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Nonexistence of Normal Quintic Abelian Surfaces in P^3

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Introduction.

Normal surfaces X_d of degree d in the complex projective 3-space P^3 have simple birational structure if d is small: X_1 and X_2 are rational, and X_3 is birationally equivalent to a ruled surface (for further details, see [B-W], [H-W]), since in general $K_{X_d} \simeq \mathcal{O}_{X_d}(d-4)$. Moreover X_4 is birationally equivalent to either a ruled or a K3 surface ([Um1]).

To the contrary, various X_d may occur if $d \geq 5$. If the singularity of X_d is mild, then X_d is birationally equivalent to a surface of general type, while X_d may be birationally equivalent to a ruled surface if it has severe singularity. Moreover there are examples of X_5 which are birationally equivalent to K3 surfaces, Enriques surfaces or general elliptic surfaces ([I], [Yan], [St], [K], [Um2], [Um3], [Um4]). This leads us to the question whether there exists an X_d which is birationally an abelian or a hyperelliptic surface or not. The purpose of this note is to answer this question in the case of $d=5$. We prove:

MAIN THEOREM. *No normal quintic surface in P^3 is birationally equivalent to an abelian or a hyperelliptic surface.*

Our proof of the theorem goes as follows. First we note that if a normal quintic surface $X = X_5$ is birationally an abelian or a hyperelliptic surface, then its minimal resolution $\tilde{X}$ is an at most 5-fold blowing-up $\mu: \tilde{X} \to \bar{X}$ of the non-singular minimal model $\bar{X}$. On the other hand, the pull-back of K_X to $\tilde{X}$ minus $K_{\tilde{X}}$ is an effective divisor $\tilde{D}$, which reflects the property of the singularity of X fairly well. Such property of $\tilde{D}$ and the condition of $\mu_* \tilde{D}$ as a divisor on an abelian or hyperelliptic surface finally lead us in every case to a contradiction.

CONJECTURE. No normal hypersurface in P^3 is birationally equivalent to an abelian surface.

Also for hyperelliptic surfaces we raise:

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PROBLEM. Are there normal hypersurfaces in P^3 which are birationally hyperelliptic surfaces?

§1. Preliminaries.

In this section we summarize some results from local theory of surface singularity, which we will use later.

Let (Y, y) be a numerically Gorenstein normal surface singularity, $\pi: \tilde{Y} \to Y$ its minimal resolution and A the exceptional set: $A = \pi^{-1}(y)$. Then there is an effective divisor D with $\operatorname{supp} D \subseteqq A$ such that $\omega_{\tilde{Y}} \equiv \mathcal{O}_{\tilde{Y}}(-D)$. Let p_a or p_g stand for the arithmetic or the geometric genus of (Y, y) respectively. By definition ([W]),

$$p_a = \sup_{\substack{D' > 0 \\ \operatorname{supp}(D') \subseteqq A}} p_a(D'), \qquad p_g = \dim R^1 \pi_* \mathcal{O}_{\tilde{Y}}.$$

It is known (cf. [A]) that the following conditions are equivalent:
(i) $D \neq 0$ (ii) $\operatorname{supp} D = A$ (iii) $p_a > 0$ (iv) $p_g > 0$.

LEMMA 1.1 (Y. Koyama). $p_a \leq -D^2/8 + 1$. *In particular, if* $D^2 \geq -7$, *then* $p_a \leq 1$.

PROOF. See [Um5].

REMARK 1.2. If $D^2 = -8$ and $p_a = 2$, then $D/2$ is an integral and the unique divisor on A whose arithmetic genus is equal to 2.

In what follows (except for Corollary 1.5) we assume moreover that $p_a = 1$, i.e. our singularity (Y, y) is elliptic ([W]). Then Yau (for the minimal good resolution) and Tomari (for any resolution) defined the elliptic sequence $\{Z_1, \cdots, Z_l\}$ as follows: Let E denote the minimal elliptic cycle of Laufer [L], i.e., E is the minimal effective divisor such that $\operatorname{supp} E \subseteqq A$ and $p_a(E) = 1$. For Z_1 we take the fundamental cycle. Suppose that we have defined $Z_1, \cdots, Z_k$. If $Z_k E < 0$, we define $\{Z_1, \cdots, Z_k\}$ as the elliptic sequence: $l = k$. Assume $Z_k E = 0$. Then let B_{k+1} denote the connected component containing E of the sum of the components A_i of A satisfying $Z_k A_i = 0$. We define Z_{k+1} to be the fundamental cycle of B_{k+1}. Since $\operatorname{supp} Z_k \supsetneqq \operatorname{supp} Z_{k+1}$, the elliptic sequence $\{Z_1, \cdots, Z_l\}$ is defined as a finite sequence. The following results for the minimal resolution will play an important role later.

THEOREM 1.3 ([T], [Yau]). (i) $D = \sum_{i=1}^{l} Z_i$. (ii) $Z_l = E$. (iii) $p_g \leq l$.

From this theorem, we obtain the following

COROLLARY 1.4. $p_g \leq -D^2$.

COROLLARY 1.5. *Let* (Y, y) *be a numerically Gorenstein normal surface singularity of geometric genus* p_g, *and* $\pi: \tilde{Y} \to Y$ *its minimal resolution. Assume that the exceptional set* $\pi^{-1}(y)$ *consists of a chain of curves* $A_0 = E, A_1, \cdots, A_m$ $(m \geq 1)$ *with* $p_a(E) = 1$, $p_a(A_i) = 0$

$(1 \leq i \leq m)$. *Then we have*

(i) $E^2 = -1$,

(ii) $m \geq p_g - 1$ *and* $A_i^2 = -2$ *for* $1 \leq i \leq p_g - 2$.

PROOF. The fundamental cycle Z_1 coincides with $\pi^{-1}(y)$ with reduced structure, hence $p_a(Z_1) = 1$, and so (Y, y) is an elliptic singularity ([W]). Then E is the minimal elliptic cycle. Theorem 1.3 implies that for every i, Z_i contains more than $p_g - i$ components. In particular, for $i = 1$, we get $m \geq p_g - 1$; for $i = 2$, $Z_1 E = 0$ and $Z_1 A_i = 0$ $(1 \leq i \leq p_g - 2)$, which proves $E^2 = -1$ and $A_i^2 = -2$ $(1 \leq i \leq p_g - 2)$.

§2. Properties of divisors on the resolution.

Let X be a normal quintic surface in $\boldsymbol{P}^3$. Let $\pi: \tilde{X} \to X$ denote the minimal resolution of X, H a general hyperplane section of X and $\tilde{H}$ its pull-back on $\tilde{X}$. Then there exists a unique effective divisor $\tilde{D}$ on $\tilde{X}$ such that $K_{\tilde{X}} = \tilde{H} - \tilde{D}$. This divisor $\tilde{D}$ is supported on the exceptional sets of π which correspond to singularities with positive geometric genus. Let $\mu: \tilde{X} = X_n \xrightarrow{\mu_n} X_{n-1} \xrightarrow{\mu_{n-1}} \cdots \xrightarrow{\mu_1} X_0 = \bar{X}$ be the sequence of blow-downs obtaining a non-singular minimal model $\bar{X}$ of $\tilde{X}$, μ_i' the induced morphism $\tilde{X} \to X_i$ $(0 \leq i \leq n)$, and E_i $(1 \leq i \leq n)$ the total transform on $\tilde{X}$ of the exceptional curve of the blow-up μ_i. In what follows we fix our notations as above and assume moreover that $\bar{X}$ is either an abelian or a hyperelliptic surface.

LEMMA 2.1. $\tilde{D}^2 = -n - 5$ *and* $1 \leq n \leq 5$. *Moreover, if* $n = 5$ *and if* Γ *is a rational curve on* $\tilde{X}$, *then either*

(i) $\tilde{H}\Gamma = 1$ *(Γ is not exceptional for π),* $\tilde{D}\Gamma = 2$, $\Gamma^2 = -1$, *or*

(ii) $\tilde{H}\Gamma = 0$ *(Γ is exceptional for π),* $\tilde{D}\Gamma = 0$, $\Gamma^2 = -2$.

PROOF. Since $\bar{X}$ has a numerically trivial canonical bundle, $-n = K_{\tilde{X}}^2 = (\tilde{H} - \tilde{D})^2 = 5 + \tilde{D}^2$, and hence $\tilde{D}^2 = -n - 5$. Since each E_i contains at least one (-1)-curve and $\tilde{X}$ is the minimal resolution, we have $\tilde{H}E_i > 0$, and so $5 = \tilde{H}^2 = \tilde{H}(\tilde{H} - \tilde{D}) = \tilde{H}K_{\tilde{X}} = \sum_{i=1}^n \tilde{H}E_i \geq n$. $n \geq 1$ since $\tilde{H} - \tilde{D} \not\equiv 0$. Note that any rational curve on $\tilde{X}$ is a component of E_i for some i since $\bar{X}$ contains no rational curve. Assume $n = 5$. Then $\tilde{H}E_i = 1$ $(1 \leq i \leq 5)$. Hence, for each i, there exists a unique component Γ_i in E_i, with multiplicity 1, such that $\tilde{H}\Gamma_i = 1$, and other components of E_i are exceptional for π and so have non-positive intersection number with $\tilde{D}$. Since Γ_i is a (-1)-curve, $-1 = K_{\tilde{X}}\Gamma_i = (\tilde{H} - \tilde{D})\Gamma_i$, hence $\tilde{D}\Gamma_i = 2$. By $-1 = K_{\tilde{X}}E_i = K_{\tilde{X}}\Gamma_i + K_{\tilde{X}}(E_i - \Gamma_i) = -1 + \tilde{D}(E_i - \Gamma_i)$, we see that any component Γ in $E_i - \Gamma_i$ satisfies $\tilde{D}\Gamma = 0$ and so $\Gamma^2 = -2$.

LEMMA 2.2. *For each i $(1 \leq i \leq n)$, the center of the blow-up μ_i lies on the singular locus of* $(\mu_{i-1}')_* \tilde{D}$.

PROOF. Since $-1 = K_{\tilde{X}}E_i = \tilde{H}E_i - \tilde{D}E_i$ and $\tilde{H}E_i > 0$, we have $\tilde{D}E_i \geq 2$, which implies the Lemma.

LEMMA 2.3. $\dim R^1\pi_*\mathcal{O}_{\tilde{X}}=5$.

PROOF. From the exact sequence accociated with the Leray spectral sequence:

$$0\to H^1(X,\mathcal{O}_X)\to H^1(\tilde{X},\mathcal{O}_{\tilde{X}})\to R^1\pi_*\mathcal{O}_{\tilde{X}}\to H^2(X,\mathcal{O}_X)\to H^2(\tilde{X},\mathcal{O}_{\tilde{X}})\to 0\,,$$

we have $\dim R^1\pi_*\mathcal{O}_{\tilde{X}}=4+q(\tilde{X})-p_g(\tilde{X})=5$.

COROLLARY 2.4. *Let $\tilde{D}=\tilde{D}_1+\cdots+\tilde{D}_s$ be the decomposition of $\tilde{D}$ into its connected components. Then*

$$\sum_{i=1}^{s} h^0(\mathcal{O}_{\tilde{D}_i})=5\,.$$

PROOF. Let y_i denote the singular point on X, which is obtained by contracting $\tilde{D}_i$. Let $Y_i\subset X$ be a Stein neighbourhood of y_i, $\tilde{Y}_i=\pi^{-1}(Y_i)$ and $\pi_i=\pi_{|\tilde{Y}_i}$. Note that $\omega_{\tilde{Y}_i}\simeq\mathcal{O}_{\tilde{Y}_i}(-\tilde{D}_i)$. Then we have $h^0(\mathcal{O}_{\tilde{D}_i})=\dim R^1(\pi_i)_*\mathcal{O}_{\tilde{Y}_i}$, so that the Corollary follows from Lemma 2.3. In fact, consider the natural exact sequence:

$$H^1(\mathcal{O}_{\tilde{Y}_i}(-\tilde{D}_i))\to H^1(\mathcal{O}_{\tilde{Y}_i})\to H^1(\mathcal{O}_{\tilde{D}_i})\to 0\,.$$

We see that the first term vanishes by Grauert-Riemenschneider's theorem. The second term is isomorphic to $R^1(\pi_i)_*\mathcal{O}_{\tilde{Y}_i}$, and the third is dual to $H^0(\mathcal{O}_{D_i})$.

LEMMA 2.5. *Let D be a connected divisor on $\tilde{X}$ with negative intersection matrix. If all irreducible components of D are rational curves, then the contraction of D is at worst rational singularity.*

PROOF. Since $\bar{X}$ contains no rational curve, the support of D is contained in a divisor which is contracted to a non-singular point (by μ). Hence the geometric genus of the contraction of D vanishes.

LEMMA 2.6. *Let C and C' be irreducible curves on an abelian surface [resp. a hyperelliptic surface] S. Then*

(i) *C^2 is an even non-negative integer,*

(ii) *$C^2=0$ if and only if C is a non-singular elliptic curve,*

(iii) *if the desingularization of C is an elliptic curve, then C itself is smooth,*

(iv) *if $CC'=0$, then $C^2=C'^2=0$ and C and C' are algebraically equivalent [resp. if $CC'=0$, then $C^2=C'^2=0$ and $C\equiv qC'$ for some positive $q\in\mathbf{Q}$],*

(v) *if C and C' are elliptic curves, then they intersect transversally.*

PROOF. Since S has trivial or numerically trivial canonical sheaf, $p_a(C)=C^2/2+1$. Moreover S contains no rational curves, whence (i) and (ii). For (iii)–(v), we first assume that S is an abelian surface. Then (iii) is a special case of [Ue, Theorem 10.3]. (iv) If $CC'=0$, then neither C nor C' is ample, and so $C^2=C'^2=0$. Moreover C' is a fiber of the quotient morphism $S\to S/C$. Hence C and C' are algebraically equivalent. Finally, if two elliptic curves C and C' intersect, then the morphism $C'\to S/C$ is finite and unramified, and hence (v). Suppose next that S is a hyperelliptic surface. Then there is a finite unramified covering $f\colon \tilde{S}\to S$ where $\tilde{S}$ is an abelian surface. Notice that any unramified cover of an elliptic curve is a disjoint union of elliptic curves. Hence (v) is clear. (iii) Let $\tilde{C}\to C$ denote the desingularization of C. Then $\tilde{S}\times_S\tilde{C}$ is the resolution of $f^{-1}(C)$. Since $\tilde{S}\times_S\tilde{C}$ is a disjoint union of non-singular elliptic curves, we see that $f^{-1}(C)$ itself is non-singular, and hence so is C. (iv) The former part is proved in the same way as in the abelian case. Moreover, $C^2=C'^2=0$ and $CC'=0$ mean that $[C]$ and $[C']$ are not linearly independent in $NS(S)\otimes \boldsymbol{Q}$.

§3. Reduction to the case with only elliptic singularities.

We use the notations in §2, and assume that $\bar{X}$ is either an abelian or a hyperelliptic surface. In this section we will prove that there exists on X no singularity with arithmetic genus greater than 1. We first notice by Lemma 2.5 that every connected component of $\tilde{D}$ contains a non-rational curve.

(3.1) Assume that there exists in $\tilde{D}$ an irreducible curve D_1 with $p_a(D_1)\geq 2$. Let D denote the connected component of $\tilde{D}$ containing D_1 and D' the sum of the other components: $\tilde{D}=D+D'$. By Lemma 1.1 and 2.1, we have $-10\leq D^2\leq -8$ and the arithmetic genus of the singularity corresponding to D is equal to 2. Hence, by Lemma 2.6 (iii), D_1 is a non-singular curve of genus 2 and the other components of D, if exist, are all non-singular rational curves. Since $0\geq D'^2\geq -2$, all singular points corresponding to D' are elliptic singularities.

Case 3.1.1. $D^2=-8$: Remark 1.2 says that $D=2D_1$ and $D_1^2=-2$. The exact sequence

$$0\to \mathcal{O}_{D_1}(-D_1)\to \mathcal{O}_{2D_1}\to \mathcal{O}_{D_1}\to 0$$

shows

$$h^0(\mathcal{O}_D)=h^0(\mathcal{O}_{D_1}(-D_1))+h^0(\mathcal{O}_{D_1})=h^0(\omega_{D_1})+1=3\,.$$

Hence, by Corollary 2.4 and its proof, it follows that D' corresponds to either one singular point with geometric genus equal to 2 or two singular points both of which have geometric genus 1. Hence $D'^2=-2$, $n=5$ (Corollary 1.4 and Lemma 2.1). By Lemma 2.6, we have $(\mu_*D)^2>0$ and $(\mu_*D)(\mu_*D')>0$ and so there is a chain of rational curves $\Gamma_1,\cdots,\Gamma_k$ on $\tilde{X}$ such that $\Gamma_i\not\subset\tilde{D}$ $(1\leq i\leq k)$ and $\Gamma_1D_1>0$, $\Gamma_kD'>0$. Both Γ_1 and Γ_k are not exceptional for π, and hence are (-1)-curves by Lemma 2.1. For $2\leq i\leq k-1$, Γ_i is either a (-1)-curve or else an exceptional curve for π, i.e., a (-2)-curve. Therefore it turns out that $k=1$: there exists a (-1)-curve Γ such that $\tilde{D}\Gamma=(2D_1+D')\Gamma\geq 3$, which contradicts Lemma 2.1.

Case 3.1.2. $D^2=-9$: Let $D=mD_1+Y$ $(Y\not\geq D_1)$. Since $2=p_a(D_1)=(D_1^2-D_1D)/2+1$,

$$D_1D=D_1^2-2\leq -3\,. \tag{1}$$

Therefore

$$-9=D^2=mD_1D+YD\leq mD_1D=m(D_1^2-2)\leq -3m\,, \tag{2}$$

and so

$$1\leq m\leq 3\,.$$

If $m=1$, then $D_1D=D_1^2+D_1Y\geq D_1^2$, which contradicts (1).

If $m=2$, then, by (1), $-3\geq D_1D=2D_1^2+D_1Y$ and so $D_1^2\leq -2$. Hence $D_1^2=-2$ by (2). Hence, using (1), we have

$$D_1Y=D_1D-2D_1^2=D_1^2-2-2D_1^2=0\,.$$

This implies $Y=0$ and so $D^2=(2D_1)^2=-8$, a contradiction.

Assume $m=3$. Then $D_1^2=-1$ by (2) and hence $D_1Y=D_1D-3D_1^2=0$ by (1), i.e. $Y=0$: $D=3D_1$. Therefore

$$h^0(\mathcal{O}_D)\leq h^0(\mathcal{O}_{D_1}(-2D_1))+h^0(\mathcal{O}_{2D_1})=h^0(\omega_{D_1})+h^0(\mathcal{O}_{D_1}(-D_1))+h^0(\mathcal{O}_{D_1})\leq 4\,.$$

Hence we have $D'\neq 0$, $n=5$, and are led to a contradiction as in Case 3.1.1.

Case 3.1.3. $D^2=-10$: In this case $D=\tilde{D}$ and $n=5$. We set $\tilde{D}=mD_1+Y$ as before, where $\tilde{D}Y=0$ and Y consists of (-2)-curves by Lemma 2.1. Note that (1) in the previous case holds as well. Hence we have

$$-10=\tilde{D}^2=mD_1\tilde{D}=m(D_1^2-2)\leq -3m\,,$$

therefore $m=1$ or 2.

If $m=1$, then $D_1\tilde{D}=-10$ and $D_1^2=-8$, which is impossible because $D_1\tilde{D}=D_1^2+D_1Y\geq D_1^2$.

If $m=2$, then $D_1\tilde{D}=-5$ and $D_1^2=-3$, hence $\tilde{D}=2D_1+Y$ and $D_1Y=1$. This implies that Y is a reduced irreducible (-2)-curve. But then we calculate

$$h^0(\mathcal{O}_{\tilde{D}})=h^0(\mathcal{O}_{D_1}(-D_1-Y))+h^0(\mathcal{O}_{D_1+Y})=h^0(\omega_{D_1})+1=3\,,$$

and so get a contradiction with Corollary 2.4.

Hence we have proved with Lemma 2.6 that every non-rational component of $\tilde{D}$ is a non-singular elliptic curve.

(3.2) Suppose that $\tilde{D}$ has a connected component D which contains two distinct non-singular elliptic curves D_1 and D_2. Then D corresponds to a singularity with $p_a=2$ (Lemma 1.1). We set $\tilde{D}=D+D'$.

Case 3.2.1. $D^2=-8$: We can show a contradiction in a similar way as in Case 3.1.1, by taking a chain of reduced curves in D connecting D_1 and D_2 instead of D_1.

Case 3.2.2. $D^2=-9$: Set $D=m_1D_1+m_2D_2+Y$, $Y\not\geq D_1, D_2$.
From

$$D_i^2=D_iD=m_iD_i^2+m_jD_1D_2+D_iY,$$

we have

$$(1-m_i)D_i^2=m_jD_1D_2+D_iY \tag{3}$$

where $\{i,j\}=\{1,2\}$. Since D is connected, the right hand side of (3) is positive, and so $m_i\geq 2$ $(i=1,2)$. Also we have

$$m_1D_1^2+m_2D_2^2=(m_1D_1+m_2D_2)D=D^2-DY\geq D^2=-9. \tag{4}$$

Consider first the case of $D_1D_2>0$. Then $D_1D_2=1$ since $p_a=2$.

We first show that $D_i^2\leq -2$ $(i=1,2)$. Assume to the contrary, say $D_1^2=-1$. Then (3) is reduced to

$$m_1-1=m_2+D_1Y, \tag{5}$$
$$(1-m_2)D_2^2=m_1+D_2Y,$$

hence

$$(1-m_2)D_2^2=m_2+1+(D_1+D_2)Y. \tag{6}$$

From (5), we get $m_1\geq m_2+1$. (6) implies $D_2^2\leq -2$, but if $D_2^2=-2$, then $m_2\geq 3$, and so $m_1\geq 4$, which contradicts (4). Therefore the unique possibility is $D_2^2=-3$, $m_1=3$, $m_2=2$ and $D_1Y=D_2Y=0$, i.e. $D=3D_1+2D_2$ with $D_1^2=-1$, $D_2^2=-3$. But then we have

$$h^0(\mathcal{O}_D)\leq h^0(\mathcal{O}_{D_1}(-2D_1-2D_2))+h^0(\mathcal{O}_{D_2}(-2D_1-D_2))+h^0(\mathcal{O}_{D_1}(-D_1-D_2))$$
$$+h^0(\mathcal{O}_{D_1+D_2})\leq 4.$$

Hence it follows $D'\neq 0$ and $n=5$, and then a contradiction as in Case 3.1.1. Thus we obtain $D_1^2, D_2^2\leq -2$.

By (4) we have $D_1^2=D_2^2=-2$, $m_1=m_2=2$ and $DY=-1$. But then (3) implies $D_1Y=D_2Y=0$ and hence $Y=0$, a contradiction.

Therefore we obtain $D_1D_2=0$, in particular $D_1Y>0$, $D_2Y>0$. Then, by (3),

$$(1-m_i)D_i^2=D_iY \qquad (i=1,2). \tag{7}$$

If $D_1Y=1$, then $m_1=2$, $D_1^2=-1$, and there exists a unique component Y_1 of Y such that $Y_1D_1=1$ and that the multiplicity of Y_1 in Y is 1. Therefore

$$-2=Y_1^2-Y_1D=Y_1(-2D_1-m_2D_2-(Y-Y_1))$$
$$=-2-Y_1(m_2D_2+(Y-Y_1))<-2.$$

Hence $D_1Y\geq 2$, $D_2Y\geq 2$.

Suppose $D_i^2=-1$. Then, since $(\mu_*D_i)^2=0$ (Lemma 2.6 (ii), (iii)), there exists a unique component Y_i of Y such that $D_iY_i=1$ and every other component of Y is disjoint from D_i. Moreover, we see from (7) that the multiplicity of Y_i in Y is m_i-1, and that $m_i\geq 3$ since $D_iY\geq 2$. We note that $(\mu_*D_1)(\mu_*D_2)>0$. Hence, if furthermore $D_1^2=D_2^2=-1$, we get $m_1=m_2\geq 3$ because in this situation $Y_1=Y_2$ in the notation above. Therefore, assuming $D_1^2\geq D_2^2$ in general, we have by (4) and (7) that there are the following possibilities:

	D_1^2	D_2^2	m_1	m_2	DY
(i)	-1	-1	3	3	-3
(ii)	-1	-1	4	4	-1
(iii)	-1	-2	3	2	-2
(iv)	-1	-2	3	3	0
(v)	-1	-3	3	2	0
(vi)	-1	-2	4	2	-1
(vii)	-1	-2	5	2	0
(viii)	-2	-2	2	2	-1

In (i) and (ii), there is a curve Y_1 of multiplicity $m_1-1\geq 2$ in Y with $D_1Y_1=D_2Y_1=1$. Note that $Y_1^2\leq -3$, since if $Y_1^2=-2$ then $0=DY_1\geq 2m_1+(m_1-1)(-2)=2$. Hence (ii) is impossible, and in (i) we have $Y_1^2=-3$ by $DY=-3$. But then $-1=DY_1\geq 2m_1+(m_1-1)(-3)=0$. In (iv) and (v), all components of Y are (-2)-curves. There is a component Y_1 with multiplicity $m_1-1=2$ in Y and $D_1Y_1=1$. Hence $(D-3D_1-2Y_1)Y_1=1$, and so there is a unique component Y_2 with multiplicity 1 in Y such that $Y_1Y_2=1$. This implies $D=3D_1+2Y_1+Y_2$, which is absurd. In (vi)–(viii), where $D_2^2=-2$ and $m_2=2$, there is a curve Y_2 of multiplicity 2 in Y with $D_2Y_2=1$. Since $DY\geq -1$, Y_2 is a (-2)-curve. Hence there is a unique curve Y_3 in $D-2D_2-2Y_2$ of multiplicity 2 in it with $Y_2Y_3=1$, Y_3 is a (-2)-curve if $Y_3\leq Y$. Proceeding in this way, we find in (vi) and (vii) an infinite sequence $Y_2, Y_3, \cdots$ in Y; in (viii) $D=2(D_2+Y_2+\cdots+D_1)$, contradicting $D^2=-9$. Therefore it only remains the case (iii). Since then also $D_2^2=-2$ and $m_2=2$, we can start from D_2 in the same way as above and deduce $D=3D_1+2Y_1+\cdots+2Y_k+2D_2$, where $k\geq 1$, $Y_1^2=-3$, $Y_2^2=\cdots=Y_k^2=-2$ and $D_1, Y_1, \cdots, Y_k, D_2$ form a chain. Hence we obtain

$$\begin{aligned} h^0(\mathcal{O}_D)&\leq h^0(\mathcal{O}_{D_1}(-2D_1-2Y_1-\cdots-2Y_k-2D_2)) \\ &\quad +h^0(\mathcal{O}_{D_1+Y_1+\cdots+Y_k+D_2}(-D_1-Y_1-\cdots-D_2))+h^0(\mathcal{O}_{D_1+Y_1+\cdots+Y_k+D_2}) \\ &\leq 4\,, \end{aligned}$$

and so $D'\neq 0$, $n=5$, hence a contradiction as in Case 3.1.1.

Case 3.2.3. $D^2=-10$: Note first that $D=\tilde{D}$ and $n=5$. Set $\tilde{D}=m_1D_1+m_2D_2+Y$ as in Case 3.2.2. Then we have as before

$$\begin{aligned} &(1-m_i)D_i^2=m_jD_1D_2+D_iY\,, \qquad \{i,j\}=\{1,2\}\,, \\ &m_1, m_2\geq 2\,, \end{aligned} \tag{8}$$

and since Y consists of (-2)-curves by Lemma 2.1,

$$m_1D_1^2+m_2D_2^2=(m_1D_1+m_2D_2)\tilde{D}=\tilde{D}^2-\tilde{D}Y=\tilde{D}^2=-10\,. \tag{9}$$

Suppose $D_1D_2>0$. Then $D_1D_2=1$ since $p_a=2$, and so (8) is rewritten as

$$\begin{aligned}&(1-m_i)D_i^2=m_j+D_iY\,, \qquad \{i,j\}=\{1,2\}\,,\\ &m_1, m_2\geq 2\,.\end{aligned} \tag{10}$$

If $D_1^2=-1$, then $m_1=m_2+1+D_1Y$ by (10). In particular $D_2^2\leq -2$. If furthermore $D_2^2\leq -3$, then $D_2^2=-3$, $m_1=4$ and $m_2=2$ by (9), which is not compatible with (10). Hence $D_2^2=-2$, and we see with (9) and (10) that only $(m_1, m_2)=(4, 3)$ is possible. The case of $D_i^2\leq -2$ is easier. Then, assuming $D_1^2\geq D_2^2$, there are two possibilities:

	D_1^2	D_2^2	m_1	m_2	D_1Y	D_2Y
(i)	-1	-2	4	3	0	0
(ii)	-2	-3	2	2	0	1

In (i) we get $Y=0$: $\tilde{D}=4D_1+3D_2$. But then $\tilde{D}$ can not be obtained from $\bar{X}$ by more than 3 blow-ups (Lemma 2.2 and Lemma 2.6 (ii), (iii)). In (ii) Y is a reduced irreducible (-2)-curve since $D_2Y=1$ and $m_2=2$. This implies

$$h^0(\mathcal{O}_{\tilde{D}})=h^0(\mathcal{O}_{D_1+D_2}(-D_1-D_2-Y))+h^0(\mathcal{O}_{D_1+D_2+Y})=h^0(\omega_{D_1+D_2})+1=3\,,$$

which is impossible by Corollary 2.4.

This proves $D_1D_2=0$ and so by (8)

$$(1-m_i)D_i^2=D_iY \qquad (i=1,2)\,.$$

Then we have, as in Case 3.2.2, $m_i\geq 2$; $m_i\geq 3$ if $D_i^2=-1$; $m_1=m_2$ if $D_1^2=D_2^2=-1$. We may assume that $D_1^2\geq D_2^2$ and that $m_1\leq m_2$ if $D_1^2=D_2^2$. Then, by (9), the possibilities are as follows:

	D_1^2	D_2^2	m_1	m_2
(i)	-1	-1	5	5
(ii)	-1	-2	4	3
(iii)	-1	-2	6	2
(iv)	-1	-3	4	2
(v)	-2	-2	2	3
(vi)	-2	-3	2	2

For (i), (ii) and (iv), let $Y_0=D_1, Y_1, \cdots, Y_k, Y_{k+1}=D_2$ denote the chain of curves in $\tilde{D}$ connecting D_1 and D_2 $(k\geq 1)$, and l_j the multiplicity of Y_j in $\tilde{D}$. Since $D_1^2=-1$, $l_1=m_1-1$. Moreover, since Y_j is a (-2)-curve for $1\leq j\leq k$, we have $l_j-1\geq l_{j+1}$ for $0\leq j\leq k$, and so $m_1-k-1\geq m_2$. Hence (i) and (ii) are impossible. In (iv) we have $k=1$, hence $\tilde{D}=4D_1+3Y_1+2D_2$. But then there exists a (-1)-curve Γ on $\tilde{X}$ such that $\Gamma Y_1>0$ and so $\tilde{D}\Gamma=3$, which contradicts Lemma 2.1. Replace D_1 and D_2 in (iii). Then for each of the remaining cases we notice that $D_1^2=-2$ and $m_1=2$. Let $Y_0, Y_1, \cdots, Y_k, Y_{k+1}$ be as above. Then we have $\tilde{D}=2D_1+2Y_1+\cdots+2Y_k+2D_2+Y'$, where Y' consists of (-2)-curves and is disjoint from $D_1, Y_1, \cdots, Y_k$. Hence only the case (vi) can occur. Since $D_2\tilde{D}=-3$, $D_2Y'=1$, i.e., Y' is reduced and irreducible. But then there exists a (-1)-curve Γ on $\tilde{X}$ such that $Y'\Gamma=1$, hence $\tilde{D}\Gamma$ is odd, contradicting Lemma 2.1.

(3.3). By what we have proved, every connected component D of $\tilde{D}$ is the union of an elliptic curve and some exceptional rational curves for μ. Let F be a non-trivial effective divisor on $\tilde{X}$ with $\operatorname{supp} F \subseteqq \operatorname{supp} D$. Then $F=\mu^*\bar{F}+\sum_{i=1}^n a_iE_i$ where $\bar{F}=\mu_*F$ and $a_i \in \boldsymbol{Z}$ $(1\le i\le n)$. Therefore

$$p_a(F)=\frac{1}{2}\left(\mu^*\bar{F}+\sum_{i=1}^n a_iE_i\right)\left(\mu^*\bar{F}+\sum_{i=1}^n a_iE_i+\sum_{i=1}^n E_i\right)+1$$

$$=\frac{1}{2}\bar{F}^2+1-\frac{1}{2}\sum_{i=1}^n a_i(a_i+1)\,.$$

Here $\bar{F}$ is a multiple of an elliptic curve and so $\bar{F}^2=0$ by Lemma 2.6. Moreover $a(a+1)\ge 0$ for any integer a. Hence $p_a(F)\le 1$, and so the singularity corresponding to D is elliptic.

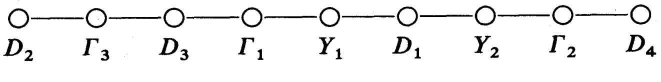
§4. The case with elliptic singularities.

We continue to use the notations in §2 and assume that $\bar{X}$ is either an abelian or a hyperelliptic surface. To complete our proof of the Theorem, we will deduce a contradiction under the assumption that every connected component D of $\tilde{D}$ corresponds to an elliptic singularity. Recall that then D consists of a non-singular elliptic curve and possibly some rational curves.

(4.1) Assume that there exists a connected component D of $\tilde{D}$ which contains a rational curve. This is equivalent to that X has a singularity with $p_g\ge 2$. Set $\tilde{D}=D+D'$ and let D_1 denote the unique elliptic curve in D. Then D_1 is the minimal elliptic cycle of D. In general, we note that $\mu_*\tilde{D}$ is numerically equivalent to $\mu_*\tilde{H}$, and hence is connected and ample. In particular $D'\ne 0$.

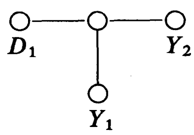
Suppose first that the rational components of D are not connected. Then $D_1^2\le -2$ and there are exactly two connected components of rational curves because the number of blow-ups n is bounded by 5 (Lemma 2.1). If one of them has length ≥ 2, then the equality holds, the other has length 1, and every rational curve in D is a (-2)-curve. It follows that D can not be contracted to a numerically Gorenstein singularity (Theorem 1.3 (ii)). Hence we have that $D=2D_1+Y_1+Y_2$, where Y_i is a rational curve with $D_1Y_1=D_1Y_2=1$, $Y_1Y_2=0$ and $D_1^2=-2$, and that the corresponding singularity has a geometric genus equal to 2 (Theorem 1.3 (i), (iii)). Hence the sum of the geometric genera of singularities corresponding to D' is equal to 3 (Lemma 2.3). Moreover, if

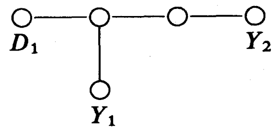
$Y_1^2 \leq -3$, then $n=5$, which contradicts Lemma 2.1. Hence we have $Y_1^2 = Y_2^2 = -2$, and so there exist rational curves Γ_1 and Γ_2 such that $Y_i \Gamma_j = \delta_{ij}$. If $n=4$, then Γ_1 and Γ_2 are (-1)-curves. There is on $\tilde{X}$ no rational curve other than Y_1, Y_2, Γ_1 and Γ_2, and so D' consists of three disjoint elliptic curves: $D' = D_2 + D_3 + D_4$, each of which meets either Γ_1 or Γ_2. We may assume that $D_2 \Gamma_1 > 0$ and $D_3 \Gamma_1 > 0$. But then $\mu_* D_2$ intersects $\mu_* D_3$ tangentially, contradicting Lemma 2.6 (v). If $n=5$, then Γ_1 and Γ_2 are also (-1)-curves by Lemma 2.1. There is another (-1)-curve Γ_3 which is disjoint from D, Γ_1 and Γ_2. Moreover $D' = D_2 + D_3 + D_4$ as before. Since $\tilde{D} \Gamma_j = 2$ and $D_i \Gamma_j \leq 1$ $(1 \leq i \leq 4,\ 1 \leq j \leq 3)$ by Lemma 2.1 and 2.6 (iii), we can assume that the dual graph of $D_1, \cdots, D_4, Y_1, Y_2, \Gamma_1, \Gamma_2, \Gamma_3$ is as follows:



But then we obtain $(\mu_* D_2 + \mu_* D_3)^2 > 0$, $(\mu_* D_4)^2 = 0$ and $(\mu_* D_2 + \mu_* D_3)(\mu_* D_4) = 0$, a contradiction to Hodge Index Theorem. It follows that there is a unique rational component Y_1 in D which intersects D_1.

Suppose that there is a rational curve in D which intersects more than two other components in D. From Lemma 2.1, we can deduce that the dual graph of D is one of the following:

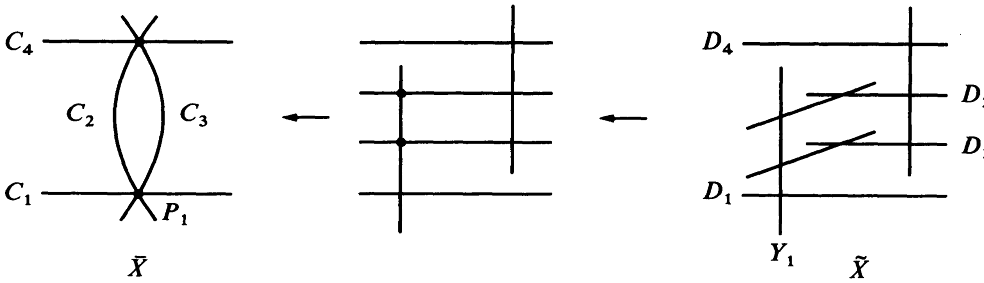
(i) 

(ii) 

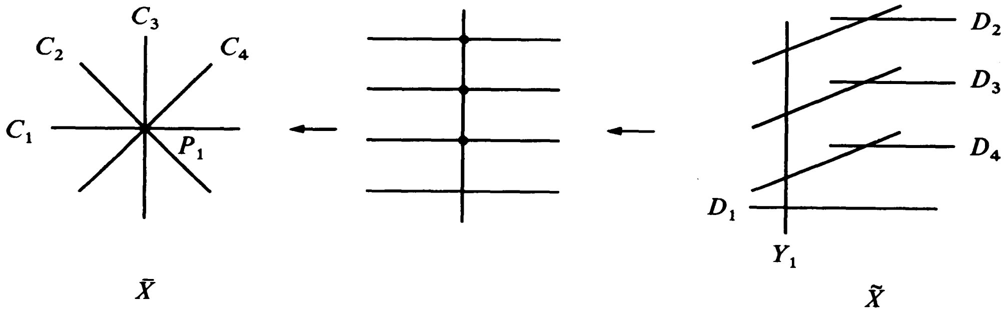
In (ii) we have $n=5$. Hence all rational components of D are (-2)-curves (Lemma 2.1), and so there is a unique rational curve Γ, which is a (-1)-curve, except the components of D. Γ intersects Y_1 or Y_2, and also every elliptic curve D_i in D'. Therefore $\mu_* D_1$ and $\mu_* D_i$ can not intersect transversally, which contradicts Lemma 2.6 (v). In (i), if $n=4$, then there is a (-1)-curve Γ, which plays a similar role as Γ in (ii), and we are led to a contradiction. Assume $n=5$. Then every rational component in $\tilde{D}$ is a (-2)-curve, hence again there is a unique (-1)-curve Γ_1 which intersects Y_1 or Y_2. If $D'\Gamma_1 > 0$, we get a contradiction as in (ii). If $D'\Gamma_1 = 0$, then there is another (-1)-curve Γ_2 such that $D_1 \Gamma_2 > 1$, $D'\Gamma_2 > 0$. But the multiplicity of D_1 in D is greater than 1 (Theorem 1.3), and so we have $D\Gamma_2 \geq 2$, hence $\tilde{D}\Gamma_2 \geq 3$, which contradicts Lemma 2.1. Therefore we conclude that D consists of a chain $D_1, Y_1, \cdots, Y_m$ $(m \geq 1)$ of an elliptic curve D_1 and rational curves Y_i.

Let $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_s$ be the decomposition of $\tilde{D}$ into its connected components with $\tilde{D}_1 = D$. Let p_i denote the geometric genus of the singularity corresponding to $\tilde{D}_i$, D_i the unique elliptic curve in $\tilde{D}_i$. We note that $D_i^2 = -1$ if $p_i \geq 2$ (Corollary 1.5 (i)), and that $D_i^2 \geq -3$ if $p_i = 1$, since then $\tilde{D}_i = D_i$ is to be contracted to a simple elliptic hypersurface singularity ([Sa]). Set $C_i = \mu_* D_i$, then C_i is also a non-singular elliptic curve. Recall that $C_1 + \cdots + C_s$ is connected and ample. In particular $s \geq 2$ and there exists i

$(2 \le i \le s)$ such that $C_1C_i > 0$. Since $D_1^2 = -1$, we see that $C_1C_i = 1$. We may assume that the point $P_1 = C_1 \cdot C_i$ is the center of the first blow-up μ_1. By our assumption, the proper transform Y_1 on $\tilde{X}$ of $\mu_1^{-1}(P_1)$ is a component of D, and so $D_i^2 \le -2$, and hence $p_i = 1$. Suppose that some p_j $(2 \le j \le s)$ is greater than 1. Then we have $C_1C_j = 0$, and so $C_iC_j > 0$ for any i such that $C_1C_i > 0$, since $C_1 + C_i$ is ample (or by Lemma 2.6 (iv)). Hence our assumption that $p_j \ge 2$ implies $D_i^2 \le -4$, which is impossible. Therefore we have $p_i = 1$ for $2 \le i \le s$, and so $p_1 + s - 1 = 5$ (Lemma 2.3). We may assume that $P_1 \in C_i$ $(1 \le i \le s_1)$ and $P_1 \notin C_i$ $(s_1 + 1 \le i \le s)$ for some s_1 $(2 \le s_1 \le s)$. Since $C_{s_1+1}, \cdots, C_s$ are disjoint from C_1, they are also disjoint from each other. Hence $C_2 \cap (\sum_{i=s_1+1}^{s} C_i)$ consists of greater than or equal to $s - s_1$ distinct points. This proves, with Corollary 1.5 (ii), $-3 \le D_2^2 \le -p_1 - (s - s_1) = s_1 - 6$. On the other hand, if $p_1 \ge 3$, then $Y_1^2 = -2$ by Corollary 1.5 (ii), and so $s_1 = 2$ (cf. Lemma 2.6 (v)), which is impossible. Hence we obtain $p_1 = 2$, $s = 4$ and $3 \le s_1 \le 4$. The last inequality implies $Y_1^2 \le -3$ and hence $n \le 4$ (Lemma 2.1). If $s_1 = 3$, then we have by Theorem 1.3 $\tilde{D}_1 = 2D_1 + Y_1$ with $Y_1^2 = -3$, $D_2^2 = D_3^2 = -3$ and $D_4^2 = -1$:



If $s_1 = 4$, then $\tilde{D}_1 = 2D_1 + Y_1$ with $Y_1^2 = -4$ and $D_2^2 = D_3^2 = D_4^2 = -2$:



In both cases we obtain $\tilde{D}^2 = -10$, contradictory to Lemma 2.1.

Thus we proved that $\tilde{D}$ has no rational components.

(4.2) Finally let us consider the case where $\tilde{D}$ consists of disjoint non-singular elliptic curves. Lemma 2.3 implies that $\tilde{D}$ has five components. Set $\tilde{D} = \sum_{i=1}^{5} D_i$ and $C = \mu_* D = \sum_{i=1}^{5} C_i$ where $C_i = \mu_* D_i$. C_i and D_i are non-singular elliptic curves $(1 \le i \le 5)$, D_i's are disjoint, but C is connected. Let P_i $(1 \le i \le n)$ denote the center of the blow-up μ_i. Then, by Lemma 2.2 and 2.6 (v), $P_1, \cdots, P_n$ are not infinitely near each other, hence we may regard them as distinct points on $\bar{X}$. Set $k_i = \mathrm{mult}_{P_i} C$. Then k_i is equal to the number of curves C_j which pass through P_i. Lemma 2.2 says $k_i \geq 2$ $(1 \leq i \leq n)$, and we may assume $5 \geq k_1 \geq k_2 \geq \cdots \geq k_n \geq 2$.

With these notations we have first from Lemma 2.1

$$\sum_{i=1}^{n} k_i = n+5 \tag{11}$$

since $-n-5 = \tilde{D}^2 = \sum_{j=1}^{5} D_j^2 = \sum_{j=1}^{5} C_j^2 - \sum_{i=1}^{n} k_i = -\sum_{i=1}^{n} k_i$. Next, let us show that $C_i C_j > 0$ for any i, j $(i \neq j)$. Let s denote the maximal number of components in C, which are disjoint each other. We may assume that $C_1, \cdots, C_s$ are disjoint. Lemma 2.6 (iv) implies $C_j \equiv q_j C_1$ for $2 \leq j \leq s$, where q_j are some positive rational numbers. Hence we obtain by (11)

$$s(5-s) \leq \left(\sum_{j=1}^{s} C_j\right)\left(\sum_{j=s+1}^{5} C_j\right)$$

$$= \sum_{i;\, P_i \in \bigcup_{j=1}^{s} C_j} (k_i - 1) \leq \sum_{i=1}^{n} (k_i - 1) = 5\,,$$

and so $s=1$, 4 or 5. If $s=5$, then C is not connected, which is excluded. If $s=4$, then C_5 meets C_1, C_2, C_3 and C_4, and hence $D_5^2 \leq -4$, which is impossible since D_5 corresponds to a simple elliptic hypersurface singularity. Therefore $s=1$ as required.

Now we shall derive a contradiction for each n $(1 \leq n \leq 5$ by Lemma 2.1) from what we have proved.

$n=1$: Clear since then $5 \geq k_1 = n+5 = 6$.

$n=2$: We have two possibilities: (i) $k_1=5$, $k_2=2$; (ii) $k_1=4$, $k_2=3$. In (i), all C_i pass through P_1 and we may assume that $P_2 \in C_1$, C_2 and $P_2 \notin C_3$, C_4, C_5. Then the intersection form of $C_1, \cdots, C_5$ is as follows:

$$(C_i C_j) = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

which is clearly non-degenerate. But the Picard number of $\bar{X}$ is not greater than 4 since $\bar{X}$ is an abelian or a hyperelliptic surface, and so we get a contradiction. In (ii), we may assume that $P_1 \notin C_1$, but then C_1 must meet every C_i $(2 \leq i \leq 5)$ away from P_1, that is at P_2, a contradiction.

$n=3$: There are two possibilities: (i) $k_1=4$, $k_2=k_3=2$; (ii) $k_1=k_2=3$, $k_3=2$. In both cases we may assume $P_1 \notin C_1$, and hence C_1 meets every C_i $(2 \leq i \leq 5)$ at P_2 or P_3, which is impossible for $k_2, k_3 < 5$ and $k_2 + k_3 < 6$.

$n=4$: We have $k_1=3$, $k_2=k_3=k_4=2$. Assuming $P_1 \notin C_1$, we see that C_1 should meet every C_i $(2 \leq i \leq 5)$ at either P_2, P_3 or P_4, which is also impossible.

$n=5$: In this last case, we have $k_1=\cdots=k_5=2$ and so the five curves $C_1,\cdots,C_5$ should meet each other at only five points $P_1,\cdots,P_5$ with multiplicity 2, which is absurd.

Thus we have completed our proof of the Main Theorem.

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