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A Sharp Symmetrization of the L^2-Well-Posed Mixed Problem for Regularly Hyperbolic Equations of Second Order

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Introduction.

We consider the mixed problem

$$
(\mathrm{P})\quad\begin{cases}
L[u]=\dfrac{\partial^2 u}{\partial t^2}-2\sum_{j=1}^{n} h_j(t,x)\dfrac{\partial^2 u}{\partial t\partial x_j}-\sum_{i,j=1}^{n} a_{ij}(t,x)\dfrac{\partial^2 u}{\partial x_i\partial x_j} \\
\qquad +a_0(t,x)\dfrac{\partial u}{\partial t}+\sum_{j=1}^{n} a_j(t,x)\dfrac{\partial u}{\partial x_j}+e_0(t,x)u=f(t,x) \\
u(0,x)=u_0(x)\,,\qquad u_t(0,x)=u_1(x) \\
B[u]|_{x_1=0}=a_{11}(t,0,x')^{-1/2}\Big\{a_{11}(t,0,x')\dfrac{\partial u}{\partial x_1} \\
\qquad +\sum_{j=2}^{n} a_{1j}(t,0,x')\dfrac{\partial u}{\partial x_j}+h_1(t,0,x')\dfrac{\partial u}{\partial t}\Big\} \\
\quad +\sum_{j=2}^{n} b_j(t,x')\dfrac{\partial u}{\partial x_j}-c(t,x')\left(1+\dfrac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{1/2} \\
\quad \cdot\Big\{\dfrac{\partial u}{\partial t}-\left(1+\dfrac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{-1}\sum_{j=2}^{n}\Big(h_j(t,0,x') \\
\qquad -\dfrac{h_1(t,0,x')}{a_{11}(t,0,x')}a_{1j}(t,0,x')\Big)\dfrac{\partial u}{\partial x_j}\Big\} \\
\quad +\gamma(t,x')u|_{x_1=0}=g(t,x') \\
(t,x)=(t,x_1,x')\in \boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^{n-1}
\end{cases}
$$

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where $x=(x_1, x_2, \cdots, x_n)$, $x'=(x_2, \cdots, x_n)$, $n \geq 2$, the coefficients h_j and a_{ij} belong to $\mathscr{B}^{l+1+\sigma}(\boldsymbol{R} \times \overline{\boldsymbol{R}^n_+})$ and are constants outside a compact set in $\boldsymbol{R} \times \overline{\boldsymbol{R}^n_+}$, the coefficients a_0, a_j and e_0 belong to $\mathscr{B}^l(\boldsymbol{R} \times \overline{\boldsymbol{R}^n_+})$ and are constants outside a compact set in $\boldsymbol{R} \times \overline{\boldsymbol{R}^n_+}$, the coefficients b_j, c and γ belong to $\mathscr{B}^{l+1+\sigma}(\boldsymbol{R} \times \boldsymbol{R}^{n-1})$ and are constants outside a compact set in $\boldsymbol{R} \times \boldsymbol{R}^{n-1}$, l is a non-negative integer, and σ is a real number $(1/2 < \sigma < 1)$.

This paper is a continuation of the previous paper [7], and the aim here is to give a complete and sharp symmetrization of the L^2-well-posed mixed problem (P) for regularly hyperbolic equations of second order in $\boldsymbol{R}^1_{t+} \times \boldsymbol{R}^n_{x+}$, by which we reduce our mixed problem to the problem for symmetric hyperbolic differential systems of first order with non-negative type boundary condition. See the introduction in [7] for the purpose of the symmetrization. In [7], we treated a symmetrization of an L^2-well-posed mixed problem (P) under the conditions (A.I) and (A.III-j):

(A.III-1)
$$\begin{cases} c(t, x') \neq 1 \\ |1+c(t, x')| - |1-c(t, x')| \\ \qquad \geq 2\{\sup_{\eta'} |\mathrm{Re}\, b(t, x', \eta')|^2 + \sup_{\eta'} |\mathrm{Im}\, b(t, x', \eta')^2\}^{1/2} \end{cases}$$

or

(A.III-2)
$$\begin{cases} \text{The functions } b_j \text{ and } c \text{ are real valued, and} \\ c(t, x') \geq \sup_{\eta'} |b(t, x', \eta')| \end{cases}$$

or

(A.III-3) (1.3) holds

or

(A.III-4)
$$\begin{cases} \mathrm{Re}\, b_j \equiv 0 \ (j=2, \cdots, n), \qquad \mathrm{Re}\, c \equiv 0 \quad \text{and} \\ 1 + \{\mathrm{Im}\, c(t, x')\}^2 > \{\mathrm{Im}\, b(t, x', \eta')\}^2 \end{cases}$$

for all $(t, x', \eta') \in \boldsymbol{R} \times \boldsymbol{R}^{n-1} \times (\boldsymbol{R}^{n-1} - \{0\})$. The condition (A.III-$j$) is a special case of the condition (A.II) ($j=1, \cdots, 4$). If b_j and c are real valued, the condition (A.III-2) is equal to the condition (A.II). If b_j and c are pure imaginary valued, the condition (A.III-4) is equal to the condition (A.II). Also, the condition (A.III-3) is equal to the uniform Lopatinski boundary condition. In [7], we could not obtain a symmetrization of the problem (P) under the conditions (A.I) and (A.II). Here, we succeed in solving the above problem, and obtain a complete and sharp symmetrization of the L^2-well-posed mixed problem (Main Theorem 1). The key points of our method are Corollary of Lemma 2.1 in [7], the condition (A.II) and Lemma 2.1. In [7], we used the another condition which gave the L^2-well-posedness, and here we knew that under the condition (A.I), the condition (A.II) held if and only if the problem (P) was L^2-well-posed. Main

Theorem 1 is useful to the L^2-well-posed mixed problem (P) in the domain $(0, T)\times\Omega$ instead of $\boldsymbol{R}^1_+\times\boldsymbol{R}^n_+$ where Ω is a domain in $\boldsymbol{R}^n$ and $\partial\Omega$ is smooth. But, it is not always useful to the problem (P) in $(0, T)\times\Omega$ where $\partial\Omega$ is non-smooth, i.e. $\partial\Omega$ has a corner. Therefore, we treat a symmetrization of an L^2-well-posed mixed problem (Main Theorem 2), which is useful to the mixed problem in a domain with a corner ([6]). Main Theorem 2 is an extension of Theorem A in [7], which was used in Reference 34 in [7]. To obtain Main Theorem 2, we use Corollary of Lemma 2.1 in [7], the condition (A.II′) and Lemma 2.2. And various applications of these two symmetrizations (mixed problems for regularly hyperbolic equations of second order, weakly hyperbolic equations of second order, the wave equation in a domain with a corner, non-linear hyperbolic equations of second order and etc.) will be considered in [6] and in future (see [3]).

Since our paper is continued from [7], the references are added to those in [7]. As for the other symmetrization, we refer the reader to [2] and [4].

The paper is organized as follows. In §1, we state the notation, the assumption and the result. In §2, we treat several lemmas. In §3, we prove Main Theorem 1. In §4, we prove Main Theorem 2.

§1. The statement of the notation, the assumption and the result.

Firstly, we introduce notations.

For $\tilde{\tau}$, $\tilde{\xi}$, $\tilde{d}(\eta')$, Q_0 and Q_1, we use the same definitions as the definitions of the notations in [7: (2.12), (2.27) and (2.28)]. Also, we use the notations in [7: p. 135 and p. 178] except for the following notations:

$$d(t, x', \eta')=\Bigg[\sum_{i,j=2}^{n} a_{ij}(t, 0, x')\eta_i\eta_j-\frac{1}{a_{11}(t, 0, x')}\Big(\sum_{j=2}^{n} a_{1j}(t, 0, x')\eta_j\Big)^2 +\Big(1+\frac{h_1(t, 0, x')^2}{a_{11}(t, 0, x')}\Big)^{-1}\Big\{\sum_{j=2}^{n}\Big(h_j(t, 0, x') -\frac{h_1(t, 0, x')}{a_{11}(t, 0, x')}a_{1j}(t, 0, x')\Big)\eta_j\Big\}^2\Bigg]^{1/2}$$

$$b(t, x', \eta')=\sum_{j=2}^{n} b_j(t, x')\eta_j/d(t, x', \eta')$$

Λ: $\Lambda\in S^1$ and its symbol $\sigma(\Lambda)=\langle\eta'\rangle$

$|u|_{k+\sigma,\boldsymbol{R}^m}$: the norm of $\mathscr{B}^{k+\sigma}(\boldsymbol{R}^m)$ $(0\le\sigma<1)$

$$K_{1+\sigma}=|a_{11}|_{x_1=0}|_{1+\sigma,\boldsymbol{R}\times\boldsymbol{R}^{n-1}}+\sum_{j=2}^{n}|b_j|_{1+\sigma,\boldsymbol{R}\times\boldsymbol{R}^{n-1}}+|c|_{1+\sigma,\boldsymbol{R}\times\boldsymbol{R}^{n-1}}$$

$$\tilde{K}_{1+\sigma}=\sum_{i,j=1}^{n}|a_{ij}|_{x_1=0}|_{1+\sigma,\boldsymbol{R}\times\boldsymbol{R}^{n-1}}+\sum_{j=1}^{n}|h_j|_{x_1=0}|_{1+\sigma,\boldsymbol{R}\times\boldsymbol{R}^{n-1}}$$
$$+\sum_{j=2}^{n}|b_j|_{1+\sigma,\boldsymbol{R}\times\boldsymbol{R}^{n-1}}+|c|_{1+\sigma,\boldsymbol{R}\times\boldsymbol{R}^{n-1}}$$

$$s(t,x,X)=s(t,x,\eta')|_{\eta'=X}\,,\qquad X=(X_2,\cdots,X_n)\,.$$

We assume the following conditions for the problem (P):

(A.I) The operator L is regularly hyperbolic on $\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$ and $a_{11}(t,x)>0$ on $\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$

and

(A.II)

$$\text{(1.1)}\qquad \begin{cases} \mathrm{Re}\,c(t,x')\geq 0 \\ \{\mathrm{Re}\,c(t,x')\}^2\geq\{\mathrm{Re}\,b(t,x',\eta')\}^2+\{\mathrm{Re}\,c(t,x') \\ \qquad\qquad\cdot\mathrm{Im}\,b(t,x',\eta')-\mathrm{Im}\,c(t,x')\cdot\mathrm{Re}\,b(t,x',\eta')\}^2 \\ 1+|c(t,x')|^2>|b(t,x',\eta')|^2 \end{cases}$$

or

(A.II′) (A.II) holds and $c(t,x')\neq 1$

for all $(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}$ and all $\eta'\in\boldsymbol{R}^{n-1}-\{0\}$.

REMARK 1. (I) We have the fact that the second inequality in (1.1) is equivalent to the following inequality

$$\text{(1.2)}\qquad \begin{aligned} &[\{\mathrm{Re}\,c(t,x')\}^2-\{\mathrm{Re}\,b(t,x',\eta')\}^2] \\ &\cdot[1+\{\mathrm{Im}\,c(t,x')\}^2-\{\mathrm{Im}\,b(t,x',\eta')\}^2] \\ &\geq\{\mathrm{Re}\,c(t,x')\cdot\mathrm{Im}\,c(t,x')-\mathrm{Re}\,b(t,x',\eta')\cdot\mathrm{Im}(b(t,x',\eta')\}^2\,. \end{aligned}$$

Now, assume the condition (A.I). Then, by (1.2), we obtain that the problem (P) is L^2-well-posed if and only if the condition (A.II) holds (see Reference 19 in [7]).

(II) Assume the condition (A.I). Then, the problem (P) satisfies the uniform Lopatinski boundary condition if and only if the following condition holds,

$$\text{(1.3)}\qquad \begin{cases} \mathrm{Re}\,c(t,x')>0 \\ \{\mathrm{Re}\,c(t,x')\}^2>\{\mathrm{Re}\,b(t,x',\eta')\}^2+\{\mathrm{Re}\,c(t,x') \\ \qquad\qquad\cdot\mathrm{Im}\,b(t,x',\eta')-\mathrm{Im}\,c(t,x')\cdot\mathrm{Re}\,b(t,x',\eta')\}^2 \end{cases}$$

for all $(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}$ and all $\eta'\in\boldsymbol{R}^{n-1}-\{0\}$.

REMARK 2. We assume that $h_j\in\mathscr{B}^{l+1+\sigma}(\boldsymbol{R}^1\times\overline{\boldsymbol{R}^n_+})$ etc. for the treatment of not only the mixed problem (P) but also the boundary value problem $L[u]=f\,(x_1>0)$ and $B[u]|_{x_1=0}=g$ (see Reference [19: p. 218] in [7]). But, the condition $h_j\in\mathscr{B}^{l+1+\sigma}(\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+})$ is replaced by the condition $h_j\in\mathscr{B}^{l+1+\sigma}(\overline{\boldsymbol{R}^1_+}\times\overline{\boldsymbol{R}^n_+})$, etc. for the mixed problem. Also, in (A.I), the condition $a_{11}(t,x)>0$ on $\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$ is replaced by the condition $a_{11}(t,0,x')>0$ on $\overline{\boldsymbol{R}^1_+}\times\boldsymbol{R}^{n-1}$ for the mixed problem.

Now, we state our results.

MAIN THEOREM 1. *Assume the conditions* (A.I) *and* (A.II). *Then, the mixed problem* (P) *is reduced to the mixed problem for a symmetric hyperbolic system of first order with non-negative type boundary condition* (*see* (3.14) *and* (3.15)).

REMARK 3. See (3.15) for the sense of non-negative type boundary condition. We treat the symmetrization of the problem (P) under (1.3) (see Theorem 3.2 in this paper).

MAIN THEOREM 2. *Assume the conditions* (A.I) *and* (A.II$'$). *Then, the mixed problem* (P) *is reduced to the mixed problem for a symmetric hyperbolic system of first order with non-negative type boundary condition* (*see* (3.14) *and* (4.10)).

REMARK 4. See (4.10) for the sense of non-negative type boundary condition. The symmetrization in Main Theorem 2 is different from the one in Main Theorem 1. This symmetrization is used to treat the mixed problem in a domain with a corner (see Remark 6 in this paper and [6]).

§2. Several lemmas.

In this section, we prepare several lemmas which are useful to obtain our results. We set

$$(2.1)\quad \begin{cases} c(t,x')=c_1+ic_2\,,\quad b_j(t,x')=b_{j1}+ib_{j2}\\ b(t,x',\eta')=b_{\mathrm{I}}+ib_{\mathrm{II}}\\ \tilde{b}(t,x',\eta')=b(t,x',\eta')\cdot d(t,x',\eta')=\tilde{b}_{\mathrm{I}}+i\tilde{b}_{\mathrm{II}}\\ \alpha_1(t,x',\eta')=b_{\mathrm{I}}-c_2\{c_1b_{\mathrm{II}}-c_2b_{\mathrm{I}}\}\\ \tilde{\alpha}_1(t,x',\eta')=\alpha_1(t,x',\eta')\cdot d(t,x',\eta')\\ \alpha_2(t,x',\eta')=c_1\{c_1b_{\mathrm{II}}-c_2b_{\mathrm{I}}\}\\ \tilde{\alpha}_2(t,x',\eta')=\alpha_2(t,x',\eta')\cdot d(t,x',\eta') \end{cases}$$

under the conditions (A.I) and (A.II) where c_1, c_2, b_{j1}, b_{j2}, b_{I}, b_{II}, $\tilde{b}_{\mathrm{I}}$ and $\tilde{b}_{\mathrm{II}}$ are real valued functions and $2\le j\le n$.

Firstly, we treat several inequalities under the condition (A.II). These results are used in §3.

LEMMA 2.1. *Assume the conditions* (A.I) *and* (A.II). *Then, we have the following inequality*

$$c_1^2\{(1+|c|^2)d^2-\tilde{b}_{\mathrm{I}}^2-\tilde{b}_{\mathrm{II}}^2\}-(\tilde{\alpha}_1^2+\tilde{\alpha}_2^2)\geq 0 \tag{2.2}$$

for all $(t, x')\in \boldsymbol{R}\times\boldsymbol{R}^{n-1}$ *and all* $\eta'\in\boldsymbol{R}^{n-1}$ *where* c_1, c_2, $\tilde{b}_{\mathrm{I}}$, $\tilde{b}_{\mathrm{II}}$, $\tilde{\alpha}_1$ *and* $\tilde{\alpha}_2$ *are same functions as in* (2.1), *and* $d=d(t, x', \eta')$.

PROOF. By simple calculations and (A.II), we have

$$\begin{aligned} &c_1^2(1+|c|^2-|b|^2)-(|\alpha_1|^2+|\alpha_2|^2) \\ &\quad=(1+|c|^2)\{c_1^2-b_{\mathrm{I}}^2-(c_1b_{\mathrm{II}}-c_2b_{\mathrm{I}})^2\}\geq 0\,. \end{aligned} \tag{2.3}$$

Therefore, we obtain Lemma 2.1. Q.E.D.

REMARK 5. Assume the conditions (A.I) and (1.3). Then, by (1.3) and (2.3), there is a positive constant D such that

$$c_1^2[(1+|c|^2)(\tilde{d}(\eta')|_{x_1=0})^2-\tilde{b}_{\mathrm{I}}^2-\tilde{b}_{\mathrm{II}}^2]-(\tilde{\alpha}_1^2+\tilde{\alpha}_2^2)\geq D(\tilde{d}(\eta')|_{x_1=0})^2 \tag{2.4}$$

for all $(t, x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}$ and all $\eta'\in\boldsymbol{R}^{n-1}$ where c_1, c_2, $\tilde{b}_{\mathrm{I}}$, $\tilde{b}_{\mathrm{II}}$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are the same functions as in (2.1), and

$$\begin{cases} \tilde{d}(\eta')|_{x_1=0}=d(t, x', \eta')\,, \\ \tilde{d}(\eta')^2>0 \text{ for all } (t, x)\in\boldsymbol{R}^1\times\overline{\boldsymbol{R}^n_+} \text{ and all } \eta'\in\boldsymbol{R}^{n-1}-\{0\} \\ \qquad\text{(see (2.6), (2.9)}\sim\text{(2.11) and (2.12) in [7])}\,. \end{cases} \tag{2.5}$$

Secondly, we treat several inequalities under the condition (A.II′). These results are used in §4.

We set

$$\begin{cases} \sqrt{1-c(t, x')^2}=A(t, x')+iB(t, x') \qquad (A, B\in\boldsymbol{R}) \\ q_0(t, x', \eta')=\dfrac{4\{b_{\mathrm{II}}(t, x', \eta')A(t, x')-b_{\mathrm{I}}(t, x', \eta')B(t, x')\}}{(|c(t, x')+1|+|c(t, x')-1|)^2} \\ \tilde{q}_0(t, x', \eta')=q_0(t, x', \eta')\cdot d(t, x', \eta') \\ \beta_1(t, x', \eta')=B(t, x')q_0(t, x', \eta')+b_{\mathrm{I}}(t, x', \eta') \\ \tilde{\beta}_1(t, x', \eta')=\beta_1(t, x', \eta')\cdot d(t, x', \eta') \\ \beta_2(t, x', \eta')=-A(t, x')q_0(t, x', \eta')+b_{\mathrm{II}}(t, x', \eta') \\ \tilde{\beta}_2(t, x', \eta')=\beta_2(t, x', \eta')\cdot d(t, x', \eta') \end{cases} \tag{2.6}$$

under the conditions (A.I) and (A.II′)

LEMMA 2.2. *For the functions* $\tilde{q}_0$, $\tilde{\beta}_1$ *and* $\tilde{\beta}_2$ *in* (2.6) *and* $d=d(t, x', \eta')$, *we have*

$$4(\tilde{\beta}_1^2+\tilde{\beta}_2^2)\leq(|c+1|-|c-1|)^2(d^2-\tilde{q}_0^2) \tag{2.7}$$

i.e.

$$4\{(B\tilde{q}_0+\tilde{b}_{\mathrm{I}})^2+(-A\tilde{q}_0+\tilde{b}_{\mathrm{II}})^2\}\leq(|c+1|-|c-1|)^2(d^2-\tilde{q}_0^2)\,. \tag{2.8}$$

PROOF. We consider the following quadratic inequality with respect to q,

$$4\{(Bq+b_{\mathrm{I}})^2+(-Aq+b_{\mathrm{II}})^2\}\leq(|c+1|-|c-1|)^2(1-q^2)\,, \tag{2.9}$$

i.e.

$$\begin{aligned} f(q)=(|c+1|+|c-1|)^2q^2+8(b_{\mathrm{I}}B-b_{\mathrm{II}}A)q \\ +4(b_{\mathrm{I}}^2+b_{\mathrm{II}}^2)-(|c+1|-|c-1|)^2\leq 0\,. \end{aligned} \tag{2.10}$$

Then, the discriminant D of the quadratic equation $f(q)=0$ with respect to q satisfies the following,

$$D/4=16(b_{\mathrm{I}}B-b_{\mathrm{II}}A)^2-4(b_{\mathrm{I}}^2+b_{\mathrm{II}}^2)\cdot 2(1+c_1^2+c_2^2+A^2+B^2)+16c_1^2\,. \tag{2.11}$$

By (1.1), (2.6), (2.11) and simple calculations, we have

$$D=64\{c_1^2-b_{\mathrm{I}}^2-(c_1b_{\mathrm{II}}-c_2b_{\mathrm{I}})^2\}\geq 0\,. \tag{2.12}$$

Also, by (A.II′) and (2.6), we get

$$\begin{aligned} &(|c+1|+|c-1|)^2\mp 4(b_{\mathrm{II}}A-b_{\mathrm{I}}B) \\ &=2\{1+c_1^2+c_2^2-b_{\mathrm{I}}^2-b_{\mathrm{II}}^2+(A\mp b_{\mathrm{II}})^2+(B\pm b_{\mathrm{I}})^2\}>0\,. \end{aligned} \tag{2.13}$$

Then, by (2.12) and (2.13), we have a real root $q_0(t, x', \eta')$ of the inequality (2.9) which satisfies $|q_0|<1$. Q.E.D.

REMARK 6. In [6], we considered a case where

$$f(0)=4(b_{\mathrm{I}}^2+b_{\mathrm{II}}^2)-(|c+1|-|c-1|)^2\leq 0\,. \tag{2.14}$$

§3. Proof of Main Theorem 1.

In this section, we shall prove Main Theorem 1 by using the results in §2 and notations in (2.1) (without writing variables sometimes).

By (A.I), (A.II) and (2.5), we have

$$\inf_{\substack{(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}\\ |\eta'|=1}}\left[\,\tilde{d}(\eta')\Big|_{x_1=0}^2-\frac{|\tilde{b}|^2}{1+|c|^2}\right]>0\,. \tag{3.1}$$

Then, by (2.5) and (3.1), there exist a positive constant δ and a function ρ such that

$$\begin{cases} \rho\in C^\infty(\boldsymbol{R})\,, & 0\leq\rho\leq 1 \\ \rho(x_1)=1(x_1\leq\delta)\,, \quad \rho(x_1)=0 & (x_1\geq 2\delta) \end{cases} \tag{3.2}$$

and

$$\inf_{\substack{(t,x)\in \boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}\\ |\eta'|=1}}\left[\tilde{d}(\eta')^2-\frac{\{\rho(x_1)\cdot|\tilde{b}|\}^2}{1+|c|^2}\right]>0\,. \tag{3.3}$$

We set

$$\begin{cases} \tilde{p}_0(t,x,\eta')=\dfrac{-\rho(x_1)\cdot\tilde{b}_{\mathrm{I}}}{\sqrt{1+|c|^2}}\,, \qquad \tilde{q}_0(t,x,\eta')=\dfrac{\rho(x_1)\cdot\tilde{b}_{\mathrm{II}}}{\sqrt{1+|c|^2}} \\ J(\eta')=\tilde{d}(\eta')^2-\{\tilde{p}_0(t,x,\eta')\}^2-\{\tilde{q}_0(t,x,\eta')\}^2 \end{cases} \tag{3.4}$$

for $(t,x)\in \boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$ and $\eta'=(\eta_2,\cdots,\eta_n)\in \boldsymbol{R}^{n-1}$. Then, by the same argument in [7: p. 164~p. 167], and (3.3), we have the following facts:

(i) There is a real and symmetric $(n-1)\times(n-1)$ matrix $M(t,x)\in\mathscr{B}^{l+1+\sigma}(\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+})$ such that for ${}^t\eta'=(\eta_2,\cdots,\eta_n)\in \boldsymbol{R}^{n-1}$,

$$J(\eta')={}^t\eta' M\eta' \quad \text{and} \quad M>0\,. \tag{3.5}$$

(ii) There exist a real $(n-1)\times(n-1)$ matrix $N(t,x)\in\mathscr{B}^{l+1+\sigma}(\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+})$ and positive constants C_1 and C_2 such that

$$\begin{cases} {}^tNMN=\mathrm{diag}(\theta_2(t,x),\cdots,\theta_n(t,x))>0 \\ C_1>|\det N(t,x)|>C_2\,. \end{cases} \tag{3.6}$$

(iii) For $J(\eta')$ in (3.4), we have

$$\begin{cases} J(\eta')={}^t\zeta'\cdot\zeta'\,, \quad \tilde{d}(\eta')=\tilde{p}_0^2+\tilde{q}_0^2+\sum_{j=2}^n \zeta_j^2 \\ \sigma_0(L)=\tilde{\xi}^2+\tilde{p}_0^2+\tilde{q}_0^2+\sum_{j=2}^n \zeta_j^2-\tilde{\tau}^2 \end{cases} \tag{3.7}$$

where

$$\zeta'={}^t(\zeta_2,\cdots,\zeta_n)=\mathrm{diag}(\sqrt{\theta_2},\cdots,\sqrt{\theta_n})\cdot N^{-1}\cdot\eta' \quad (\eta'={}^t(\eta_2,\cdots,\eta_n))\,. \tag{3.8}$$

We set

$$\begin{cases} Q_{21}=\tilde{p}_0\left(t,x,\dfrac{\partial}{\partial x'}\right)=\sum_{j=2}^n p_j\dfrac{\partial}{\partial x_j}\,, \qquad p_j=\dfrac{-\rho(x_1)b_{j1}}{\sqrt{1+|c|^2}} \\ Q_{22}=\tilde{q}_0\left(t,x,\dfrac{\partial}{\partial x'}\right)=\sum_{j=2}^n q_j\dfrac{\partial}{\partial x_j}\,, \qquad q_j=\dfrac{\rho(x_1)b_{j2}}{\sqrt{1+|c|^2}} \end{cases} \tag{3.9}$$

and

$$(3.10)\quad \begin{cases} ((p_{ij}(t, x)) = \mathrm{diag}(\sqrt{\theta_2}, \cdots, \sqrt{\theta_n}) \cdot N^{-1} \\ Q_{j+1} = \sum_{l=2}^{n} p_{jl}(t, x) \dfrac{\partial}{\partial x_l} \qquad (j=2, \cdots, n) \end{cases}$$

where $\partial/\partial x' = (\partial/\partial x_2, \cdots, \partial/\partial x_n)$.

For the solution u of the problem (P), we set

$$(3.11)\quad U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{n+1} \\ U_{n+2} \end{pmatrix} = \begin{pmatrix} Q_0 u - Q_1 u + z(Q_{21}u - iQ_{22}u) \\ z(Q_0 u + Q_1 u) + Q_{21}u + iQ_{22}u \\ Q_3 u \\ \vdots \\ Q_{n+1} u \\ u \end{pmatrix}$$

where

$$(3.12)\quad z(t, x') = \frac{\sqrt{1 + |c(t, x')|^2}}{c(t, x') + 1}.$$

We have

$$(3.13)\quad \begin{cases} \hat{\alpha}_k = \hat{\alpha}_k\left(t, x', \dfrac{\partial}{\partial x'}\right) = \sum_{j=2}^{n} \hat{\alpha}_{jk}(t, x') \dfrac{\partial}{\partial x_j} \qquad (k=1, 2) \\ \hat{\alpha}_{j1}(t, x') = b_{j1} - c_2(c_1 \cdot b_{j2} - c_2 \cdot b_{j1}) \\ \hat{\alpha}_{j2}(t, x') = c_1(c_1 \cdot b_{j2} - c_2 \cdot b_{j1}) . \end{cases}$$

Then, by the same argument in [7: p. 164$\sim$p. 167], we have:

(Main Theorem 1) *Assume the conditions* (A.I) *and* (A.II) *for the problem* (P). *Then, U in* (3.11) *satisfies the following system*:

$$(3.14)\quad \begin{cases} MU_t = A_1 U_{x_1} + \sum_{j=2}^{n} A_j U_{x_j} + EU + F(t, x) \\ U(0, x) = U_0(x) \\ PU|_{x_1=0} = G(t, x') \\ (t, x) = (t, x_1, x') \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^{n-1} \end{cases}$$

where

$$M = \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} \mathrm{diag}(1, 1, 1+|z|^2, \cdots, 1+|z|^2, 1)$$
$$- \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}} \mathrm{diag}(-1, 1, 1-|z|^2, \cdots, 1-|z|^2, 1),$$

$$A_1=\sqrt{a_{11}(t,x)}\,\mathrm{diag}(-1,1,1-|z|^2,\cdots,1-|z|^2,1)\,,$$

$$A_j=\frac{a_{1j}(t,x)}{a_{11}(t,x)}A_1+\left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{-1/2}\cdot(h_j(t,x)$$

$$-\frac{h_1(t,x)}{a_{11}(t,x)}a_{1j}(t,x))\mathrm{diag}(1,1,1+|z|^2,\cdots,1+|z|^2,1)$$

$$+\left(\begin{array}{cc|ccc} 0 & p_j-iq_j & & 0 & \\ p_j+iq_j & 0 & & & \\ \hline & & -2(\mathrm{Re}\,z\cdot p_j+\mathrm{Im}\,z\cdot q_j) & & \\ & & & \ddots & 0 \\ 0 & & 0 & & -2(\mathrm{Re}\,z\cdot p_j+\mathrm{Im}\,z\cdot q_j) \\ & & & & 1 \end{array}\right)$$

$$+\sum_{l=2}^{n}p_{lj}\;\begin{array}{c}\\ \\ \\ \\ \\ \\ (l+1)\text{ row}\end{array}\overset{(l+1)\text{ column}}{\begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & z & & \cdot \\ \cdot & & & & & \cdot & 0 & & \cdot \\ \cdot & & & & & \cdot & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot & \mathbf{0} & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & & \cdot \\ 1 & \bar{z} & 0 & \cdot & \cdot & \cdot & 0 & & \cdot \\ & & & & & & & & \cdot \\ & & & \mathbf{0} & & & & & \cdot \\ & & & & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}}\qquad (j=2,\cdots,n)\,,$$

$E\cdots$ *an* $(n+2)\times(n+2)$ *matrix which has the property that for* $E=(e_{ij})$, e_{ij} *belongs to* $\mathscr{B}^1(\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+})$,

$$F={}^t(f,zf,0,\cdots,0)\,,$$

$$P=\left(1,\frac{c-1}{\sqrt{1+|c|^2}},-\frac{2(\tilde{\alpha}_{21}+i\tilde{\alpha}_{22})}{(c+1)(1+|c|^2)},\cdots,-\frac{2(\tilde{\alpha}_{n1}+i\tilde{\alpha}_{n2})}{(c+1)(1+|c|^2)},0\right)$$

$$\left(\begin{array}{l}(\tilde{\alpha}_{21}+i\tilde{\alpha}_{22},\cdots,\tilde{\alpha}_{n1}+i\tilde{\alpha}_{n2})=(\hat{\alpha}_{21}+i\hat{\alpha}_{22},\cdots,\hat{\alpha}_{n1}+i\hat{\alpha}_{n2})\cdot\\ \qquad\cdot\left\{N\cdot\mathrm{diag}\left(\frac{1}{\sqrt{\theta_2}},\cdots,\frac{1}{\sqrt{\theta_n}}\right)\right\}\Big|_{x_1=0}\qquad(\tilde{\alpha}_{j1},\tilde{\alpha}_{j2}\in\boldsymbol{R}:j=2,\cdots,n)\end{array}\right)$$

$$G=-\frac{2g(t,x')}{c(t,x')+1},$$

and the following inequality holds,

(3.15) $$\langle A_1|_{x_1=0}\tilde{U},\tilde{U}\rangle\ge -C(K_{1+\sigma})\langle\!\langle\tilde{u}\rangle\!\rangle_{1/2}^2 \qquad \text{for all} \quad \tilde{U}\in\operatorname{Ker}P$$

where $u\in\mathscr{E}_t^0(H_2(\boldsymbol{R}_+^n))\cap\mathscr{E}_t^1(H_1(\boldsymbol{R}_+^n))\cap\mathscr{E}_t^2(L^2(\boldsymbol{R}_+^n))$ *is the solution of the problem* (P), U *is the same vector as in* (3.11), $\tilde{u}=u|_{x_1=0}$, $\tilde{U}=U|_{x_1=0}$ *and* C *is a positive constant depending on* $K_{1+\sigma}$.

PROOF. By the same argument as in [7: p. 164~p. 167], (3.9), (3.10), (3.11), (3.12) and (3.13), we have (3.14). Therefore, we have only to prove (3.15).

From now on, we shall prove (3.15).

We denote $h|_{x_1=0}$ by $\tilde{h}$. Let $\tilde{U}\in\operatorname{Ker}P$. Then, we get

(3.16) $$\tilde{U}_1=\frac{1-c}{\sqrt{1+|c|^2}}\tilde{U}_2+\frac{2}{(c+1)(1+|c|^2)}(\hat{\alpha}_1+i\hat{\alpha}_2)\tilde{u}\,.$$

We set

(3.17) $$I=\frac{1}{\sqrt{a_{11}(t,0,x')}}[A_1|_{x_1=0}\tilde{U},\tilde{U}] \qquad \text{for} \quad \tilde{U}\in\operatorname{Ker}P\,.$$

Then, we get

(3.18) $$I=\left\{-|\tilde{U}_1|^2+|\tilde{U}_2|^2+\frac{2c_1}{|c+1|^2}\sum_{j=2}^{n}|\tilde{Q}_{j+1}\tilde{u}|^2+|\tilde{u}|^2\right\}$$

where

(3.19) $$\tilde{Q}_{j+1}=\sum_{l=2}^{n}p_{jl}(t,0,x')\frac{\partial}{\partial x_l} \qquad (j=2,\cdots,n)\,.$$

By (3.16), we have for $c_1(t,x')>0$,

(3.20) $$|\tilde{U}_1|^2\le|\tilde{U}_2|^2+\frac{2}{c_1|c+1|^2(1+|c|^2)}|(\hat{\alpha}_1+i\hat{\alpha}_2)\tilde{u}|^2\,.$$

Also, by (3.4), (3.7), (3.8), (3.9), (3.10) and (3.19), we obtain

(3.21) $$\sum_{j=2}^{n}|\tilde{Q}_{j+1}\tilde{u}|^2=\left[\left\{d\left(t,x',\frac{\partial v}{\partial x'}\right)^2-|\tilde{Q}_{21}v|^2-|\tilde{Q}_{22}v|^2\right\}\right.$$
$$\left.+\left\{d\left(t,x',\frac{\partial w}{\partial x'}\right)^2-|\tilde{Q}_{21}w|^2-|\tilde{Q}_{22}w|^2\right\}\right]$$

where

$$
(3.22)\qquad \begin{cases} v=\mathrm{Re}\tilde{u}\,, \quad w=\mathrm{Im}\tilde{u}\,, \quad \dfrac{\partial v}{\partial x'}=\left(\dfrac{\partial v}{\partial x_2}, \cdots, \dfrac{\partial v}{\partial x_n}\right) \\ \tilde{Q}_{21}=\sum_{j=2}^{n} p_j(t, 0, x')\dfrac{\partial}{\partial x_j}, \qquad \tilde{Q}_{22}=\sum_{j=2}^{n} q_j(t, 0, x')\dfrac{\partial}{\partial x_j} \end{cases}
$$

and etc. Therefore, by (3.2), (3.4), (3.9), (3.12), (3.18), (3.20) and (3.21), we get

$$
\begin{aligned}
(3.23)\qquad I\geq \frac{2}{c_1|c+1|^2(1+|c|^2)} & \left\{\left[c_1\left((1+|c|^2)d\left(t, x', \frac{\partial v}{\partial x'}\right)^2\right.\right.\right. \\
& -\left(\sum_{j=2}^{n}\mathrm{Re}b_j\frac{\partial v}{\partial x_j}\right)^2-\left.\left(\sum_{j=2}^{n}\mathrm{Im}b_j\frac{\partial v}{\partial x_j}\right)^2\right)-|\hat{\alpha}_1 v|^2 \\
& \left.-|\hat{\alpha}_2 v|^2\right]+\left[c_1^2\left((1+|c|^2)d\left(t, x', \frac{\partial w}{\partial x'}\right)^2\right.\right. \\
& -\left(\sum_{j=2}^{n}\mathrm{Re}b_j\frac{\partial w}{\partial x_j}\right)^2-\left.\left(\sum_{j=2}^{n}\mathrm{Im}b_j\frac{\partial w}{\partial x_j}\right)^2\right)-|\hat{\alpha}_1 w|^2 \\
& \left.\left.-|\hat{\alpha}_2 w|^2\right]-2(\hat{\alpha}_2 v\cdot\hat{\alpha}_1 w-\hat{\alpha}_1 v\cdot\hat{\alpha}_2 w)\right\}.
\end{aligned}
$$

Then, by (2.2), (3.13) and (3.23), we have

$$
(3.24)\qquad I\geq\frac{-4}{c_1|c+1|^2(1+|c|^2)}(\hat{\alpha}_2 v\cdot\hat{\alpha}_1 w-\hat{\alpha}_1 v\cdot\hat{\alpha}_2 w)
$$

for $c_1(t, x')>0$. Also, by (A.II), we obtain

$$
(3.25)\qquad \left\{(t, x')\in \boldsymbol{R}\times\boldsymbol{R}^{n-1}\;\middle|\;\sum_{j=2}^{n}(\mathrm{Re}b_j)^2=0\right\}\supseteq\{(t, x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}\mid c_1=0\}\,.
$$

Then, by (3.13), (3.24) and (3.25), we get

$$
(3.26)\qquad I\geq\frac{-4}{|c+1|^2(1+|c|^2)}(\hat{\hat{\alpha}}_2 v\cdot\hat{\alpha}_1 w-\hat{\alpha}_1 v\cdot\hat{\hat{\alpha}}_2 w)
$$

for $c_1(t, x')>0$ and $c_1(t, x')=0$ where $\hat{\hat{\alpha}}_2=c_1^{-1}\cdot\hat{\alpha}_2$. By (A.I), (3.26), v, $w\in \mathscr{E}_t^0(H_{3/2}(\boldsymbol{R}_{x'}^{n-1}))\cap\mathscr{E}_t^1(H_{1/2}(\boldsymbol{R}_{x'}^{n-1}))$ and the fact that $C_0^\infty(\boldsymbol{R}_{x'}^{n-1})$ is dense in $H_{3/2}(\boldsymbol{R}_{x'}^{n-1})$, we have

$$
\begin{aligned}
(3.27)\qquad \langle A_1|_{x_1=0}\tilde{U}, \tilde{U}\rangle &= \int_{\boldsymbol{R}^{n-1}}\sqrt{a_{11}(t, 0, x')}\,I\,dx' \\
&\geq\int_{\boldsymbol{R}^{n-1}}\{k[\hat{\alpha}_1, \hat{\hat{\alpha}}_2]v\cdot w+(\hat{\alpha}_1 k\cdot\hat{\hat{\alpha}}_2 v-\hat{\hat{\alpha}}_2 k\cdot\hat{\alpha}_1 v)w\}dx'
\end{aligned}
$$

where

$$k=\frac{4\sqrt{a_{11}(t, 0, x')}}{|c(t, x')+1|^2(1+|c(t, x')|^2)}. \tag{3.28}$$

LEMMA 3.1 ([5]). *Assume that* $a\in\mathscr{B}^{\sigma}(\boldsymbol{R}^{n-1}_{x'})$ $(1/2<\sigma<1)$ *and* $T\in\boldsymbol{S}^{1/2}$ *whose symbol is equal to* $T(\eta')$ $(\eta'=(\eta_2, \cdots, \eta_n))$. *Then, we have*

$$\langle\!\langle [a, T]\theta\rangle\!\rangle_0\leq C|a|_{\sigma,\boldsymbol{R}^{n-1}}\langle\!\langle\theta\rangle\!\rangle_0 \tag{3.29}$$

where C *is a positive constant.*

Also, for $a\in\mathscr{B}^{\sigma}(\boldsymbol{R}^{n-1}_{x'})$ $(1/2<\sigma<1)$, we have

$$\left\langle a\frac{\partial}{\partial x_j}v, w\right\rangle=\left\langle\Lambda^{-1/2}\left[a, \frac{\partial}{\partial x_j}\Lambda^{-1/2}\right]\Lambda^{1/2}v, \Lambda^{1/2}w\right\rangle \tag{3.30}$$
$$+\left\langle\Lambda^{-1/2}\frac{\partial}{\partial x_j}\Lambda^{-1/2}a\Lambda^{1/2}v, \Lambda^{1/2}w\right\rangle \qquad (j=2, \cdots, n).$$

By the results in [1: p. 214], we obtain

$$\langle\!\langle\Lambda^{1/2}h\rangle\!\rangle_0^2\leq C\langle\!\langle\tilde{u}\rangle\!\rangle_{1/2}^2 \qquad \text{for} \quad h=v, w \tag{3.31}$$

where C is a positive constant. Then, by (3.27), Lemma 3.1, (3.30) and (3.31), we obtain (3.15).

Therefore, we get Main Theorem 1. Q.E.D.

REMARK 7. We have that M, $A_j\in\mathscr{B}^{l+1+\sigma}(\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+})$, z, $P\in\mathscr{B}^{l+1+\sigma}(\boldsymbol{R}\times\boldsymbol{R}^{n-1})$ and $|z|\leq 1$.

REMARK 8. To obtain the energy inequality of higher order, we have to estimate U_{x_1} for U in (3.11). Operating the differential operator Q_{j+1} $(j=2, \cdots, n)$ for $U_1=Q_0u-Q_1u+z(Q_{21}u-iQ_{22}u)$ where $U={}^t(U_1, U_2, \cdots, U_{n+2})$ in (3.11), we can obtain the similar result as Corollary of Theorem 5.1 in [7].

Now, we have the following result for the problem (P) under (1.3).

THEOREM 3.2. *Assume the conditions* (A.I) *and* (1.3) *for the problem* (P). *Then,* (3.14) *holds and we have the following inequality,*

$$\langle A_1|_{x_1=0}\tilde{U}, \tilde{U}\rangle\geq C_1\langle\tilde{U}, \tilde{U}\rangle-C_2\langle\!\langle\tilde{u}\rangle\!\rangle_0^2 \qquad \textit{for all} \quad \tilde{U}\in\operatorname{Ker}P \tag{3.32}$$

where $u\in\mathscr{E}^0_t(H_2(\boldsymbol{R}^n_+))\cap\mathscr{E}^1_t(H_1(\boldsymbol{R}^n_+))\cap\mathscr{E}^2_t(L^2(\boldsymbol{R}^n_+))$ *is the solution of the problem* (P), U *is the same vector as in* (3.11), $\tilde{u}=u|_{x_1=0}$, $\tilde{U}=U|_{x_1=0}$ *and* C_1 *and* C_2 *are positive constants depending on* $\tilde{K}_1$ $(\sigma=0)$, *and* $A_1=\sqrt{a_{11}(t, x)}\operatorname{diag}(-1, 1, 1-|z|^2, \cdots, 1-|z|^2, 1)$.

PROOF. We have only to prove (3.32). We have for $\tilde{U}\in\operatorname{Ker}P$,

$$(3.33)\qquad |\tilde{U}_1|^2 \leq \frac{|1-c|^2+2c_1(1-\varepsilon)}{1+|c|^2}|\tilde{U}_2|^2+\frac{2}{|c+1|^2(1+|c|^2)^2} \cdot\left\{\frac{|1-c|^2}{c_1}\left(1+\frac{\varepsilon}{1-\varepsilon}\right)+2\right\}|(\hat{\alpha}_1+i\hat{\alpha}_2)\tilde{u}|^2$$

where ε is a sufficiently small positive constant. Therefore, by (2.4), (3.18), (3.21) and (3.33), there is a sufficiently small positive constant ε_0 such that for all ε $(0<\varepsilon\leq\varepsilon_0)$

$$(3.34)\qquad I \geq \frac{2c_1\varepsilon}{1+|c|^2}|\tilde{U}_2|^2+\frac{1}{c_1|c+1|^2(1+|c|^2)}\Bigg[\Bigg\{c_1^2\Bigg((1+|c|^2) \cdot d\left(t,x',\frac{\partial v}{\partial x'}\right)^2-\left(\sum_{j=2}^n \mathrm{Re}\, b_j\frac{\partial v}{\partial x_j}\right)^2-\left(\sum_{j=2}^n \mathrm{Im}\, b_j\frac{\partial v}{\partial x_j}\right)^2\Bigg) -|\hat{\alpha}_1 v|^2-|\hat{\alpha}_2 v|^2\Bigg\}+\Bigg\{c_1^2\Bigg((1+|c|^2)d\left(t,x',\frac{\partial w}{\partial x'}\right)^2 -\left(\sum_{j=2}^n \mathrm{Re}\, b_j\frac{\partial w}{\partial x_j}\right)^2-\left(\sum_{j=2}^n \mathrm{Im}\, b_j\frac{\partial w}{\partial x_j}\right)^2\Bigg)-|\hat{\alpha}_1 w|^2 -|\hat{\alpha}_2 w|^2\Bigg\}-4(\hat{\alpha}_2 v\cdot\hat{\alpha}_1 w-\hat{\alpha}_1 v\cdot\hat{\alpha}_2 w)\Bigg]+|\hat{u}|^2 .$$

Then, by (2.4) and (3.34), we have

$$(3.35)\quad \langle A_1|_{x_1=0}\tilde{U},\tilde{U}\rangle \geq C_0\langle\tilde{U},\tilde{U}\rangle+\int_{\mathbf{R}^{n-1}}\{k[\hat{\alpha}_1,\hat{\alpha}_2]v\cdot w+(\hat{\alpha}_1 k\cdot\hat{\alpha}_2 v-\hat{\alpha}_2 k\cdot\hat{\alpha}_1 v)w\}dx'$$

where v and w are the same functions as in (3.22), k is the same function as in (3.28) and C_0 is a positive constant. By (3.35), we get Theorem 3.2. Q.E.D.

§4. Proof of Main Theorem 2.

In this section, we shall prove Main Theorem 2 by using the results in §2 and notations in (2.1) and (2.6) (without writing variables sometimes).

By (A.I), (A.II′), (2.5) and (2.13), we obtain

$$(4.1)\qquad \inf_{\substack{(t,x')\in\mathbf{R}\times\mathbf{R}^{n-1}\\ |\eta'|=1}}[\tilde{d}(\eta')|_{x_1=0}^2-\{\tilde{q}_0(t,x',\eta')\}^2]>0 .$$

Then, by (2.5) and (4.1), there exist a positive constant δ and a function $\rho(x_1)$ such that ρ satisfies (3.2) and

$$(4.2)\qquad \inf_{\substack{(t,x)\in\mathbf{R}\times\overline{\mathbf{R}^n_+}\\ |\eta'|=1}}[\tilde{d}(\eta')^2-\{\rho(x_1)\cdot\tilde{q}_0(t,x',\eta')\}^2]>0 .$$

We set

$$J(\eta')=\tilde{a}(\eta')^2-\{\rho(x_1)\cdot\tilde{q}_0(t,x',\eta')\}^2 \tag{4.3}$$

for $(t,x)\in \boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$ and $\eta'=(\eta_2,\cdots,\eta_n)\in\boldsymbol{R}^{n-1}$. Then, by the same argument as in [7: p. 164~167], we can obtain the similar results (M, N, $J(\eta')$, ζ' etc.) as in (3.5), (3.6), (3.7) and (3.8).

We set

$$\begin{cases} q_j(t,x)=\dfrac{4\rho\cdot(A\cdot b_{j2}-B\cdot b_{j1})}{(|c+1|+|c-1|)^2} & (j=2,\cdots,n) \\ Q_2=\rho(x_1)\tilde{q}_0\left(t,x',\dfrac{\partial}{\partial x'}\right)=\sum_{j=2}^{n}q_j(t,x)\dfrac{\partial}{\partial x_j} \end{cases} \tag{4.4}$$

and

$$\begin{cases} (p_{ij}(t,x))=\mathrm{diag}(\sqrt{\theta_2},\cdots,\sqrt{\theta_n})\cdot N^{-1} \\ Q_{j+1}=\sum_{l=2}^{n}p_{jl}(t,x)\dfrac{\partial}{\partial x_l} \qquad (j=2,\cdots,n)\,. \end{cases} \tag{4.5}$$

For the solution u of the problem (P), we set

$$U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{n+1} \\ U_{n+2} \end{pmatrix}=\begin{pmatrix} Q_0u-Q_1u-izQ_2u \\ z(Q_0u+Q_1u)+iQ_2u \\ Q_3u \\ \vdots \\ Q_{n+1}u \\ u \end{pmatrix} \tag{4.6}$$

where

$$z(t,x')=\frac{\sqrt{1-c(t,x')^2}}{c(t,x')+1}\,. \tag{4.7}$$

Also, we set

$$\begin{cases} \hat{\beta}_{j1}(t,x')=B(t,x')q_j(t,0,x')+b_{j1}(t,x') \\ \hat{\beta}_{j2}(t,x')=-A(t,x')q_j(t,0,x')+b_{j2}(t,x') \\ \qquad (j=2,\cdots,n)\,. \end{cases} \tag{4.8}$$

Then, we have

$$
(4.9)\qquad \begin{cases} \hat{\beta}_k=\tilde{\beta}_k\left(t, x', \dfrac{\partial}{\partial x'}\right)=\sum_{j=2}^{n}\beta_{jk}(t, x')\dfrac{\partial}{\partial x_j} \qquad (k=1, 2) \\ \hat{\beta}_1=\dfrac{4}{(|c+1|+|c-1|)^2}\left\{-c_1c_2\sum_{j=2}^{n} b_{j2}\dfrac{\partial}{\partial x_j}+(1+c_2^2)\sum_{j=2}^{n} b_{j1}\dfrac{\partial}{\partial x_j}\right\} \\ \hat{\beta}_2=\dfrac{4c_1}{(|c+1|+|c-1|)^2}\left\{c_1\sum_{j=2}^{n} b_{j2}\dfrac{\partial}{\partial x_j}-c_2\sum_{j=2}^{n} b_{j1}\dfrac{\partial}{\partial x_j}\right\}. \end{cases}
$$

Then, we can obtain the similar result as Main Theorem 1 with the following P_1 and (4.10) instead of P in Main Theorem 1 (see (3.14) and (3.15)) and (3.15):

$$
P_1=\left(1, -z, -\frac{2}{1+c}(\tilde{\beta}_{21}+i\tilde{\beta}_{22}), \cdots, -\frac{2}{1+c}(\tilde{\beta}_{n1}+i\tilde{\beta}_{n2}), 0\right)
$$

$$
\left(\begin{array}{l} (\tilde{\beta}_{21}+i\tilde{\beta}_{22}, \cdots, \tilde{\beta}_{n1}+i\tilde{\beta}_{n2})=(\hat{\beta}_{21}+i\hat{\beta}_{22}, \cdots, \hat{\beta}_{n1}+i\hat{\beta}_{n2})\cdot \\ \cdot\left\{N\cdot \mathrm{diag}\left(\dfrac{1}{\sqrt{\theta_2}}, \cdots, \dfrac{1}{\sqrt{\theta_n}}\right)\right\}\Big|_{x_1=0} \qquad (\tilde{\beta}_{j1}, \tilde{\beta}_{j2}\in \boldsymbol{R}: j=2, \cdots, n) \end{array}\right)
$$

and the following inequality holds,

$$
(4.10)\qquad \langle A_1|_{x_1=0}\tilde{U}, \tilde{U}\rangle \geq -C(K_{1+\sigma})\langle\!\langle \tilde{u}\rangle\!\rangle_{1/2}^2 \qquad \text{for all} \quad \tilde{U}\in \mathrm{Ker}\,P_1
$$

where $u\in\mathscr{E}_t^0(H_2(\boldsymbol{R}_+^n))\cap\mathscr{E}_t^1(H_1(\boldsymbol{R}_+^n))\cap\mathscr{E}_t^2(L^2(\boldsymbol{R}_+^n))$ is the solution of the problem (P), U is the same vector as in (4.6), $\tilde{u}=u|_{x_1=0}$, $\tilde{U}=U|_{x_1=0}$ and C is a positive constant depending on $K_{1+\sigma}$ and $A_1=\sqrt{a_{11}(t, x)}\,\mathrm{diag}(-1, 1, 1-|z|^2, \cdots, 1-|z|^2, 1)$.

From now on, we shall prove (4.10).

We denote $h|_{x_1=0}$ by $\tilde{h}$. Let $\tilde{U}\in\mathrm{Ker}\,P_1$. Then, we have

$$
(4.11)\qquad \tilde{U}_1=z\tilde{U}_2+\frac{2}{c+1}(\hat{\beta}_1+i\hat{\beta}_2)\tilde{u}\,.
$$

And, by (3.17), we obtain

$$
(4.12)\qquad I=\left\{-|\tilde{U}_1|^2+|\tilde{U}_2|^2+(1-|z|^2)\sum_{j=2}^{n}|\tilde{Q}_{j+1}\tilde{u}|^2+|\tilde{u}|^2\right\}
$$

where

$$
(4.13)\qquad \tilde{Q}_{j+1}=\sum_{l=2}^{n}p_{jl}(t, 0, x')\frac{\partial}{\partial x_l} \qquad (j=2, \cdots, n)\,.
$$

By (4.11), we get for $c_1(t, x')>0$,

$$
(4.14)\qquad |\tilde{U}_1|^2\leq\frac{|c-1|}{|c+1|}|\tilde{U}_2|^2+\sqrt{2}\,\frac{\sqrt{|c+1|-|c-1|}}{\sqrt{|c+1|}}|\tilde{U}_2|\cdot 2\sqrt{2}
$$

$$\cdot\frac{\sqrt{|c-1|}}{\sqrt{|c+1|-|c-1|}}\cdot\frac{|(\hat{\beta}_1+i\hat{\beta}_2)\tilde{u}|}{|c+1|}+\frac{4}{|c+1|^2}|(\hat{\beta}_1+i\hat{\beta}_2)\tilde{u}|^2$$

$$\leq|\tilde{U}_2|^2+\frac{|c+1|+|c-1|}{|c+1|c_1}|(\hat{\beta}_1+i\hat{\beta}_2)\tilde{u}|^2 .$$

Also, by (3.2), (4.3), (4.4), $J(\eta')=\sum_{j=2}^{n}\zeta_j^2$ and (4.13), we have

$$\text{(4.15)}\quad \sum_{j=2}^{n}|\tilde{Q}_{j+1}\tilde{u}|^2=\left\{d\left(t,x',\frac{\partial v}{\partial x'}\right)^2-|\tilde{Q}_2 v|^2\right\}+\left\{d\left(t,x',\frac{\partial w}{\partial x'}\right)^2-|\tilde{Q}_2 w|^2\right\}$$

where

$$\text{(4.16)}\quad \begin{cases} v=\mathrm{Re}\,\tilde{u}\quad w=\mathrm{Im}\,\tilde{u}\,,\quad \tilde{Q}_2=\displaystyle\sum_{j=2}^{n}q_j(t,0,x')\frac{\partial}{\partial x_j} \\ \dfrac{\partial v}{\partial x'}=\left(\dfrac{\partial v}{\partial x_2},\ \cdots,\ \dfrac{\partial v}{\partial x_n}\right) \end{cases}$$

and etc. Therefore, by (4.7), (4.12), (4.14) and (4.15), we obtain

$$\text{(4.17)}\quad I\geq\frac{|c+1|-|c-1|}{|c+1|}\left\{d\left(t,x',\frac{\partial v}{\partial x'}\right)^2-|\tilde{Q}_2 v|^2+d\left(t,x',\frac{\partial w}{\partial x'}\right)^2-|\tilde{Q}_2 w|^2\right\}$$

$$-\frac{|c+1|+|c-1|}{|c+1|c_1}|(\hat{\beta}_1+i\hat{\beta}_2)(v+iw)|^2$$

$$=\frac{|c+1|+|c-1|}{4|c+1|c_1}\left[\left\{(|c+1|-|c-1|)^2\cdot\left(d\left(t,x',\frac{\partial v}{\partial x'}\right)^2-|\tilde{Q}_2 v|^2\right)\right.\right.$$

$$\left.-4(|\hat{\beta}_1 v|^2+|\hat{\beta}_2 v|^2)\right\}+\left\{(|c+1|-|c-1|)^2\left(d\left(t,x',\frac{\partial w}{\partial x'}\right)^2-|\tilde{Q}_2 w|^2\right)\right.$$

$$\left.\left.-4(|\hat{\beta}_1 w|^2+|\hat{\beta}_2 w|^2)\right\}-8(\hat{\beta}_2 v\cdot\hat{\beta}_1 w-\hat{\beta}_1 v\cdot\hat{\beta}_2 w)\right].$$

Then, by (2.7), (4.4), (4.9), (4.16) and (4.17), we get

$$\text{(4.18)}\quad I\geq-2\left\{\frac{|c+1|+|c-1|}{|c+1|c_1}\right\}(\hat{\beta}_2 v\cdot\hat{\beta}_1 w-\hat{\beta}_1 v\cdot\hat{\beta}_2 w)$$

for $c_1(t,x')>0$. By (A.II′), (3.25), (4.9) and (4.18), we obtain

$$\text{(4.19)}\quad I\geq-2\left\{\frac{|c+1|+|c-1|}{|c+1|}\right\}\{\hat{\hat{\beta}}_2 v\cdot\hat{\beta}_1 w-\hat{\beta}_1 v\cdot\hat{\hat{\beta}}_2 w\}$$

for $c_1(t,x')>0$ and $c_1(t,x')=0$ where $\hat{\hat{\beta}}_2=c_1^{-1}\cdot\hat{\beta}_2$. By (A.I), (4.19), $v,w\in$

$\mathscr{E}_t^0(H_{3/2}(\boldsymbol{R}_{x'}^{n-1}))\cap\mathscr{E}_t^1(H_{1/2}(\boldsymbol{R}_{x'}^{n-1}))$ and the fact that $C_0^\infty(\boldsymbol{R}_{x'}^{n-1})$ is dense in $H_{3/2}(\boldsymbol{R}_{x'}^{n-1})$, we get

$$(4.20)\qquad \langle A_1|_{x_1=0}\tilde{U}, \tilde{U}\rangle=\int_{\boldsymbol{R}^{n-1}}\sqrt{a_{11}(t, 0, x')}I dx'$$
$$\geq\int_{\boldsymbol{R}^{n-1}}\{k[\hat{\beta}_1, \hat{\beta}_2]v\cdot w+(\hat{\beta}_1 k\cdot\hat{\beta}_2 v-\hat{\beta}_2 k\cdot\hat{\beta}_1 v)w\}dx'$$

where

$$(4.21)\qquad k(t, k')=\frac{2\sqrt{a_{11}(t, 0, x')}\{|c(t, x')+1|+|c(t, x')-1|\}}{|c(t, x')+1|}.$$

Therefore, by Lemma 3.1, (3.30), (3.31) and (4.20), we get Main Theorem 2.

Q.E.D.

REMARK 9. To obtain the energy inequality of higher order, we have to estimate U_{x_1} for U in (4.6). Operating the differential operator Q_{j+1} $(j=2, \cdots, n)$ for $U_1=Q_0 u-Q_1 u-izQ_2 u$ where $U={}^t(U_1, U_2, \cdots, U_{n+2})$ in (4.6), we can obtain the similar result as Corollary of Theorem 5.1 in [7].

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