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Gevrey Classes on Compact Real Analytic Riemannian Manifolds

Keiko FUJITA and Mitsuo MORIMOTO

Sophia University

Introduction.

On the Euclidean space Gevrey classes on open sets and closed sets with boundary are studied by many mathematicians. In the present paper, we define Gevrey classes on compact real analytic Riemannian manifolds (without boundary) and study their properties. Special attention will be paid for the cases of the sphere and the Lie sphere.

Let X be a compact real analytic Riemannian manifold with a Riemannian metric g. We denote by $\mathscr{E}(X)$ the space of infinitely differentiable functions on X equipped with the usual topology, by $\mathscr{A}(X)$ the space of analytic functions on X equipped with the usual inductive limit topology, and by $\mathscr{B}(X)=\mathscr{A}'(X)$ the space of hyperfunctions on X. We denote by $\|\cdot\|_{L^2}$ the L^2-norm with respect to the measure $d\mu$ corresponding to g. Let $s>0$, $h>0$ and let Δ_X be the Laplace-Beltrami operator corresponding to g on X. We define Gevrey classes $\mathscr{E}_{\{s\}}(X)$ and $\mathscr{E}_{(s)}(X)$ by

$$\mathscr{E}_{\{s\}}(X)=\operatorname*{ind\,lim}_{h\to\infty}\left\{f\in\mathscr{E}(X);\ \sup_k\frac{1}{(2k)!^s h^{2k}}\|\Delta_X^k f\|_{L_2}<\infty\right\},$$

$$\mathscr{E}_{(s)}(X)=\operatorname*{proj\,lim}_{h\to 0}\left\{f\in\mathscr{E}(X);\ \sup_k\frac{1}{(2k)!^s h^{2k}}\|\Delta_X^k f\|_{L_2}<\infty\right\},$$

respectively. $\mathscr{E}_{\{s\}}(X)$ is a DFS space and $\mathscr{E}_{(s)}(X)$ is an FS space. We denote their dual spaces by $\mathscr{E}'_{\{s\}}(X)$ and $\mathscr{E}'_{(s)}(X)$, respectively.

According to Roumieu [9], the definition of Gevrey classes on $\boldsymbol{R}^{n+1}$ with compact support by means of the Laplacian and the supremum norm is equivalent to the usual definition given in Komatsu [2].

In our definition, the L^2-norm may be replaced by the sup-norm. In fact, for $s=1$, Lions-Magenes [3] proved the equivalence and their argument is still valid for $s>0$. Moreover, [3] showed that $\mathscr{E}_{\{1\}}(X)$ is equal to $\mathscr{A}(X)$. Further, Hashizume-Minemura [1] characterized the spaces $\mathscr{A}(X)$ and $\mathscr{B}(X)$ by the growth behavior of the coefficients

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of eigenfunction expansion of the Laplace-Beltrami operator on X relying on the equality $\mathscr{A}(X)=\mathscr{E}_{\{1\}}(X)$.

First, we extend results in [1] to the spaces $\mathscr{E}_{\{s\}}(X)$, $\mathscr{E}_{(s)}(X)$, $\mathscr{E}'_{\{s\}}(X)$ and $\mathscr{E}'_{(s)}(X)$ (Theorems 3 and 5). The spaces are characterized by the growth behavior of the coefficients of eigenfunction expansion (Corollaries 4 and 6), which deduces that $\mathscr{E}_{\{s\}}(X)$, $\mathscr{E}_{(s)}(X)$, $s>0$ and $\mathscr{E}'_{\{s\}}(X)$, $\mathscr{E}'_{(s)}(X)$, $s>1$ are subspaces of $\mathscr{B}(X)$.

Second, we investigate the case $X=S^n$, where S^n is the n-dimensional unit sphere and $n\in \boldsymbol{N}$. We call $\tilde{S}^n=\{z\in \boldsymbol{C}^{n+1}; z_1^2+z_2^2+\cdots+z_{n+1}^2=1\}$ the complex sphere. $\mathscr{O}(\tilde{S}^n)$ denotes the space of holomorphic functions on $\tilde{S}^n$ equipped with the topology of uniform convergence on compact sets. $\mathscr{O}(\tilde{S}^n)$ is an FS space. An element of $\mathscr{O}(\tilde{S}^n)$ is called an entire function on $\tilde{S}^n$. The spaces $\mathscr{E}_{\{s\}}(S^n)$, $\mathscr{E}_{(s)}(S^n)$, $\mathscr{E}'_{\{s\}}(S^n)$ and $\mathscr{E}'_{(s)}(S^n)$ are characterized by the growth behavior of spherical harmonic expansion on S^n (Corollary 8). A result in Morimoto [6] and Corollary 8 imply that $\mathscr{E}_{\{s\}}(S^n)$ and $\mathscr{E}_{(s)}(S^n)$, $0<s<1$, are spaces of entire functions of the minimal exponential type on $\tilde{S}^n$. We prove the linear topological isomorphism $\mathscr{E}_{(1)}(S^n)\simeq\mathscr{O}(\tilde{S}^n)$ (Theorem 9).

Third, we study the case $X=\Sigma^{n+1}=\{e^{i\theta}\omega; \theta\in \boldsymbol{R}, \omega\in S^n\}$, where Σ^{n+1} is the Lie sphere (see Morimoto [4]). The covering mapping $p: S^1\times S^n\mapsto\Sigma^{n+1}$ is locally isometric, where $S^1\times S^n$ is the Riemannian product of the unit circle and the n-dimensional unit sphere. For the Lie sphere, similar results to the case $X=S^n$ are valid.

1. Gevrey classes on compact real analytic Riemannian manifolds.

Let X be a compact real analytic Riemannian manifold with a Riemannian metrics g and Δ_X the Laplace-Beltrami operator corresponding to g. $L^2(X)$ denotes the space of square integrable functions on X with respect to the measure $d\mu$ corresponding to g. Choosing suitable g, we may assume $\int_X d\mu=1$. For $f\in L^2(X)$ we put $\|f\|_{L_2}^2=\int_X f(w)\overline{f(w)}d\mu$. As is well-known, the eigenvalues of Δ_X are non-negative and we can choose eigenfunctions φ_k, $k=0,1,2,\cdots$, so that they form a complete orthonormal basis of $L^2(X)$ and that the corresponding eigenvalues λ_k satisfy $0=\lambda_0\leq\lambda_1\leq\cdots\leq\lambda_k\leq\cdots$.

Let $s>0$. We take $\lambda_k^{1/(2s)}\geq 0$.

LEMMA 1. *For any $t>0$, the series $\sum_{k=0}^{\infty}\exp(-t\lambda_k^{1/(2s)})$ is convergent.*

Proof is similar to Lemma 1.2 in [1] and is omitted.

For $f(w)=\sum_{k=0}^{\infty}a_k\varphi_k(w)\in\mathscr{E}(X)$ and $m\in \boldsymbol{N}$, we define

$$\Delta_X^{(m/2)}f(w)=\sum_{k=0}^{\infty}a_k\lambda_k^{(m/2)}\varphi_k(w).$$

Put

$$|f|_{s,h}=\sup_k \frac{1}{(2k)!^s h^{2k}}\|\Delta_X^k f\|_{L_2}, \quad \|f\|_{s,h}=\sup_m \frac{1}{m!^s h^m}\|\Delta_X^{(m/2)} f\|_{L_2}.$$

LEMMA 2. *For $f\in\mathscr{E}(X)$, we have*

$$|f|_{s,h}\le \|f\|_{s,h}\le 2^{(s/2)}|f|_{s,h}.$$

Proof is similar to Lemma 1.5 in [1] and is omitted.

Let $h>0$ and put

$$\mathscr{E}_{s,h}(X)=\{f\in\mathscr{E}(X);\ \|f\|_{s,h}<\infty\}=\{f\in\mathscr{E}(X);\ |f|_{s,h}<\infty\}.$$

$\mathscr{E}_{s,h}(X)$ is a Banach space.

We define Gevrey classes $\mathscr{E}_{\{s\}}(X)$ and $\mathscr{E}_{(s)}(X)$ by

$$\mathscr{E}_{\{s\}}(X)=\operatorname*{ind\,lim}_{h\to\infty}\mathscr{E}_{s,h}(X), \qquad \mathscr{E}_{(s)}(X)=\operatorname*{proj\,lim}_{h\to 0}\mathscr{E}_{s,h}(X).$$

Since

$$\|\varphi_k\|_{t,h}=\sup_m \frac{1}{m!^t h^m}\lambda_k^{(m/2)}\le \exp\left(\frac{t\lambda_k^{1/(2t)}}{h^{1/t}}\right), \tag{1}$$

φ_k's belong to any $\mathscr{E}_{t,h}(X)$, $t>0$, $h>0$.

We employ the following spaces of sequences of numbers:

$$\mathscr{F}_{s,h}(X)=\left\{(a_k)_{k\ge 0};\ a_k\in \boldsymbol{C},\ \sum_{k=0}^{\infty}|a_k|\exp\left(\frac{1}{h}\ \lambda_k^{1/(2s)}\right)<\infty\right\},$$

$$\mathscr{F}_{\{s\}}(X)=\operatorname*{ind\,lim}_{h\to\infty}\mathscr{F}_{s,h}(X), \qquad \mathscr{F}_{(s)}(X)=\operatorname*{proj\,lim}_{h\to 0}\mathscr{F}_{s,h}(X).$$

$\mathscr{F}_{s,h}(X)$ is a Banach space, $\mathscr{F}_{\{s\}}(X)$ is a DFS space and $\mathscr{F}_{(s)}(X)$ is an FS space.

For $f\in\mathscr{E}_{(s)}(X)$ or $\mathscr{E}_{\{s\}}(X)$ we define $\Phi(f)=(a_k)_{k\ge 0}$, where $a_k=\langle f,\overline{\varphi_k}\rangle=\int_X f(w)\overline{\varphi_k(w)}\,d\mu$. The following theorem generalizes Proposition 1.7 in [1].

THEOREM 3. *The mapping Φ is a linear topological isomorphism of $\mathscr{E}_{\{s\}}(X)$ onto $\mathscr{F}_{\{s\}}(X)$ and of $\mathscr{E}_{(s)}(X)$ onto $\mathscr{F}_{(s)}(X)$.*

PROOF. Let $f\in\mathscr{E}_{s,h}(X)$.

$$\|f\|_{s,h}=\sup_m \frac{1}{m!^s h^m}\left\|\sum_{k=0}^{\infty}a_k\lambda_k^{m/2}\varphi_k\right\|_{L_2}=\sup_m \frac{1}{m!^s h^m}\left\{\sum_{k=0}^{\infty}|a_k|^2\lambda_k^m\right\}^{1/2}$$

$$\ge \sup_m \frac{1}{m!^s h^m}|a_k|\lambda_k^{m/2}\ge\left(\frac{\exp((\sqrt{\lambda_k}/h)^{1/s})}{2(\sqrt{\lambda_k}/h)^{1/s}+4}\right)^s|a_k|$$

$$\geq \frac{|a_k|}{4^s}\exp\left(\frac{s}{2}\left(\frac{\sqrt{\lambda_k}}{h}\right)^{1/s}\right).$$

Putting $t'=\frac{s}{2}(1/h)^{1/s}$, we have

$$|a_k| \leq 4^s \|f\|_{s,h} \exp(-t'\lambda_k^{1/(2s)}). \tag{2}$$

Thus, for any $t>0$ with $t<t'$ there is $C_{s,h,t}\geq 0$ such that

$$\sum_{k=0}^{\infty} |a_k| \exp(t\lambda_k^{1/(2s)}) \leq 4^s \|f\|_{s,h} \sum_{k=0}^{\infty} \exp((t-t')\lambda_k^{1/(2s)}) \leq C_{s,h,t}\|f\|_{s,h}, \tag{3}$$

where we used Lemma 1 in the last inequality.

Φ is injective by the complete orthogonality of φ_k's.

We prove only that Φ is a linear topological isomorphism of $\mathscr{E}_{(s)}(X)$ onto $\mathscr{F}_{(s)}(X)$. If $f\in\mathscr{E}_{(s)}(X)$, then $\|f\|_{s,h}<\infty$ for any $h>0$. By (3), $\Phi(f)$ belongs to $\mathscr{F}_{(s)}(X)$. This shows that $\Phi(\mathscr{E}_{(s)}(X))\subset\mathscr{F}_{(s)}(X)$. Φ is continuous by (3).

Assume $(a_k)_{k\geq 0}\in\mathscr{F}_{(s)}(X)$. Then $(a_k)_{k\geq 0}\in\mathscr{F}_{s,t}(X)$ for any $t>0$. By (1) we have

$$\left\| \sum_{k=k_0}^{k_0+l} a_k\varphi_k \right\|_{s,h} \leq \sum_{k=k_0}^{k_0+l} |a_k| \exp\left(\frac{s\lambda_k^{1/(2s)}}{h^{1/s}}\right) \qquad \text{for any } h.$$

Thus $f_N=\sum_{k=0}^{N} a_k\varphi_k$ converges to a function f in the topology of any $\mathscr{E}_{s,h}(X)$. The function $f\in\mathscr{E}_{(s)}(X)$ satisfies $\Phi(f)=(a_k)_{k\geq 0}$. Thus Φ is surjective.

Let $(a_k)_{k\geq 0}\in\mathscr{F}_{(s)}(X)$ and $f=\sum_{k=0}^{\infty} a_k\varphi_k\in\mathscr{E}_{(s)}(X)$. For any $h>0$ we have

$$\|f\|_{s,h} \leq \sum_{k=0}^{\infty} |a_k| \, \|\varphi_k\|_{s,h} \leq \sum_{k=0}^{\infty} |a_k| \exp\left(\frac{s\lambda_k^{1/(2s)}}{h^{1/s}}\right).$$

Thus Φ^{-1} is also continuous. q.e.d.

COROLLARY 4. *Let $a_k=\langle f, \varphi_k\rangle$. Then we have the following relations:*

(i) $f\in\mathscr{E}_{\{s\}}(X) \Leftrightarrow \exists t>0,\ \lim_{k\to\infty} |a_k|\exp(t\lambda_k^{1/(2s)})=0,$

(ii) $f\in\mathscr{E}_{(s)}(X) \Leftrightarrow \forall t>0,\ \lim_{k\to\infty} |a_k|\exp(t\lambda_k^{1/(2s)})=0.$

The series $f(z)=\sum_{k=0}^{\infty} a_k\varphi_k(z)$, $z\in X$, converges in the topology of respective spaces.

Put

$$\mathscr{G}_{s,h}(X)=\left\{(a_k)_{k\geq 0};\ a_k\in \boldsymbol{C},\ \sum_{k=0}^{\infty} |a_k|\exp(-h\lambda_k^{1/(2s)})<\infty\right\},$$

$$\mathscr{G}_{\{s\}}(X)=\operatorname*{proj\,lim}_{h\to 0}\mathscr{G}_{s,h}(X), \qquad \mathscr{G}_{(s)}(X)=\operatorname*{ind\,lim}_{h\to\infty}\mathscr{G}_{s,h}(X).$$

$\mathscr{G}_{s,h}(X)$ is a Banach space, $\mathscr{G}_{\{s\}}(X)$ is an FS space and $\mathscr{G}_{(s)}(X)$ is a DFS space.

Let $T \in \mathscr{E}'_{\{s\}}(X)$ or $\mathscr{E}'_{(s)}(X)$. $\langle T, \varphi_k \rangle$ is well-defined since φ_k belongs to $\mathscr{E}_{\{s\}}(X)$ and $\mathscr{E}_{(s)}(X)$. Thus, we define $\Psi(T) = (a_k)_{k \geq 0}$, where $a_k = \langle T, \varphi_k \rangle$. The following theorem generalizes Theorem 1.8 in [1].

THEOREM 5. *The mapping Ψ is a linear topological isomorphism of $\mathscr{E}'_{\{s\}}(X)$ onto $\mathscr{G}_{\{s\}}(X)$ and of $\mathscr{E}'_{(s)}(X)$ onto $\mathscr{G}_{(s)}(X)$.*

PROOF. We prove only that Ψ is a linear topological isomorphism of $\mathscr{E}'_{(s)}(X)$ onto $\mathscr{G}'_{(s)}(X)$. Let $T \in \mathscr{E}'_{(s)}(X)$. By the continuity of T, there is $h > 0$ such that

$$\|T\|^*_{s,h} = \sup\{|\langle T, f \rangle| \, ; \, \|f\|_{s,h} \leq 1\} < \infty \, .$$

Therefore,

$$|a_k| = |\langle T, \varphi_k \rangle| \leq \|T\|^*_{s,h} \cdot \|\varphi_k\|_{s,h} \leq \|T\|^*_{s,h} \exp\left(\frac{s\lambda_k^{1/(2s)}}{h^{1/s}}\right).$$

Thus, for any $t > s/h^{1/s}$ we have

$$\sum_{k=0}^{\infty} |a_k| \exp(-t\lambda_k^{1/(2s)}) \leq \|T\|^*_{s,h} \sum_{k=0}^{\infty} \exp\left(\left(\frac{s}{h^{1/s}} - t\right)\lambda_k^{1/(2s)}\right) < \infty$$

by Lemma 1. This shows that $\Psi(\mathscr{E}'_{(s)}(X)) \subset \mathscr{G}_{(s)}(X)$ and Ψ is continuous.

We prove the injectivity of Ψ. Assume that $\Psi(T) = 0$ for $T \in \mathscr{E}'_{(s)}(X)$; that is, $a_k = \langle T, \varphi_k \rangle = 0$, $k = 0, 1, 2, \cdots$. Let $f \in \mathscr{E}_{(s)}(X)$. Corollary 4 implies

$$\langle T, f \rangle = \left\langle T, \sum_{k=0}^{\infty} \langle f, \overline{\varphi_k} \rangle \varphi_k \right\rangle = \sum_{k=0}^{\infty} a_k \langle f, \overline{\varphi_k} \rangle = 0 \, ,$$

which means $T = 0$.

Assume $(a_k)_{k \geq 0} \in \mathscr{G}_{(s)}(X)$. Then there is $t > 0$ such that $(a_k)_{k \geq 0} \in \mathscr{G}_{s,t}(X)$. Let $f \in \mathscr{E}_{(s)}(X)$. Then f belongs to any $\mathscr{E}_{s,h}(X)$. Putting $t' = \frac{1}{2}(1/h)^{1/s}$, we have

$$|\langle f, \overline{\varphi_k} \rangle| \leq 4^s \|f\|_{s,h} \exp(-t' \lambda_k^{1/(2s)}) \tag{4}$$

by (2). Therefore, if $t' > t > 0$, then we have

$$\begin{aligned} \left| \sum_{k=0}^{\infty} a_k \langle f, \overline{\varphi_k} \rangle \right| &\leq \sum_{k=0}^{\infty} |a_k| \, |\langle f, \overline{\varphi_k} \rangle| \\ &\leq 4^s \|f\|_{s,h} \sum_{k=0}^{\infty} |a_k| \exp(-t' \lambda_k^{1/(2s)}) < \infty \end{aligned} \tag{5}$$

by (4) and Lemma 1. Thus, $\sum_{k=0}^{\infty} a_k \overline{\varphi_k}$ converges to a functional $T \in \mathscr{E}'_{(s)}(X)$ in the topology of $\mathscr{E}'_{(s)}(X)$. For $f \in \mathscr{E}_{(s)}(X)$, we have $\langle T, f \rangle = \sum_{k=0}^{\infty} a_k \langle f, \overline{\varphi_k} \rangle$. It is clear that T satisfies $\langle T, \varphi_k \rangle = a_k$; that is, $\Psi(T) = (a_k)_{k \geq 0}$. Thus Ψ is surjective and Ψ^{-1} is continuous by (5). q.e.d.

COROLLARY 6. *Let $a_k=\langle T, \varphi_k\rangle$. Then we have the following relations*:

(i) $T\in\mathscr{E}'_{\{s\}}(X)\Leftrightarrow\forall t>0,\ \lim_{k\to\infty}|a_k|\exp(-t\lambda_k^{1/(2s)})=0$,

(ii) $T\in\mathscr{E}'_{(s)}(X)\Leftrightarrow\exists t>0,\ \lim_{k\to\infty}|a_k|\exp(-t\lambda_k^{1/(2s)})=0$.

Further, we have $T=\sum_{k=0}^{\infty}a_k\overline{\varphi_k}$, *i.e.*

$$\langle T, f\rangle=\sum_{k=0}^{\infty}\int_X a_k\overline{\varphi_k(w)}f(w)d\mu(w)\,,$$

where f is a test function in respective spaces.

From Corollaries 4 and 6, we have the following remark.

REMARK. For $1<s<t$, we have

$$\begin{aligned}\cdots\subset\mathscr{E}_{\{1/t\}}(X)\subset\mathscr{E}_{(1/s)}(X)\subset\mathscr{E}_{\{1/s\}}(X)\subset\mathscr{E}_{(1)}(X)\subset\mathscr{A}(X)\subset\mathscr{E}_{(s)}(X)\subset\mathscr{E}_{\{s\}}(X)\\ \subset\mathscr{E}_{(t)}(X)\subset\mathscr{E}(X)\subset L^2(X)\subset\mathscr{E}'(X)\subset\mathscr{E}'_{(t)}(X)\subset\mathscr{E}'_{\{s\}}(X)\subset\mathscr{E}'_{(s)}(X)\\ \subset\mathscr{B}(X)\subset\mathscr{E}'_{(1)}(X)\subset\mathscr{E}'_{\{1/s\}}(X)\subset\mathscr{E}'_{(1/s)}(X)\subset\mathscr{E}'_{\{1/t\}}(X)\subset\cdots\,.\end{aligned}$$

2. The sphere and the complex sphere.

In this section, we consider the case where $X=S^n$. Taguchi [8] studied the case $n=1$. We denote Δ_X by Δ_S. In our previous papers, Δ_S was denoted by $-\Delta_S$.

The cross norm $L(z)$ on C^{n+1} corresponding to the Euclidean norm $\|x\|$ is the Lie norm given by

$$L(z)=L(x+iy)=[\|x\|^2+\|y\|^2+2\sqrt{\|x\|^2\|y\|^2-(x\cdot y)^2}]^{1/2}\,,$$

where $x\cdot y=x_1y_1+x_2y_2+\cdots+x_{n+1}y_{n+1}$. Put

$$\tilde{S}(r)=\{z\in\tilde{S}^n;\ L(z)<r\}\,,\qquad 0<r\le\infty$$

and

$$\tilde{S}[r]=\{z\in\tilde{S}^n;\ L(z)\le r\}\,,\qquad 0\le r<\infty\,.$$

Note that $\tilde{S}^n(\infty)=\tilde{S}^n$ and $\tilde{S}^n[1]=S^n$. For $1<r\le\infty$ we denote by $\mathscr{O}(\tilde{S}^n(r))$ the space of holomorphic functions on $\tilde{S}^n(r)$ equipped with the topology of uniform convergence on compact sets. $\mathscr{O}(\tilde{S}^n(r))$ is an FS space. Now we set

$$\mathscr{O}(\tilde{S}^n[r])=\operatorname{ind}\lim_{r'>r}\mathscr{O}(\tilde{S}^n(r'))\,,\qquad 1\le r<\infty\,.$$

$\mathscr{O}(\tilde{S}^n[r])$ is a DFS space. Since $\{\tilde{S}^n(r);\ r>1\}$ is a fundamental system of complex neighborhoods of S^n, we have $\mathscr{A}(S^n)=\mathscr{O}(\tilde{S}^n[1])$. We denote by

$$\mathrm{Exp}(\tilde{S}^n; (0)) = \{f \in \mathcal{O}(\tilde{S}^n); \forall \varepsilon > 0, \exists C \geq 0 \text{ s.t. } |f(z)| \leq Ce^{\varepsilon L(z)}, z \in \tilde{S}^n\}$$

the space of entire functions of minimal exponential type on $\tilde{S}^n$.

We denote by $\mathscr{P}^k(C^{n+1})$ the space of k-homogeneous polynomials with complex coefficients of $n+1$ variables. Put

$$\mathscr{P}_\Delta^k(C^{n+1}) = \{F \in \mathscr{P}^k(C^{n+1}); \Delta_z F = 0\}$$

where Δ_z is the complex Laplacian; $-\Delta_z = \partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \cdots + \partial^2/\partial z_{n+1}^2$. It is the space of $\mathscr{H}^k$-homogeneous complex harmonic polynomials. Let

$$\mathscr{H}^k(S^n) = \{P|_{S^n}; P \in \mathscr{P}_\Delta^k(C^{n+1})\}$$

be the space of k-spherical harmonics. We know that the restriction mapping $f \mapsto f|_{S^n}$ is a linear isomorphism of $\mathscr{P}_\Delta^k(C^{n+1})$ onto $\mathscr{H}^k(S^n)$. $N(k, n)$ denotes the dimension of the space $\mathscr{H}^k(S^n)$;

$$N(0, 1) = 1,$$

$$N(k, n) = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} = O(k^{n-1}), \qquad (k, n) \neq (0, 1). \tag{6}$$

It is well-known that

$$L^2(S^n) = \bigoplus_{k=0}^{\infty} \mathscr{H}^k(S^n).$$

For spherical harmonics see Müller [7], for example.

Fix an orthonormal basis of $\mathscr{H}^k(S^n)$: $\varphi_{k,j}, j = 1, 2, \cdots, N(k, n)$.

We renumber $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots$ and $\varphi_0, \varphi_1, \cdots, \varphi_k, \cdots$ in Section 1 to $0 \leq \lambda_{0,1} \leq \lambda_{1,1} \leq \lambda_{1,2} \leq \cdots \leq \lambda_{1,N(1,n)} \leq \lambda_{2,1} \leq \cdots \leq \lambda_{k,j} \leq \cdots$ and $\varphi_{0,1}, \varphi_{1,1}, \varphi_{1,2}, \cdots, \varphi_{1,N(1,d)}, \varphi_{2,1}, \cdots, \varphi_{k,j}, \cdots$. In the case $X = S^n$, $\lambda_{k,j} = k(k+n-1)$, $1 \leq j \leq N(k, n)$, and we write dS for $d\mu$. Let $f \in L^2(S^n)$ and $a_{k,j} = \langle f, \overline{\varphi_{k,j}} \rangle = \int_{S^n} f(\omega)\overline{\varphi_{k,j}(\omega)}dS$. Then $S_k(\omega) = \sum_{j=1}^{N(k,n)} a_{k,j}\varphi_{k,j}(\omega)$ is independent of the choice of orthonormal basis $\{\varphi_{k,j}\}$ of $\mathscr{H}^k(S^n)$. In fact, we have

$$S_k(\omega) = S_k(f; \omega) = N(k, n) \int_{S^n} f(\tau) P_{k,n}(\omega \cdot \tau) dS(\tau), \tag{7}$$

where $P_{k,n}(t)$ is the Legendre polynomial of degree k and of dimension $n+1$. We call $S_k(\omega)$ the *k-spherical harmonic component* of f.

We shall show that all results in Section 1 expressed in terms of $\{a_k\}$ can be reformulated in terms of $\{S_k\}$.

Let $f \in \mathscr{E}(S^n)$ and $f(\omega) = \sum_{k=0}^{\infty} S_k(\omega)$ the spherical harmonic expansion of f. Then we define

$$\Delta_S^{m/2} f = \sum_{k=0}^{\infty} \{k(k+n-1)\}^{m/2} S_k$$

and

$$\| f \|_{s,h} = \sup_m \frac{1}{m!^s h^m} \left\{ \sum_{k=0}^{\infty} \{k(k+n-1)\}^m \| S_k \|_{L^2}^2 \right\}^{1/2} .$$

Especially, for $S_k \in \mathscr{H}^k(S^n)$

$$\| S_k \|_{s,h} = \sup_m \frac{1}{m!^s h^m} \{k(k+n-1)\}^{m/2} \leq \exp\left(s \frac{(k(k+n-1))^{1/(2s)}}{h^{1/s}} \right).$$

Put

$$\tilde{P}_{k,n}(z, \omega) = (\sqrt{z^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}} \cdot \omega \right).$$

Then

$$\tilde{S}_k(z) = N(k, n) \int_{S^n} S_k(\tau) \tilde{P}_{k,n}(z, \tau) dS(\tau) \tag{8}$$

is the k-homogeneous harmonic extension of $S_k(\omega)$. Note that

$$| \tilde{P}_{k,n}(z, \omega) | \leq L(z)^k , \qquad z \in C^{n+1} , \quad \omega \in S^n \tag{9}$$

(Lemma 5.5 in [5]) and that

$$P_{k,n}(\tfrac{1}{2}(r^2 + 1/r^2)) \geq C_n k^{-n/2} r^{2k} \qquad \text{for} \quad r > 1 , \tag{10}$$

where C_n is a constant independent of $k = 0, 1, 2, \cdots$ (Lemma 8 in [6]).

We have

$$| S_k |_{S^n} \leq \sqrt{N(k, n)} \| S_k \|_{L^2} , \qquad \| S_k \|_{L^2} \leq | S_k |_{S^n} ,$$

where $| f |_{S^n} = \sup_{\omega \in S^n} | f(\omega) |$ (Proposition 1.1 in [5]).

By the same argument as in Theorem 3, we have the following theorem:

THEOREM 7. *Put*

$$\mathscr{F}^{s,h}(S^n) = \left\{ (S_k)_{k \geq 0} ; \, S_k \in \mathscr{H}^k(S^n) , \, \sum_{k=0}^{\infty} | S_k |_{S^n} \exp(\tfrac{1}{h} k^{1/s}) < \infty \right\} .$$

Then we have the following linear topological isomorphisms:

(i) $\mathscr{E}_{\{s\}}(S^n) = \operatorname{ind}\lim_{h \to \infty} \mathscr{F}^{s,h}(S^n)$,

(ii) $\mathscr{E}_{(s)}(S^n) = \operatorname{proj}\lim_{h \to 0} \mathscr{F}^{s,h}(S^n)$.

COROLLARY 8. *Let S_k be the k-spherical harmonic component of $f \in L^2(S^n)$. Then we have the following relations*:

(i) $f \in \mathscr{E}_{\{s\}}(S^n) \Leftrightarrow \exists t>0\,,\ \lim_{k\to\infty} |S_k|_{S^n} \exp(tk^{1/s})=0\,,$

(ii) $f \in \mathscr{E}_{(s)}(S^n) \Leftrightarrow \forall t>0\,,\ \lim_{k\to\infty} |S_k|_{S^n} \exp(tk^{1/s})=0\,.$

The series $f(\omega)=\sum_{k=0}^{\infty} S_k(\omega)$, $\omega\in S^n$, *converges in the topology of respective spaces.*

By identifying $F\in\mathcal{O}(\tilde{S}^n)$ with $F|_{S^n}$, we consider

$$\mathrm{Exp}(\tilde{S}^n\,;\,(0)) \hookrightarrow \mathcal{O}(\tilde{S}^n) \hookrightarrow \mathscr{A}(S^n)\,.$$

Then we have the following theorem.

THEOREM 9. (i) $\mathscr{E}_{\{1\}}(S^n)=\mathscr{A}(S^n)$.

(ii) *The restriction mapping* $\alpha\colon \mathcal{O}(\tilde{S}^n)\to\mathscr{E}_{(1)}(S^n)$ *is a linear topological isomorphism.*

(iii) *Let* $0<s<1$. *Then* $\mathscr{E}_{\{s\}}(S^n)$ *and* $\mathscr{E}_{(s)}(S^n)$ *are subspaces of* $\mathrm{Exp}(\tilde{S}^n\,;\,(0))$.

PROOF. (i) This is a special case of the general result in [3].

(ii) Suppose $f\in\mathcal{O}(\tilde{S}^n)$. Then we have

$$\begin{aligned} S_k(w) &= S_k(f\,;\,w) \\ &= \frac{N(k,n)}{P_{k,n}(\frac{1}{2}(r^2+1/r^2))}\int_{\partial\tilde{S}^n[r]} f(z)P_{k,n}(\bar{z}\cdot w)d\mu_r(z)\,, \qquad w\in\tilde{S}^n(r)\,, \end{aligned} \tag{11}$$

where $r\geq 1$, $\partial\tilde{S}^n[r]=\{z\in\tilde{S}^n\,;\,L(z)=r\}$ and $d\mu_r(z)$ is the $O(n+1)$-invariant measure on $\partial\tilde{S}^n[r]$ (Lemma 3.3 in [10], in which n is assumed $n\geq 2$, but (11) is also true for $n=1$).

Let $h>0$. Then by (9) we have

$$|S_k(\omega)|_{S^n}\leq |f(z)|_{\partial\tilde{S}^n[r]}\frac{N(k,n)r^k}{P_{k,n}(\frac{1}{2}(r^2+1/r^2))}$$

where $|f(z)|_{\partial\tilde{S}^n[r]}=\sup_{z\in\partial\tilde{S}^n[r]}|f(z)|$. By (6) and (10), for $r>\exp(1/h)$ we have

$$C_{h,r}=\sum_{k=0}^{\infty}\frac{N(k,n)\exp(k/h)}{P_{k,n}(\frac{1}{2}(r^2+1/r^2))}<\infty\,.$$

Therefore, we have

$$\|f\|_{\mathscr{F}^{1,h}(S^n)}\leq C_{h,r}|f|_{\partial\tilde{S}^n[r]}\,,$$

where $\|f\|_{\mathscr{F}^{1,h}(S^n)}$ denotes the norm of the space $\mathscr{F}^{1,h}(S^n)$. Thus $f=\sum_{k=0}^{\infty}S_k$ is in $\mathscr{E}_{(1)}(S^n)$ and α is continuous.

Conversely, suppose $f=\sum_{k=0}^{\infty}S_k\in\mathscr{E}_{(1)}(S^n)$. By (8) and (9) we have

$$|\tilde{S}_k(z)|\leq N(k,n)|S_k|_{S^n}L(z)^k\,,\qquad z\in\tilde{S}^n\,.$$

For any $r>1$ and $h>0$ satisfying $r<\exp(1/h)$, there is $C'_{h,r}\geq 0$ such that

$$\sum_{k=0}^{\infty} |\tilde{S}_k(z)|_{\partial\tilde{S}^n[r]} \leq C_{h,r} \|f\|_{\mathscr{F}^{1,h}(S^n)} . \tag{12}$$

Therefore $F(z)=\sum_{k=0}^{\infty}\tilde{S}_k(z)$ defines an entire function on $\tilde{S}^n$ and $\alpha(F)=f$; that is, α is surjective. By (12) we have

$$|F(z)|_{\partial\tilde{S}^n[r]} \leq C'_{h,r} \|f\|_{\mathscr{F}^{1,h}(S^n)} .$$

This shows that α is injective and that α^{-1} is continuous.

(iii) is clear by Corollary 9 and Theorem 11 in [6] which are valid for $n\geq 1$.

q.e.d.

Similar to (7), we define the k-spherical harmonic component S_k of $T\in\mathscr{E}'_{(s)}(S^n)$ by

$$S_k(\omega)=S_k(T;\omega)=N(k,n)\langle T_\tau, P_{k,n}(\omega\cdot\tau)\rangle .$$

Then we have

$$\langle T,f\rangle=\sum_{k=0}^{\infty} S_k(T;\omega)S_k(f;\omega) .$$

By the same argument as in Theorem 5, we have the following theorem:

THEOREM 10. *Put*

$$\mathscr{G}^{s,h}(S^n)=\left\{(S_k)_{k\geq 0};\ S_k\in\mathscr{H}^k(S^n),\ \sum_{k=0}^{\infty}|S_k|_{S^n}\exp(-hk^{1/s})<\infty\right\} .$$

Then we have the following linear topological isomorphisms:

(i) $\mathscr{E}'_{\{s\}}(S^n)=\text{proj}\lim_{h\to 0}\mathscr{G}^{s,h}(S^n)$,

(ii) $\mathscr{E}'_{(s)}(S^n)=\text{ind}\lim_{h\to\infty}\mathscr{G}^{s,h}(S^n)$.

COROLLARY 11. *Let S_k be the k-spherical harmonic component of $T\in\mathscr{E}'_{(s)}(S^n)$. Then we have the following relations*:

(i) $T\in\mathscr{E}'_{\{s\}}(S^n)\Leftrightarrow\forall t>0,\ \lim_{k\to\infty}|S_k|_{S^n}\exp(-tk^{1/s})=0$,

(ii) $T\in\mathscr{E}'_{(s)}(S^n)\Leftrightarrow\exists t>0,\ \lim_{k\to\infty}|S_k|_{S^n}\exp(-tk^{1/s})=0$.

Further, we have

$$\langle T,f\rangle=\sum_{k=0}^{\infty}\int_{S^n} S_k(\omega)f(\omega)dS(\omega) ,$$

where f is a test function in respective spaces.

3. The Lie sphere.

In this section, we consider the case $X=\Sigma^{n+1}$, where $\Sigma^{n+1}=\{e^{i\theta}\omega;\ \theta\in\boldsymbol{R},\ \omega\in S^n\}$. We review some results in [4]. First we note that

$$\tilde{V}=\{z\in\boldsymbol{C}^{n+1};\ z^2\neq 0\}$$

is a complexification of Σ^{n+1}. (Lemmas 1.2 and 1.5 in [4]).

For $A, B>0$ with $AB>1$ and $R>1$ put

$$\tilde{V}(A, B;\ R)=\{z\in\tilde{V};\ B^{-2}<|z^2|<A^2,\ L(z)^2<|z^2|R^2\}$$

and for $A, B>0$ with $AB>1$ and $R\geq 1$ put

$$\tilde{V}[A, B;\ R]=\{z\in\tilde{V};\ B^{-2}\leq|z^2|\leq A^2,\ L(z)^2\leq|z^2|R^2\}.$$

We consider $\tilde{V}=\tilde{V}(\infty, \infty;\ \infty)$. Note that $\Sigma^{n+1}=\tilde{V}[1, 1;\ 1]$.

Since $\{\tilde{V}(A, B;\ R);\ A>1, B>1, R>1\}$ is a fundamental system of complex neighborhoods of Σ^{n+1} (Lemma 1.6 in [4]), we have $\mathscr{A}(\Sigma^{n+1})=\mathcal{O}(\tilde{V}[1, 1;\ 1])$.

The inner product of the Hilbert space $L^2(\Sigma^{n+1})$ is given by

$$(f, g)_{L^2(\Sigma^{n+1})}=\langle f, \bar{g}\rangle_{\Sigma^{n+1}}=\frac{1}{\pi}\int_0^\pi\int_{S^n} f(e^{i\theta}\omega)\overline{g(e^{i\theta}\omega)}d\theta dS(\omega),$$

for $f, g\in L^2(\Sigma^{n+1})$. The Laplace-Beltrami operator Δ_Σ on Σ^{n+1} is represented by Δ_S;

$$\Delta_\Sigma=-\frac{\partial^2}{\partial\theta^2}+\Delta_S,$$

and its eigenvalues are

$$\lambda_{m,k}=m^2+k(k+n-1), \qquad m=0, \pm 1, \pm 2, \cdots, \quad k=0, 1, 2, \cdots.$$

Put

$$\Lambda=\{(m, k)\in\boldsymbol{Z}\times\boldsymbol{Z}_+;\ m\equiv k\ (\mathrm{mod}\ 2)\}.$$

For $(m, k)\in\Lambda$ we define

$$\mathscr{H}^{m,k}(\Sigma^{n+1})=\{e^{im\theta}S_k(\omega);\ S_k\in\mathscr{H}^k(S^n)\}.$$

By Theorem 2.1 in [4], we have

$$L^2(\Sigma^{n+1})=\bigoplus_{(m,k)\in\Lambda}\mathscr{H}^{m,k}(\Sigma^{n+1}).$$

For $f\in L^2(\Sigma^{n+1})$ we define the (m, k)-component $S_{m,k}(f;\ \omega)$ by

$$S_{m,k}(f;\omega)=\frac{N(k,n)}{\pi}\int_0^\pi\int_{S^n} f(e^{i\theta}\tau)e^{-im\theta}P_{k,n}(\omega\cdot\tau)d\theta dS(\tau)$$
$$=N(k,n)\langle f(e^{i\theta}\tau), e^{-im\theta}P_{k,n}(\omega\cdot\tau)\rangle_{\Sigma^{n+1}}\,.$$

Since $|e^{im\theta}S_{m,k}|_{\Sigma^{n+1}}=|S_{m,k}|_{S^n}$, by the same argument as in Theorem 3, we have the following theorem:

THEOREM 12. *Put*

$$\mathscr{F}^{s,h}(\Sigma^{n+1})=\left\{(S_{m,k})_\Lambda\,;\ S_{m,k}\in\mathscr{H}^k(S^n)\,,\ \sum_{(m,k)\in\Lambda}|S_{m,k}|_{S^n}\exp(\tfrac{1}{h}(|m|+k)^{1/s})<\infty\right\}.$$

Then we have the following linear topological isomorphisms:

(i) $\mathscr{E}_{\{s\}}(\Sigma^{n+1})=\operatorname{ind}\lim_{h\to\infty}\mathscr{F}^{s,h}(\Sigma^{n+1})$,

(ii) $\mathscr{E}_{(s)}(\Sigma^{n+1})=\operatorname{proj}\lim_{h\to 0}\mathscr{F}^{s,h}(\Sigma^{n+1})$.

COROLLARY 13. *Let $S_{m,k}$ be the (m, k)-component of $f\in L^2(\Sigma^{n+1})$. Then we have the following relations*:

(i) $f\in\mathscr{E}_{\{s\}}(\Sigma^{n+1})\Leftrightarrow\exists t>0\,,\ \lim_{|m|+k\to\infty}|S_{m,k}|_{S^n}\exp(t(|m|+k)^{1/s})=0$,

(ii) $f\in\mathscr{E}_{(s)}(\Sigma^{n+1})\Leftrightarrow\forall t>0\,,\ \lim_{|m|+k\to\infty}|S_{m,k}|_{S^n}\exp(t(|m|+k)^{1/s})=0$.

The series $f(e^{i\theta}\omega)=\sum_{(m,k)\in\Lambda}e^{im\theta}S_{m,k}(\omega)$, $e^{i\theta}\omega\in\Sigma^{n+1}$, converges in the topology of respective spaces.

By identifying $F\in\mathscr{O}(\tilde V)$ with $F|_{\Sigma^{n+1}}$ we consider

$$\operatorname{Exp}(\tilde V;(0))\hookrightarrow\mathscr{O}(\tilde V)\hookrightarrow\mathscr{A}(\Sigma^{n+1})\,,$$

where

$$\operatorname{Exp}(\tilde V;(0))=\{f\in\mathscr{O}(\tilde V);\ \forall\varepsilon>0,\exists C\geq 0 \text{ s.t. } |f(z)|\leq Ce^{\varepsilon L(z)},\ z\in\tilde V\}$$

is the space of entire functions of minimal exponential type on $\tilde V$.

THEOREM 14. (i) $\mathscr{E}_{\{1\}}(\Sigma^{n+1})=\mathscr{A}(\Sigma^{n+1})$.

(ii) *The restriction mapping $\alpha\colon\mathscr{O}(\tilde V)\to\mathscr{E}_{(1)}(\Sigma^{n+1})$ is a linear topological isomorphism.*

(iii) *Let $0<s<1$. Then $\mathscr{E}_{\{s\}}(\Sigma^{n+1})$ and $\mathscr{E}_{(s)}(\Sigma^{n+1})$ are subspaces of* $\operatorname{Exp}(\tilde V;(0))$.

(i) is a special case of the general result in [3]. By the similar argument to Theorem 9, Theorem 3.1 and its Corollary 2 (i) in [4] imply (ii). (iii) is clear by Corollary 13.

Similarly, we define the (m, k)-component of a functional $T\in\mathscr{E}'_{(s)}(\Sigma^{n+1})$ by

$$S_{m,k}(T;\,\tau)=N(k,\,n)\langle T(\theta,\,\omega),\,e^{-im\theta}P_{k,n}(\omega\cdot\tau)\rangle_{\Sigma^{n+1}}\,.$$

Then for $f\in\mathscr{E}_{(s)}(\Sigma^{n+1})$ we have

$$\langle T,f\rangle=\sum_{(m,k)\in\Lambda}\int_{S^n}S_{m,k}(T;\,\tau)S_{m,k}(f;\,\tau)dS(\tau)\,.$$

By the same argument as in Theorem 5, we have the following theorem:

THEOREM 15. *Put*

$$\mathscr{G}^{s,h}(\Sigma^{n+1})=\left\{(S_{m,k})_\Lambda;\,S_{m,k}\in\mathscr{H}^k(S^n),\,\sum_{(m,k)\in\Lambda}|S_{m,k}|_{S^n}\exp(-h(|m|+k)^{1/s})<\infty\right\}.$$

Then we have the following linear topological isomorphisms:

(i) $\mathscr{E}'_{\{s\}}(\Sigma^{n+1})=\operatorname{proj}\lim_{h\to 0}\mathscr{G}^{s,h}(\Sigma^{n+1})$,

(ii) $\mathscr{E}'_{(s)}(\Sigma^{n+1})=\operatorname{ind}\lim_{h\to\infty}\mathscr{G}^{s,h}(\Sigma^{n+1})$.

COROLLARY 16. *Let $S_{m,k}$ be the (m,k)-component of $T\in\mathscr{E}'_{(s)}(\Sigma^{n+1})$. Then we have the following relations*:

(i) $T\in\mathscr{E}'_{\{s\}}(\Sigma^{n+1})\Leftrightarrow\forall t>0\,,\quad\lim_{|m|+k\to\infty}|S_{m,k}|_{S^n}\exp(-t(|m|+k)^{1/s})=0$,

(ii) $T\in\mathscr{E}'_{(s)}(\Sigma^{n+1})\Leftrightarrow\exists t>0\,,\quad\lim_{|m|+k\to\infty}|S_{m,k}|_{S^n}\exp(-t(|m|+k)^{1/s})=0$.

Further, we have

$$\langle T,f\rangle_{\Sigma^{n+1}}=\sum_{(m,k)\in\Lambda}\langle S_{m,k}(T;\,\omega),\,S_{m,k}(f;\,\omega)\rangle_{\Sigma^{n+1}}\,,$$

where f is a test function in respective spaces.

4. Appendix.

Consider a sequence $\{M_p\}$ which satisfies the following conditions:

$$M_p^2\le M_{p-1}M_{p+1}\,,\quad M_0=M_1\,,\quad\lim_{p\to\infty}\frac{M_p}{M_{p+1}}=0\,.$$

Put

$$M_k(h)=\sup_p\frac{1}{M_ph^p}\lambda_k^{p/2}\,.$$

ASSUMPTION. *For any $h<h'$, $\sum_{k=0}^{\infty}M_k(h')/M_k(h)$ is convergent.*

If $M_p=p!^s$, $s>0$, then we have

$$\frac{1}{4^s}\exp\left(\frac{s}{2}\left(\frac{\lambda_k}{h}\right)^{1/(2s)}\right)\leq M_k(h)\leq\exp\left(2s\left(\frac{\lambda_k}{h}\right)^{1/s}\right).$$

Hence $M_k(h)$ and $\exp(t\lambda_k^{1/s})$ are equivalent and Assumption is valid by Lemma 1.

Let $h>0$ and put

$$\mathscr{E}_{M,h}(X)=\{f\in\mathscr{E}(X);\,|f|_{M,h}<\infty\},$$

where

$$|f|_{M,h}=\sup_k\frac{1}{M_{2k}h^{2k}}\|\Delta_X^k f\|_{L_2}.$$

We define the spaces of ultradifferentiable functions on X by

$$\mathscr{E}_{\{M_p\}}(X)=\operatorname{ind}\lim_{h\to\infty}\mathscr{E}_{M,h}(X),\qquad \mathscr{E}_{(M_p)}(X)=\operatorname{proj}\lim_{h\to 0}\mathscr{E}_{M,h}(X).$$

Similarly, we define the spaces $\mathscr{F}_{\{M_p\}}(X)$, $\mathscr{F}_{(M_p)}(X)$, $\mathscr{G}_{\{M_p\}}(X)$ and $\mathscr{G}_{(M_p)}(X)$ as in Section 1. Then under Assumption, which plays the role of Lemma 1, we can modify the results in Section 1 as follows.

THEOREM 3′. *The mapping Φ is a linear topological isomorphism of $\mathscr{E}_{\{M_p\}}(X)$ onto $\mathscr{F}_{\{M_p\}}(X)$ and of $\mathscr{E}_{(M_p)}(X)$ onto $\mathscr{F}_{(M_p)}(X)$.*

COROLLARY 4′. *Let $a_k=\langle f,\varphi_k\rangle$. Then we have the following relations:*

(i) $f\in\mathscr{E}_{\{M_p\}}(X)\Leftrightarrow\exists t>0,\ \lim_{k\to\infty}|a_kM_k(t)|=0$,

(ii) $f\in\mathscr{E}_{(M_p)}(X)\Leftrightarrow\forall t>0,\ \lim_{k\to\infty}|a_kM_k(t)|=0$.

The series $f(z)=\sum_{k=0}^{\infty}a_k\varphi_k(z)$, $z\in X$, converges in the topology of respective spaces.

Because $M_p(h)<\infty$ for any h, φ_k belongs to $\mathscr{E}_{(M_p)}(X)$. Thus we can define $a_k=\langle T,\varphi_k\rangle$ for $T\in\mathscr{E}'_{(M_p)}(X)$.

THEOREM 5′. *The mapping Ψ is a linear topological isomorphism of $\mathscr{E}'_{\{M_p\}}(X)$ onto $\mathscr{G}_{\{M_p\}}(X)$ and of $\mathscr{E}'_{(M_p)}(X)$ onto $\mathscr{G}_{(M_p)}(X)$.*

COROLLARY 6′. *Let $a_k=\langle T,\varphi_k\rangle$. Then we have the following relations:*

(i) $T\in\mathscr{E}'_{\{M_p\}}(X)\Leftrightarrow\forall t>0,\ \lim_{k\to\infty}\left|a_k\frac{1}{M_k(t)}\right|=0$,

(ii) $T\in\mathscr{E}'_{(M_p)}(X)\Leftrightarrow\exists t>0,\ \lim_{k\to\infty}\left|a_k\frac{1}{M_k(t)}\right|=0$.

Further, we have

$$\langle T, f\rangle = \sum_{k=0}^{\infty} \int_X a_k \overline{\varphi_k(w)} f(w) d\mu(w) ,$$

where f is a test function in respective spaces.

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Present Addresses:
KEIKO FUJITA
DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, SAGA UNIVERSITY,
SAGA, 840 JAPAN.

MITSUO MORIMOTO
DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY,
KIOICHO, CHIYODA-KU, TOKYO, 102 JAPAN.