i) *and* (ii).

If we assume $\mathrm{Re}\,\lambda>0$ *in addition to* (a) *or* (b), *then* $\mathscr{E}^{-+}_{-\lambda,l}\subset L^2(X, \delta_l)$, *the* L^2*-space on the associated homogeneous vector bundle.*

(2) *Assume* $p=1$ *and* $\boldsymbol{F}=\boldsymbol{R}$.

(i) $\mathscr{I}_{\lambda,l}=\mathscr{E}^{++}_{-\lambda,l}$ *and* $\ker\mathscr{P}_{\lambda,l}=V^{--}_{\lambda,l}$ *for* $\lambda+\rho+l\in -2N$.

(ii) $\mathscr{I}_{\lambda,l}=\mathscr{E}^{-+}_{-\lambda,l}$ *and* $\ker\mathscr{P}_{\lambda,l}=V^{+-}_{\lambda,l}$ *for* $\lambda+\rho+1-l\in -2N$.

(iii) $\mathscr{I}_{\lambda,l}=\mathscr{E}_{\lambda,l}(X)$ *and* $\ker\mathscr{P}_{\lambda,l}=\{0\}$ *for* $\lambda+\rho\notin -N$.

(3) *Let* $p=1$ *and* $\boldsymbol{F}=\boldsymbol{C}$ *or* $\boldsymbol{H}$.

(i) $\mathscr{I}_{\lambda,l}=\mathscr{E}^{++}_{-\lambda,l}$ *and* $\ker\mathscr{P}_{\lambda,l}=V^{--}_{\lambda,l}$ *for* $\lambda+\rho+|l|\in -2N$.

(ii) $\mathcal{I}_{\lambda,l}=\mathcal{E}_{-\lambda,l}^{-+}$ *and* $\ker\mathcal{P}_{\lambda,l}=V_{\lambda,l}^{+-}$ *for* $\lambda+\rho+|l|\in 2\boldsymbol{Z}$ *and* $-|l|+2\leq\lambda+\rho\leq |l|+d-2$.

(iii) $\mathcal{I}_{\lambda,l}=\mathcal{E}_{\lambda,l}(X)$ *and* $\ker\mathcal{P}_{\lambda,l}=\{0\}$ *for* $\lambda+\rho-d-|l|+2\notin -2\boldsymbol{N}$.

If $0<\lambda\leq-\rho+|l|+d-2$ *and* $\lambda+\rho+|l|\in 2\boldsymbol{Z}$, *then* $\mathcal{E}_{-\lambda,l}^{-+}\subset L^2(X,\delta_l)$.

PROOF. By Lemma 4.3, $\mathcal{P}_{\lambda,l}(\mathcal{E}_l^{m,n}(\Sigma))=\{0\}$ if and only if

(a) $A_\lambda^{+-}(m,n)\in 2\boldsymbol{N}$ (and $\lambda+\rho-d-|l|+2\leq 0$ if $p=1$), or

(b) $\lambda+\rho+|l|\in-2\boldsymbol{N}$ and $A_\lambda^{--}(m,n)\geq 0$ (and $\lambda+\rho+|l|\leq 0$ if $p=1$).

Therefore the theorem follows from Theorem 6.1 and Theorem 6.2. □

REMARK 6.5. If $q=1$ and $\lambda\leq\rho-d-|l|$ in (1)(ii) of the theorem above, then $\mathcal{I}_{\lambda,l}=\mathcal{E}_{-\lambda,l}^{-+}=\mathcal{E}_{\lambda,l}(X)$. Moreover, if $q=1$ and $0<\lambda\leq\rho-d-|l|$, $\lambda-\rho\in l-dq+2\boldsymbol{Z}$, then the discrete series representation $\mathcal{E}_{-\lambda,l}^{-+}=\mathcal{E}_{\lambda,l}(X)$ is isomorphic to the complementary series representation $\mathcal{E}_{\lambda,l}(G/P)$.

REMARK 6.6. Let $p>1$ and $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$. Discrete series representations $\mathcal{E}_{-\lambda,l}^{-+}\subset L^2(X,\delta_l)$ for the range of the parameter $\lambda\in\rho+|l|-dq+2\boldsymbol{N}$ are those of the type studied by Schlichtkrull [16], where he constructed a part of discrete series representations for homogeneous vector bundles on general semisimple symmetric spaces. Kobayashi [9] studied algebraic structures of a part of the discrete series representations for $U(p,q;\boldsymbol{F})/U(p-m,q;\boldsymbol{F})$.

REMARK 6.7. Assume $\mathrm{Re}\lambda\geq 0$. The representations $\mathcal{E}_{\pm\lambda,l}(G/P)$ and $\mathcal{E}_{\lambda,l}(X)$ have the same composition factors. From Theorem 6.1, Theorem 6.4, and Remark 5.4, we can read off relations between the images and the kernels of the Poisson transformations $\mathcal{P}_{\pm\lambda,l}$, the intertwining operator $A_{\lambda,l}$, and the boundary value map from $\mathcal{E}_{\lambda,l}(X)$ to $\mathcal{E}_{\lambda,l}(G/P)$.

For example, we illustrate the barriers and the Hasse diagrams for the case $\boldsymbol{F}=\boldsymbol{C}$, $\boldsymbol{H}$ and $\lambda\in|l|+\rho+2\boldsymbol{N}$ in the Diagram 6.8. The arrows on each barrier point in the direction of the submodule defined by the barrier. $\beta_{\lambda,l}$ denotes the boundary map, which is given by $\beta_{\lambda,l}f(x)=\lim_{t\to\infty}e^{(\lambda-\rho)(t)}f(xa_t)$ for $f\in\mathcal{E}_{\lambda,l}(X)$ that is K-finite and for $\mathrm{Re}\lambda>0$. Notice that there are no K-types in the regions between A_λ^{-+} and $A_{-\lambda}^{+-}$, between A_λ^{++} and $A_{-\lambda}^{--}$, and between A_λ^{+-} and $A_{-\lambda}^{-+}$.

REMARK 6.9. We can determine all invariant subspaces of the eigenspace and the image of the Poisson transformation in the case $\boldsymbol{F}=\boldsymbol{C}$, and $p=1$ or $q=1$. The difference is that the barriers $A_\lambda^{\pm\pm}$ must be taken not on E_l, but on $\{(m,n)\in\boldsymbol{Z}\times\boldsymbol{N}; m+n\equiv l \bmod 2, n-m\geq l, m+n\geq -l\}$ for the case $p=1$ and $q>1$, and we interchange the roles of m and n for the case $q=1$ and $p>1$. We refer [3, Section 4.5] for the parametrization of K-types in $\mathcal{E}_l(\Sigma)$ for these cases.

We have the following corollary for the restrictions of the discrete series representations with respect to the inclusions $U(p,q)\subset O(2p,2q)$ and $Sp(p,q)\subset U(2p,2q)$.

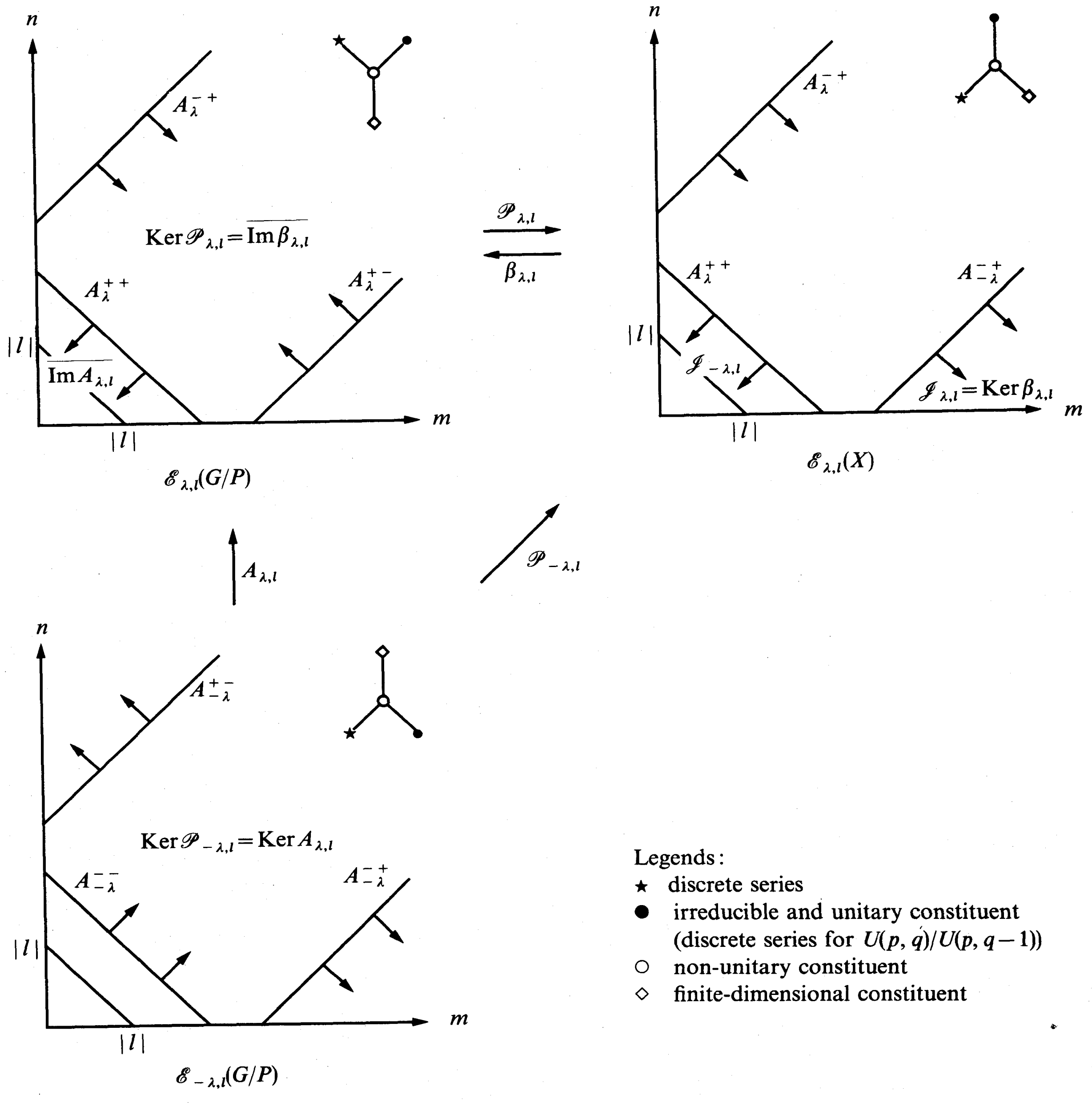

DIAGRAM 6.8

COROLLARY 6.10. (Vilenkin-Klimyk [25, Theorem 1,2]) *Let* $p>2$.
(1) *Let* $\varepsilon=0$ *or* 1 *and* $\lambda>0$, $\lambda-\rho\equiv 2 \bmod 2$. *Then as a* $U(p,q)$*-module, the space* $\mathscr{E}^{-+}_{-\lambda,\varepsilon}(X(2p,2q;\boldsymbol{R}))$ *decomposes into a direct sum*

$$\mathscr{E}^{-+}_{-\lambda,\varepsilon}(X(2p,2q;\boldsymbol{R})) \simeq \sum_{l\in\varepsilon+2\boldsymbol{Z}} \mathscr{E}^{-+}_{-\lambda,l}(X(p,q;\boldsymbol{C})) \tag{6.5}$$

of mutually inequivalent irreducible representations.

(2) *Let* $l \in \boldsymbol{Z}$ *and* $\lambda > 0$, $\lambda \equiv \rho + l \bmod 2$. *Then as a* $Sp(p, q)$*-module, the space* $\mathscr{E}^{-+}_{-\lambda,l}(X(2p, 2q; \boldsymbol{C}))$ *decomposes into a direct sum*

$$\mathscr{E}^{-+}_{-\lambda,l}(X(2p, 2q; \boldsymbol{C})) \simeq \sum_{k \in |l| + 2N} \mathscr{E}^{-+}_{-\lambda,k}(X(p, q; \boldsymbol{H})) \tag{6.6}$$

of mutually inequivalent irreducible representations.

REMARK 6.11. Kobayashi [8] showed Corollary 6.10 (2) for $p = q = 1$, where the right-hand side of (6.6) consists of discrete series for the group $Sp(1, 1)$. For $p, q > 1$, Kobayashi [10, Theorem 6.1] announced a proof of the classification of the discrete series for $U(p, q; \boldsymbol{F})/U(p-1, q; \boldsymbol{F})$ in terms of the derived functor modules and a proof of Corollary 6.10 by a different method from that of ours and Vilenkin-Klimyk [25].

Appendix. The Jacobi functions.

In this section we review some results on the Jacobi functions, which are introduced by Flensted-Jensen and Koornwinder. We refer [2, Appendix] and [11] for details.

Let α, β and s be complex numbers and t a real number. The *Jacobi functions* are defined by

$$\varphi_s^{(\alpha,\beta)}(t) = {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1-is), \tfrac{1}{2}(\alpha+\beta+1+is); \alpha+1; -\sinh^2 t), \tag{A.1}$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Let $\rho_{\alpha,\beta} = \alpha+\beta+1$ and

$$\Delta_{\alpha,\beta}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}, \qquad t > 0. \tag{A.2}$$

Let $L_{\alpha,\beta}$ be the differential operator that is given by

$$L_{\alpha,\beta} = \frac{1}{\Delta_{\alpha,\beta}(t)} \frac{d}{dt}\left(\Delta_{\alpha,\beta} \frac{d}{dt}\right), \qquad t > 0. \tag{A.3}$$

The function $\varphi_s^{(\alpha,\beta)}(t)$ is the unique even function v on $\boldsymbol{R}$ such that $v(0) = 1$ and

$$(L_{\alpha,\beta} + s^2 + \rho_{\alpha,\beta}^2)v = 0. \tag{A.4}$$

The asymptotic behavior of $\varphi_s^{(\alpha,\beta)}$ as $t \to \infty$ is given by

$$\lim_{t\to\infty} e^{(\rho_{\alpha,\beta} - is)t} \varphi_s^{(\alpha,\beta)}(t) = c_{\alpha,\beta}(s), \tag{A.5}$$

where

$$c_{\alpha,\beta}(s) = \frac{2^{\rho_{\alpha,\beta} - is}\Gamma(\alpha+1)\Gamma(is)}{\Gamma(\frac{1}{2}(is+\rho_{\alpha,\beta}))\Gamma(\frac{1}{2}(is+\alpha-\beta+1))}. \tag{A.6}$$

Define the *Jacobi transform* $f \mapsto f^\wedge_{\alpha,\beta}$ by

$$f^{\wedge}_{\alpha,\beta}(s)=\int_0^\infty f(t)\varphi_s^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt \tag{A.7}$$

for $f\in\mathscr{D}_{\text{even}}(\boldsymbol{R})$ (the space of even C^∞-functions with compact support on $\boldsymbol{R}$) and complex number s.

Since $\varphi_s^{(\alpha,\beta)}=\varphi_{-s}^{(\alpha,\beta)}$, we may assume $\operatorname{Im} s\geq 0$. Then the function $\varphi_s^{(\alpha,\beta)}(t)$ is square integrable with respect to the measure $\Delta_{\alpha,\beta}(t)dt$ on $\boldsymbol{R}_+$ if and only if s lies in the set

$$D_{\alpha,\beta}=\{i(|\beta|-\alpha-1-2j)\ ;j\in\boldsymbol{N},|\beta|-\alpha-1-2j>0\}\,. \tag{A.8}$$

The set $D_{\alpha,\beta}$ coincides with the set of the poles of the function $s\mapsto(c_{\alpha,\beta}(s))^{-1}$ for $\operatorname{Im} s\geq 0$.

If $\alpha>-1$ and $\beta\in\boldsymbol{R}$, then the Jacobi transform is inverted by the formula

$$\begin{aligned} f(t)=&\frac{1}{2\pi}\int_0^\infty f^{\wedge}_{\alpha,\beta}(s)\varphi_s^{(\alpha,\beta)}(t)|c_{\alpha,\beta}(s)|^{-2}ds \\ &-i\sum_{s\in D_{\alpha,\beta}} f^{\wedge}_{\alpha,\beta}(s)\varphi_s^{(\alpha,\beta)}(t)\operatorname*{Res}_{\mu=s}((c_{\alpha,\beta}(\mu)c_{\alpha,\beta}(-\mu))^{-1})\,. \end{aligned} \tag{A.9}$$

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