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Geometric Approach to Rigidity of Horocycles

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1. Introduction.

Horocycles are defined as submanifolds associated with the geodesic flow on the unit tangent bundle of a two-dimensional compact connected Riemannian manifold with negative curvature. The geodesic flow and the horocycles themselves are interesting subjects of investigations in the theory of dynamical systems. Our interest in this paper is to characterize a manifold with negative curvature via horocycles and we call this characterization rigidity of horocycles.

Let N be a two-dimensional compact connected orientable Riemannian manifold with variable negative curvature, and let M be its unit tangent bundle. The expanding horocycles on M are obtained associated with the geodesic flow g. Let N' and M' be defined similarly. The result is the following:

THEOREM A. *If a homeomorphism $\varphi : M \to M'$ maps every expanding horocycle on M onto an expanding horocycle on M' and preserves their orientations, then N and N' are homothetic.*

In this theorem to be homothetic means that there exists a diffeomorphism from N to N' and the difference between the metric on N' and the image of the metric on N by this diffeomorphism is a constant multiple. The orientation of expanding horocycles is defined in §2.

From a point of view of characterization of a manifold with negative curvature via horocycle flows, *i.e.*, rigidity of horocycle flows, there exists a typical result by M. Ratner [R]. Let N_c, N'_c be complete 2-dimensional Riemannian manifolds with constant negative curvature, and let M_c, M'_c be their unit tangent bundles, respectively. If there is a measurable isomorphism between horocycle flows on M_c and M'_c, then N_c and N'_c are isometric. This result provided the motivation of this work.

In this paper we give the proof of the following theorem.

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THEOREM B. *If a homeomorphism $\varphi : M \to M'$ maps every expanding horocycle on M onto an expanding horocycle on M' and preserves their orientations, then there exists a constant $c \in \boldsymbol{R}$ and a homeomorphism $\psi : M \to M'$ such that*

$$\psi \circ g_r(x) = g'_{cr} \circ \psi(x) \qquad \textit{for all} \quad x \in M .$$

In §2 we state properties of Anosov flows, and then give the proof of Theorem B in §3.

We may deduce Theorem A from the following result by J. P. Otal and Theorem B.

THEOREM C ([O]). *If a homeomorphism $\psi : M \to M'$ satisfies $\psi \circ g_r = g'_{cr} \circ \psi$ for some constant c, then N and N' are homothetic. In particular, if $c = 1$ then N is isometric to N'.*

This result is contained in the latter part of the proof of Otal's spectral rigidity theorem.

The rigidity theorem proved by Ratner is extended to the case of variable negative curvature by J. Feldman and D. Ornstein [FO] in the following manner. The expanding horocycle flow on M is defined by using the parameterization with respect to Margulis' measure [M]. If there exists a measurable isomorphism ϕ between expanding horocycle flows $\bar{h}^-$ on M and $\bar{h}^{-\prime}$ on M', then the map ϕ is almost everywhere of the form $\phi = \bar{h}_\sigma^{-\prime} \phi_0$, where σ is some constant and ϕ_0 is a homeomorphism from M to M' and carries g_r to g'_r. We may deduce the following theorem from the assertion above and Theorem C.

THEOREM D ([FO]), [O]). *Suppose there exists a measurable isomorphism between expanding horocycle flows $\bar{h}^-$ on M and $\bar{h}^{-\prime}$ on M'. Then N and N' are isometric.*

The author is grateful for obtaining information on the geometric background in §2 and the existence of Theorem C from M. Kanai.

2. Preliminaries.

2.1. Anosov background. Let N be a two-dimensional compact connected orientable Riemannian manifold with variable negative curvature, and let M be its unit tangent bundle. Since N has negative curvature, the geodesic flow g_r defined on M is an Anosov flow; that is, it satisfies the following conditions:

(A) g_r is a differentiable flow without fixed points,

(B) there is a g_r-invariant splitting of the tangent bundle $TM = E^- \oplus E^0 \oplus E^+$,

(C) E^0 is the line bundle tangent to the orbits of the flow g_r,

(D) E^- and E^+ satisfy:
there exist positive constants λ, μ such that for $r > 0$

$$|dg_{-r}X| \leq \lambda |X| e^{-\mu r} \quad \text{if} \quad X \in E^- ,$$

$$|dg_r Y| \leq \lambda |Y| e^{-\mu r} \quad \text{if} \quad Y \in E^+ .$$

We remark that the constants λ and μ can be chosen independent of the point.

From the general theory of Anosov flow, the following four submanifolds are defined associated with the geodesic flow g_r:

the unstable manifold at a point $x \in M$

$$W^-(x) = \left\{ y \in M \,\middle|\, \lim_{r \to \infty} d(g_{-r}x, g_{-r}y) = 0 \right\},$$

the stable manifold at a point $x \in M$

$$W^+(x) = \left\{ y \in M \,\middle|\, \lim_{r \to \infty} d(g_r x, g_r y) = 0 \right\},$$

the weak unstable manifold at a point $x \in M$

$$W^{-0}(x) = \bigcup_{r \in \boldsymbol{R}} W^-(g_r x),$$

the weak stable manifold at a point $x \in M$

$$W^{+0}(x) = \bigcup_{r \in \boldsymbol{R}} W^+(g_r x),$$

where d is the distance measured by the Riemannian metric. The subbundle E^- is completely integrable and the corresponding maximal integral submanifold passing through x coincides with the unstable manifold $W^-(x)$. Moreover, the unstable manifolds $W^-(x)$ form a foliation on M denoted by W^-. Precisely, W^- is a partition of M into submanifolds $\{W^-(x)\}$ such that

(E) for any $x \in M$, $W^-(x)$ is a complete connected submanifold whose tangent field is E_y^-, $y \in W^-(x)$,

(F) for any $x \in M$, there exist a neighborhood U of x and a continuous map $J^- : U \to C^r(S, M)$ such that for each $y \in U$, $J^-(y)$ embeds S in $W^-(y)$ and $J^-(y)(0) = y$, where $S = \{x \in \boldsymbol{R} : |x| \leq 1\}$, $C^r(S, M)$ is the set of all C^r-maps from S to M equipped with the C^r-topology, and the value of $r > 0$ depends on the differentiability of the Anosov flow g. Moreover, each $W^-(x)$ is an immersed copy of $\boldsymbol{R}$.

In condition (E) we say that a submanifold $W^-(x)$ is complete if it is complete in the induced Riemannian metric on $W^-(x)$.

The existence of the continuous map J^- in (F) implies that the tangent fields of the foliation W^- are continuous and the leaf passing through $x \in M$ depends continuously on the point x. For simplicity, we call this property the continuity of the foliation W^-. In the same way we can define the foliations W^+, W^{-0} and W^{+0} in M which consist of the stable manifolds $W^+(x)$, the weak unstable manifolds $W^{-0}(x)$ and the weak stable manifolds $W^{+0}(x)$, respectively. In this paper we call the unstable manifold $W^-(x)$ the expanding horocycle or the h^--leaf and the stable manifold $W^+(x)$ the contracting horocycle or the h^+-leaf.

By using an orientation of E^-, we can define an orientation of expanding horocycles, that is, let $X: M \to TM$ be a nondegenerate section of E^- and then the section X orients each expanding horocycle. Moreover, we can choose the immersion $J^-(x)$: $\boldsymbol{R} \to W^-(x)$, $x \in M$ such that $(\dot{J}^-(x)(0)) \cdot (X(x)) > 0$ where $\dot{J}^-(x)(0) = d/dr|_{r=0}(J^-(x)(r))$. Then the orientation of $W^-(x)$ corresponds with the one of the real line $\boldsymbol{R}$. It is clear that we can give an order to points on $W^-(x)$.

From the definition we immediately have that the foliation W^- is g_r-invariant, *i.e.*, $g_r(W^-(x)) = W^-(g_r x)$ for all $x \in M$, $r \in \boldsymbol{R}$. By the condition (D) we obtain that the foliation W^- is expanding and the foliation W^+ is contracting with respect to the geodesic flow g_r, that is,

$$d^-(g_{-r}x_1, g_{-r}x_2) \le \lambda e^{-\mu r} d^-(x_1, x_2) \qquad \text{for any } x \in M,\ x_1, x_2 \in W^-(x),\ r \ge 0\,,$$

$$d^+(g_r y_1, g_r y_2) \le \lambda e^{-\mu r} d^+(y_1, y_2) \qquad \text{for any } y \in M,\ y_1, y_2 \in W^+(y),\ r \ge 0\,,$$

where d^- and d^+ are the distances measured along the leaves of W^- and W^+ by the induced Riemannian metric on each of them, respectively.

We define neighborhoods of $x \in M$ in the unstable manifold and the weak stable manifold, respectively, in the following way:

$$B_\varepsilon^-(x) = \{ y \in W^-(x) \mid d_x^-(x, y) < \varepsilon \}\,,$$

$$B_\varepsilon^{+0}(x) = \{ y \in W^{+0}(x) \mid d_x^{+0}(x, y) < \varepsilon \}\,,$$

where d_x^- and d_x^{+0} are the distances measured by the induced Riemannian metrics on $W^-(x)$ and $W^{+0}(x)$, respectively. Since the subbundles of the Anosov splitting $TM = E^- \oplus E^0 \oplus E^+$ are transverse, we can introduce the following canonical coordinates [M].

LEMMA 2.1. *For sufficiently small $\eta > 0$, there exists $\zeta = \zeta(\eta)$, $0 < \zeta < \eta$ such that if $d(x, y) < 2\zeta$ for $x, y \in M$ then $In(x, y) \equiv B_\eta^-(x) \cap B_\eta^{+0}(y)$ consists of one point. Moreover, $In(\cdot, \cdot)$ is continuous on $\{(x, y) \in M \times M : d(x, y) < 2\zeta\}$.*

The points x and y which satisfy the condition in Lemma 2.1 are connected by the segments of the geodesic, the h^--leaf $W^-(x)$ and the h^+-leaf $W^+(y)$. Let H^- and H^+ be nondegenerate sections of E^- and E^+, respectively. The section H^- defines the 1-parameter group of transformations h^- of W^-. (For the definition of a 1-parameter group of transformations, we refer the reader to the chapter I of [KN].) Similarly we obtain the 1-parameter group of transformations h^+ of W^+ from the section H^+. Then, by Lemma 2.1, we can choose continuous functions $r(x, y)$, $s(x, y)$ and $t(x, y)$, and by using them we can describe $y = h^+_{t(x,y)} g_{r(x,y)} h^-_{s(x,y)} x$ where we set $h^-_{s(x,y)} x = In(x, y)$.

The following is a well-known property of the foliations associated with the geodesic flow defined on the unit tangent bundle of a compact surface with negative curvature. (See Theorem 15 in [A].)

THEOREM 2.1. *Each leaf of the foliations W^- and W^+ is dense in M.*

2.2. Geometric background. The arguments described below in which we introduce an affine connection on M are due to §2 in [K]. It is well-known that the unit tangent bundle of a Riemannian manifold is a contact manifold. Then we can choose the canonical contact form θ of the unit tangent bundle M and state the relations between the contact form θ and the Anosov splitting $TM=E^-\oplus E^0\oplus E^+$ which are necessary for the following arguments.

(a) $d\theta(V_1^-, V_2^-)=d\theta(V_1^+, V_2^+)=0$ for all $V_1^-, V_2^-\in E^-$ and $V_1^+, V_2^+\in E^+$,

(b) $d\theta$ is nondegenerate on $E^-\oplus E^+$.

By using the contact form θ we can define the pseudo-Riemannian metric $\boldsymbol{g}$ on M: for $X, Y\in TM$

$$\boldsymbol{g}(X, Y)=d\theta(X, IY)+\theta(X)\theta(Y),$$

where I is the continuous map defined by $I|E^\pm=\pm id$ and $I|E^0=0$.

We need that the splitting of the unit tangent bundle $TM=E^-\oplus E^0\oplus E^+$ is C^1-differentiable in order to define an affine connection on M by using this pseudo-Riemannian metric $\boldsymbol{g}$. It is shown that if N is of dimension two, then the splitting of TM is C^1-differentiable. For detailed arguments of this fact we refer the reader to [HP]. Therefore in our context we can define a unique affine connection ∇ on M satisfying the following conditions:

(i) $\nabla\boldsymbol{g}=0$,

(ii) $\boldsymbol{T}(X, Y)=d\theta(X, Y)G$ for $X, Y\in TM$,

where $\boldsymbol{T}$ denotes the torsion tensor of ∇ and $G=d/dr|_{r=0}g_r$. We state some properties of the affine connection ∇:

(c) $\nabla_G X=[G, X]$ for any vector field X on M,

(d) the vector field $\nabla_X V^-$ and $\nabla_X V^+$ are sections of E^- and E^+, respectively, for any vector fields X, any sections V^- of E^- and V^+ of E^+.

We describe the property of the foliations associated with the geodesic flow by using the vector fields generating them.

LEMMA 2.2. *If V^+ is a section of E^+ such that $[V^-, V^+]\equiv 0$ for any section V^- of E^-, then V^+ has to vanish everywhere.*

PROOF. By the condition (ii) which the affine connection satisfies and the definition of the torsion tensor, we obtain the equation:

$$\nabla_{V^-}V^+ - \nabla_{V^+}V^- - [V^-, V^+] = T(V^-, V^+) = d\theta(V^-, V^+)G .$$

Then $d\theta(V^-, V^+)G=0$ from the assumption and the property (d). Since $d\theta$ is nondegenerate on $E^- \oplus E^+$, V^+ has to vanish everywhere. □

Let X and Y be vector fields on M which generate local 1-parameter groups of local transformations x_s and y_t of M, respectively. (See [KN].) If $x_{s_0} \circ y_{t_0} = y_{t_0} \circ x_{s_0}$ for some $s_0, t_0 \in \boldsymbol{R}$, then we say that x_s commutes with y_t. Let $\hat{h}_s^-$ and $\hat{h}_t^+$ denote local 1-parameter groups of local transformations generated by sections $\hat{H}^-$ of E^- and $\hat{H}^+$ of E^+, respectively. Then, by the calculation in the proof of the lemma described above and Corollary 1.8 in [KN], $\hat{h}_s^-$ does not commute with $\hat{h}_t^+$ when $|s|$ and $|t|$ are sufficiently small.

3. Proofs.

We proceed with the proof of Theorem B under the assumption that the homeomorphism φ maps every expanding horocycle on M onto one on M' and preserves their orientations.

Let $h^{-\prime}$ and $h^{+\prime}$ be 1-parameter groups of transformations of W^- and W^+ generated by nondegenerate sections $H^{-\prime}$ and $H^{+\prime}$ of E^- and E^+ on M', respectively. We then have the following:

LEMMA 3.1. *There exist $\tilde{r}>0$ and continuous functions $o(r)$, $p(x, r)$ such that*

$$\varphi(g_r x) = g'_{o(r)} h^{-\prime}_{p(x,r)} \varphi(x) \qquad \textit{for all} \quad 0<r\le\tilde{r} \quad \textit{and} \quad x\in M .$$

PROOF. Let x be any point in M and let $\eta>0$ be a sufficiently small constant in Lemma 2.1. Take $\tilde{r}>0$ so that we can apply Lemma 2.1 to the images of x and $y=g_r x$, $0<r<\tilde{r}$ under the mapping φ. We can obtain such a constant $\tilde{r}$ because the spaces M and M' are compact and the mapping φ is continuous. In the following discussion we set $r\le\tilde{r}$. By the assumption the expanding horocycles through x and y, *i.e.*, $W^-(x)$ and $W^-(y)$ are mapped onto the expanding horocycles $W^-(\varphi(x))$ and $W^-(\varphi(y))$, respectively. If we set $z=g_r w$ for $w\in W^-(x)$, then $z\in W^-(y)$ since $W^-(y)= W^-(g_r x)=g_r W^-(x)$.

By the remark made above following Lemma 2.1 we obtain the unique representation $\varphi(z)=h^{+\prime}_{q(w,r)} g'_{o(w,r)} h^{-\prime}_{p(w,r)} \varphi(w)$ where we use $o(w, r)$, $p(w, r)$ and $q(w, r)$ instead of $r(\varphi(w), \varphi(z))$, $s(\varphi(w), \varphi(z))$ and $t(\varphi(w), \varphi(z))$ in the remark since the mapping φ is continuous and the point z depends on the point w and the parameter r continuously. In the representation above we introduce the following notations. Let $\tilde{h}^{+\prime}_{q(w,r)}$ denote the segment of $W^+(\varphi(z))$ between the points $\varphi(z)$ and $g'_{o(w,r)} h^{-\prime}_{p(w,r)} \varphi(w)$, and let $l^+(\tilde{h}^{+\prime}_{q(w,r)})$ denote the length of the segment $\tilde{h}^{+\prime}_{q(w,r)}$ measured by the induced Riemannian metric on each leaf of W^+.

Take any three points x_1, x_2, x_3 which lie on the h^--leaf $W^-(x)$ in order. We set

$y_1=g_rx_1$, $y_2=g_rx_2$, $y_3=g_rx_3$ for fixed r with $0<r<\tilde{r}$. By the same arguments as above, we have $y_1, y_2, y_3 \in W^-(y)$ and the unique representations

$$\varphi(y_1)=h^{+'}_{q(x_1,r)}g'_{o(x_1,r)}h^{-'}_{p(x_1,r)}\varphi(x_1)\,,$$

$$\varphi(y_2)=h^{+'}_{q(x_2,r)}g'_{o(x_2,r)}h^{-'}_{p(x_2,r)}\varphi(x_2)\,,$$

$$\varphi(y_3)=h^{+'}_{q(x_3,r)}g'_{o(x_3,r)}h^{-'}_{p(x_3,r)}\varphi(x_3)\,.$$

We assert that if $l^+(\tilde{h}^{+'}_{q(x_1,r)})=l^+(\tilde{h}^{+'}_{q(x_2,r)})=l^+(\tilde{h}^{+'}_{q(x_3,r)})=0$ is not satisfied, either $o(x_1,r)<o(x_2,r)<o(x_3,r)$ or $o(x_1,r)>o(x_2,r)>o(x_3,r)$ must be obtained. This is shown in the following way. Consider the case $o(x_2,r)<o(x_1,r)<o(x_3,r)$. Since r is fixed, $o(x_i,r)$, $i=1,2,3$ change continuously along the expanding horocycle $W^-(x)$, *i.e.*, $o(w,r)$ is continuous with respect to $w\in W^-(x)$, so there exists the point $x_4\in W^-(x)$ between x_2 and x_3 with $o(x_1,r)=o(x_4,r)$. Setting $y_4=g_rx_4$, we have the following unique representation,

$$\varphi(y_4)=h^{+'}_{q(x_4,r)}g'_{o(x_4,r)}h^{-'}_{p(x_4,r)}\varphi(x_4)\,.$$

Since $o(x_1,r)=o(x_4,r)$ and the geodesic flow g preserves the foliation W^- and since the points $h^{-'}_{p(x_1,r)}\varphi(x_1)$ and $h^{-'}_{p(x_4,r)}\varphi(x_4)$ belong to the h^--leaf $W^-(\varphi(x))$, if $l^+(\tilde{h}^{+'}_{q(x_1,r)})\neq 0$, there would be a commutation relation among the h^--leaf and the h^+-leaf through $g'_{o(x_1,r)}h^{-'}_{p(x_1,r)}\varphi(x_1)$ and the h^--leaf and the h^+-leaf through $\varphi(y_4)$. This contradicts the remark made above following Lemma 2.2. By using the same arguments we can prove the assertion in the other cases. Now we obtain that for fixed r if $l^+(\tilde{h}^{+'}_{q(x_i,r)})=0$, $x_i\in W^-(x)$, $i=1,2,3$ is not satisfied, then $o(w,r)$, $w\in W^-(x)$ is monotone along $W^-(x)$.

On the other hand $o(x,r)$ for fixed r is a continuous function of x, so it always has maximum and minimum on the compact space M. This is inconsistent with the result in the preceding paragraph if $l^+(\tilde{h}^{+'}_{q(x_i,r)})\neq 0$, $x_i\in W^-(x)$, $i=1,2,3$. Therefore we obtain $l^+(\tilde{h}^{+'}_{q(w,r)})=0$ for any $w\in W^-(x)$.

As the foliation W^- is g-invariant, the value of $o(x,r)$ is constant on the h^--leaf $W^-(x)$, *i.e.*, $o(x,r)=o(w,r)$ for all $w\in W^-(x)$. Since each h^--leaf is dense in M and $o(x,r)$ is continuous in x, $o(x,r)$ is constant for all $x\in M$ and fixed r, that is, $o(x,r)$ depends only on r. The same arguments as above are valid for all $r\in \boldsymbol{R}$ with $0<r\leq\tilde{r}$. Finally, we have $\varphi(g_rx)=g'_{o(r)}h^{-'}_{p(x,r)}\varphi(x)$ for $x\in M$ and $0<r\leq\tilde{r}$. □

LEMMA 3.2. *The function o is a linear function of r and its domain can be extended to* $\boldsymbol{R}$, *i.e., there exists a constant* $c>0$ *such that*

$$\varphi(g_rx)=g'_{cr}\hat{h}^r_x(\varphi(x))\qquad \textit{for all } x\in M \textit{ and } r\in\boldsymbol{R}\,,$$

where $\hat{h}^r_x$ *is a transformation defined on each expanding horocycle and depends on x and r.*

PROOF. At first we show the linearity of r with $0<r<\tilde{r}$. Take $x\in M$ and r_0 with $0<r_0<\tilde{r}$ and map the three points x, $g_{r_0/2}x$ and $g_{r_0}x$ by φ. There exist the h^--leaves $W^-(x)$, $W^-(g_{r_0/2}x)$ and $W^-(g_{r_0}x)$ on M passing through x, $g_{r_0/2}x$ and $g_{r_0}x$, respectively. By Lemma 3.1 we have the following relation between $\varphi(x)$ and $\varphi(g_{r_0/2}x)$:

$$\varphi(g_{r_0/2}x)=g'_{o(r_0/2)}h^{-'}_{p(x,r_0/2)}\varphi(x)\,, \tag{3.1}$$

and the one between $\varphi(g_{r_0/2}x)$ and $\varphi(g_{r_0}x)$:

$$\varphi(g_{r_0/2}(g_{r_0/2}x))=g'_{o(r_0/2)}h^{-'}_{p(g_{r_0/2}x,r_0/2)}\varphi(g_{r_0/2}x)\,. \tag{3.2}$$

Since the mapping φ maps every h^--leaf on M onto an $h^{-'}$-leaf on M', the leaves $W^-(x)$, $W^-(g_{r_0/2}x)$ and $W^-(g_{r_0}x)$ are mapped by φ to the leaves $W^-(\varphi(x))$, $W^-(\varphi(g_{r_0/2}x))$ and $W^-(\varphi(g_{r_0}x))$, respectively. By the invariance of the foliation W^- in M' under the geodesic flow g' and the relations (3.1) and (3.2), we have the following equations:

$$g'_{o(r_0/2)}W^-(\varphi(x))=W^-(\varphi(g_{r_0/2}x))\,, \qquad g'_{o(r_0/2)}W^-(\varphi(g_{r_0/2}x))=W^-(\varphi(g_{r_0}x))\,. \tag{3.3}$$

On the other hand, from Lemma 3.1 we have the following relation between $\varphi(x)$ and $\varphi(g_{r_0}x)$:

$$\varphi(g_{r_0}x)=g'_{o(r_0)}h^{-'}_{p(x,r_0)}\varphi(x)\,.$$

Since the geodesic flow g' preserves the foliation W^- in M', we have

$$g'_{o(r_0)}W^-(\varphi(x))=W^-(\varphi(g_{r_0}x))\,. \tag{3.4}$$

From (3.3) and (3.4) we obtain the property of the function o described by $2o(r_0/2)=o(r_0)$. By the same arguments as above, we can establish the following property of the function o: for any r_0 with $0<r_0<\tilde{r}$ and $p\in N$

$$po\left(\frac{r_0}{p}\right)=o(r_0)\,. \tag{3.5}$$

Moreover, taking $q\in N$ which satisfies $0<(q/p)r_0<\tilde{r}$ and applying the relation (3.5) to $r'_0=(q/p)r_0$ and q, we have the relation $qo(r'_0/q)=o(r'_0)$. Therefore $o((q/p)r_0)=(q/p)o(r_0)$ for any r_0 with $0<r_0<\tilde{r}$ and $p, q\in N$ with $0<(q/p)r_0<\tilde{r}$. From this relation and the continuity of the function o with respect to r we may deduce that $o(rr_0)=r\,o(r_0)$ for $r\in \boldsymbol{R}$ with $0<rr_0<\tilde{r}$. In particular, setting $r_0=1/n$ and $o(1/n)=c$ for $n\in N$ with $0<1/n<\tilde{r}$, we obtain

$$o\left(\frac{r}{n}\right)=cr \qquad \text{for} \quad r\in \boldsymbol{R} \quad \text{with} \quad 0<\frac{r}{n}<\tilde{r}\,. \tag{3.6}$$

Next we extend the domain of $o(r)$ to $\boldsymbol{R}$. Take $x\in M$ and $r\in \boldsymbol{R}$ and map the two points x and $g_r x$ by φ. There is no loss of generality in assuming $r>0$. Indeed, if $r<0$ we can reverse the roles of x and $g_r x$. In order to use the relation (3.6), we decompose the parameter r as follows:

$$r=\frac{r_1}{n}+\frac{r_2}{n}+\cdots+\frac{r_l}{n}\,, \qquad 0<\frac{r_i}{n}<\tilde{r}\,, \quad i=1,\cdots,l\,.$$

According to the decomposition of the parameter r, the points $x_j=g_{\Sigma^j_{i=1}(r_i/n)}x$, $j=1,\cdots,l$

are obtained on the geodesic passing through x. We set $x_0=x$ and have the expanding horocycles $W^-(x_j)$ passing through x_j, $j=0, \cdots, l$ in M. From Lemma 3.1 and (3.6) we see that the following relation holds between $\varphi(x_j)$ and $\varphi(x_{j+1})$ which are the images of the points x_j and $x_{j+1}=g_{r_{j+1/n}}x_j$, $j=0, \cdots, l-1$ by φ:

$$\varphi(x_{j+1})=\varphi(g_{r_{j+1/n}}x_j)=g'_{cr_{j+1}}h^{-\prime}_{p(x_j,r_{j+1/n})}\varphi(x_j)\,. \tag{3.7}$$

Since the mapping φ maps every h^--leaf on M onto an h^--leaf on M', the leaves $W^-(x_j)$, $j=0, \cdots, l$ are mapped to $W^-(\varphi(x_j))$ by φ. By the invariance of the foliation W^- in M' under the geodesic flow g', (3.7) gives the relations

$$g'_{cr_{j+1}}W^-(\varphi(x_j))=W^-(\varphi(x_{j+1})) \qquad \text{for} \quad j=0, \cdots, l-1\,.$$

Let $\tilde{g}'$ denote the geodesic passing through $\varphi(g_r x)$. The intersection point $g'_{-cr_l}\varphi(g_r x)$ of $\tilde{g}'$ and $W^-(\varphi(x_{l-1}))$ is uniquely determined by the relation $W^-(\varphi(g_r x))=g'_{cr_l}W^-(\varphi(x_{l-1}))$ and by the way the parameter r was decomposed. The intersection point $g'_{-c(r_{l-1}+r_l)}\varphi(g_r x)$ of $\tilde{g}'$ and $W^-(\varphi(x_{l-2}))$ is also uniquely determined. By using the same argument many times, we have the intersection points $g'_{-c\Sigma_{i=j+1}^l r_i}\varphi(g_r x)$ of the geodesic $\tilde{g}'$ and the h^--leaves $W^-(\varphi(x_j))$, successively. At last we obtain the intersection point $g'_{-cr}\varphi(g_r x)$ of $\tilde{g}'$ and $W^-(\varphi(x))$ and define the transformation $\hat{h}^r_x$ by $\hat{h}^r_x : \varphi(x) \to g'_{-cr}\varphi(g_r x)$. Since the segments between $\varphi(x_j)$ and $h^{-\prime}_{p(x_j,r_{j+1/n})}\varphi(x_j)$, $j=0, \cdots, l-1$ in (3.7) are in the h^--leaf $W^-(\varphi(x_j))$, respectively, and the geodesic flow g' preserves the foliation W^- in M', the transformation $\hat{h}^r_x$ is a mapping on the expanding horocycle $W^-(\varphi(x))$. Finally, using $\hat{h}^r_x$ we obtain $\varphi(g_r x)=g'_{cr}\hat{h}^r_x(\varphi(x))$, and therefore, $o(r)=cr$ for all $r \in \boldsymbol{R}$. □

By the result of Lemma 3.2 we can define the mapping φ_r by $\varphi_r(x)=(g'_{-cr}\varphi g_r)(x)=\hat{h}^r_x(\varphi(x))$. It is clear that $\varphi_r : M \to M'$ is continuous by its construction. Our final aim is to define, using the mapping φ_r, a homeomorphism which maps the geodesic flow g on M to the geodesic flow g' on M' and changes the parameter of g' by the constant $c>0$ in Lemma 3.2.

LEMMA 3.3. *The continuous mapping*

$$\varphi_\infty(x)=\lim_{r\to\infty}\varphi_r(x) \qquad \textit{for all} \quad x \in M$$

is well-defined.

PROOF. Fix r_0 with $0<r_0<\tilde{r}$ for $\tilde{r}$ in Lemma 3.1 and get $\varphi_{r_0}(x)=\hat{h}^{r_0}_x\varphi(x)$. By the compactness of M' and the continuity of the mappings φ and φ_{r_0}, there exists a constant A such that

$$d^-(\varphi(x), \hat{h}^{r_0}_x\varphi(x))<A \qquad \text{for all} \quad x \in M\,.$$

We note that d^- denotes the distances measured along the leaves of W^- in M and M' by the induced Riemannian metrics. It will be clear from the context whether we use the distance d^- in M or M' in what follows. Fix x in M. Let $\{x_n\}$ be the sequence of points in M defined by $x_n = g_{nr_0}x$ and $\{x'_n\}$ be the sequence of points in M' defined by $x'_n = \varphi_{nr_0}(x) = \hat{h}_x^{nr_0}\varphi(x)$ for $n \in N$. Note that the points of the sequence $\{x'_n\}$ are on the expanding horocycle $W^-(\varphi(x))$. We will prove the existence of $\varphi_\infty(x)$ by showing the existence of the limit of the sequence of points $\{x'_n\}$ and by showing that $\lim_{r\to\infty}\varphi_r(x)$ coincides with the limit of $\{x'_n\}$.

As we observed in the proof of Lemma 3.2, the expanding horocycle $W^-(x_n)$ passing through x_n is mapped by φ to the expanding horocycle $W^-(\varphi(x_n))$ passing through $\varphi(x_n)$ and the relation $g'_{cr_0}W^-(\varphi(x_n)) = W^-(\varphi(x_{n+1}))$ holds. We remark that

$$\varphi(x_{n+1}) = g'_{cr_0}\hat{h}_{x_n}^{r_0}(\varphi(x_n)) \quad \text{and} \quad \varphi_{r_0}(x_n) = \hat{h}_{x_n}^{r_0}\varphi(x_n)\,. \tag{3.8}$$

Next we investigate the relation between x'_m and x'_n. There is no loss of generality in assuming $n > m$. Take the h^--leaf $W^-(\varphi(x_m))$ passing through $\varphi(x_m)$ and watch the intersection points $g'_{-cr_0}\varphi(x_{m+1}), g'_{-2cr_0}\varphi(x_{m+2}), \cdots, g'_{-(n-m)cr_0}\varphi(x_n)$ of the h^--leaf $W^-(\varphi(x_m))$ and the geodesics passing through $\varphi(x_{m+1}), \varphi(x_{m+2}), \cdots, \varphi(x_n)$, respectively. The relation (3.8) holds for all $\varphi(x_m), \varphi(x_{m+1}), \cdots, \varphi(x_n)$. Since there exists the estimation $d^-(\varphi(x_i), \varphi_{r_0}(x_i)) < A$ on each h^--leaf $W^-(\varphi(x_i))$, $i = m, \cdots, n-1$ and the foliation W^- in M' is expanding with respect to the geodesic flow on M', we can obtain the estimation on the h^--leaf $W^-(\varphi(x_m))$ that

$$\begin{aligned} d^-(\varphi(x_m), g'_{-(n-m)cr_0}\varphi(x_n)) &< A + A\lambda e^{-\mu cr_0} + \cdots + A\lambda e^{-(n-m-1)\mu cr_0} \\ &< A\left(1 + \frac{\lambda e^{-\mu cr_0}}{1 - e^{-\mu cr_0}}\right), \end{aligned}$$

where λ and μ are the constants coming from the definition of Anosov flow.

From the definition of the sequence of points $\{x'_n\}$ we have $x'_m = g'_{-mcr_0}\varphi(x_m)$ and $x'_n = g'_{-mcr_0}(g'_{-(n-m)cr_0}\varphi(x_n))$. By the expanding property of the foliation W^- in M' with respect to the geodesic flow on M' we obtain

$$\begin{aligned} d^-(x'_m, x'_n) &< \lambda e^{-m\mu cr_0} d^-(\varphi(x_m), g'_{-(n-m)\mu cr_0}\varphi(x_n)) \\ &< \lambda e^{-m\mu cr_0} A\left(1 + \frac{\lambda e^{-\mu cr_0}}{1 - e^{-\mu cr_0}}\right). \end{aligned}$$

Therefore for any $\varepsilon > 0$ if we take the constant N_0 such that

$$\frac{\lambda A\left(1 + \dfrac{\lambda e^{-\mu cr_0}}{1 - e^{-\mu cr_0}}\right)}{\varepsilon} < e^{\mu N_0 cr_0}\,,$$

then we have $d^-(x'_m, x'_n) < \varepsilon$ for $n \geq N_0$ and $m \geq N_0$. The sequence of points $\{x'_n\}$ is a Cauchy sequence and converges to some point of M'.

Next we will show that $\lim_{r\to\infty}\varphi_r(x)$ coincides with the limit of $\{x'_n\}$. Let x'_∞ be the limit of $\{x'_n\}$. Then, for a given $\varepsilon > 0$, let N_1 be such that if $n > N_1$ $d^-(x'_n, x'_\infty) < (\varepsilon/2)$. Set

$x_r = g_r x$ and $x'_r = \hat{h}^r_x \varphi(x)$ for $r \in \boldsymbol{R}$. We want to show that for the given ε there exists an integer N_2 such that if $r > N_2 r_0$ and $n > N_2$ then $d^-(x'_r, x'_n) < (\varepsilon/2)$ holds. By setting $N_3 = \max\{N_1, N_2\}$, we would then have $d^-(x'_r, x'_\infty) < \varepsilon$ if $r > N_3 r_0$.

We will find such an integer N_2 using the same arguments as in the proof of the fact that the sequence of points $\{x'_n\}$ is a Cauchy sequence. Decompose the parameter r such that $r = kr_0 + r_1$, $0 < r_1 < r_0$, $k \in \boldsymbol{N}$ and investigate the relation between x'_r and x'_n. There is no loss of generality in supposing that $k > n$. The relation (3.8) holds for all $\varphi(x_n)$, $\varphi(x_{n+1})$, $\cdots$, $\varphi(x_k)$. Using the same arguments and calculations as in the proof of the estimate for $d^-(\varphi(x_m), g'_{-(n-m)cr_0}\varphi(x_n))$, we have on the h^--leaf $W^-(\varphi(x_n))$

$$d^-(\varphi(x_n), g'_{-(k-n)cr_0 - cr_1}\varphi(x_r)) < A + A\lambda e^{-\mu c r_0} + \cdots + A\lambda e^{-(k-n)\mu c r_0}$$
$$< A\left(1 + \frac{\lambda e^{-\mu c r_0}}{1 - e^{-\mu c r_0}}\right).$$

From $x'_n = g'_{-ncr_0}\varphi(x_n)$, $x'_r = g'_{-ncr_0}(g'_{-(k-n)cr_0 - cr_1}\varphi(x_r))$ and the expanding property of the foliation W^- in M' with respect to the geodesic flow we get

$$d^-(x'_n, x'_r) < \lambda e^{-n\mu c r_0} d^-(\varphi(x_n), g'_{-(k-n)cr_0 - cr_1}\varphi(x_r))$$
$$< \lambda e^{-n\mu c r_0} A\left(1 + \frac{\lambda e^{-\mu c r_0}}{1 - e^{-\mu c r_0}}\right).$$

Therefore for the given $\varepsilon > 0$ if we take N_2 satisfying

$$\frac{2\lambda A\left(1 + \dfrac{\lambda e^{-\mu c r_0}}{1 - e^{-\mu c r_0}}\right)}{\varepsilon} < e^{\mu N_2 c r_0},$$

then $d^-(x'_r, x'_n) < \varepsilon/2$ for $r > N_2 r_0$ and $n > N_2$. We conclude that $\lim_{r\to\infty}\varphi_r(x)$ exists uniquely. Moreover, since an h^--leaf is a complete submanifold in M', $\lim_{r\to\infty}\varphi_r(x)$ lies in $W^-(\varphi(x))$.

Since the arguments by which we showed the existence of $\lim_{r\to\infty}\varphi_r(x)$ can be applied to any $x \in M$, we can define the mappings φ_∞ and $\hat{h}^\infty_x$ by

$$\varphi_\infty(x) = \hat{h}^\infty_x \varphi(x) = \lim_{r\to\infty} \varphi_r(x) \qquad \text{for all} \quad x \in M.$$

Finally we state that the mapping φ_∞ is continuous. Let $\{r_n\}$ be a monotonically increasing sequence with $r_n \to \infty$ $(n \to \infty)$ and consider a sequence of mappings $\{\varphi_{r_n}\}$. Since the way N_0, N_1, N_2 and N_3 were determined in the arguments which showed the existence of $\varphi_\infty(x)$ is independent of the point, the sequence of mappings $\{\varphi_{r_n}\}$ converges to φ_∞ in the sense that

$$\sup_{s \in M} d^-(\varphi_\infty(x), \varphi_{r_n}(x)) \to 0 \qquad (n \to \infty),$$

and therefore, we conclude that the mapping φ_∞ is continuous. $\square$

LEMMA 3.4. *The mapping φ_∞ is a homeomorphism which maps the geodesic flow g_r on M to the geodesic flow g'_{cr} on M', where $c>0$ is the constant in Lemma* 3.2.

PROOF. It is easy to see that the mapping φ_∞ maps the geodesic flow g_r on M to the geodesic flow g'_{cr} on M': for any $x\in M$ and $r\in \boldsymbol{R}$

$$\begin{aligned}\varphi_\infty(g_r x)&=\lim_{t\to\infty}(g'_{-ct}\circ\varphi\circ g_t)(g_r x)\\&=g'_{cr}\lim_{t\to\infty}(g'_{-c(t+r)}\circ\varphi\circ g_{t+r})(x)\\&=g'_{cr}\varphi_\infty(x)\,.\end{aligned}$$

Next we prove that the mapping φ_∞ is one-to-one. Take two points $x, y\in M$, $x\neq y$ and suppose that $\varphi_\infty(x)=\varphi_\infty(y)$. By the definition of the mapping φ_∞, we have $\varphi_\infty(x)=\hat{h}_x^\infty\varphi(x)$ and $\varphi_\infty(y)=\hat{h}_y^\infty\varphi(y)$. Since the mapping φ_∞ maps the geodesic flow g_r to the geodesic flow g'_{cr}, we obtain

$$\hat{h}^\infty_{g_r x}\varphi(g_r x)=\varphi_\infty(g_r x)=g'_{cr}\varphi_\infty(x)=\varphi_\infty(g_r y)=\hat{h}^\infty_{g_r y}\varphi(g_r y)\,. \tag{3.9}$$

In the equation above $\varphi(g_r x)\neq\varphi(g_r y)$ from the injectivity of the mapping φ. The calculations made in the proof of Lemma 3.3 show that the transformation $\hat{h}_x^\infty$ is bounded on the h^--leaf $W^-(\varphi(x))$. That is to say, there exists a constant $B>0$ such that

$$\sup_{x\in M} d^-(\varphi(x), \hat{h}_x^\infty\varphi(x))<B\,.$$

Therefore we obtain $d^-(\varphi(g_r x), \varphi(g_r y))<2B$ from (3.9). Since the mapping φ is a homeomorphism which maps every h^--leaf on M onto an h^--leaf on M' and preserves their orientations, for $g_r x$ and $g_r y$ there must be a constant $C>0$ such that $d^-(g_r x, g_r y)<C$ holds for every $r\in\boldsymbol{R}$. It contradicts the expanding property of the foliation W^- with respect to the geodesic flow. Therefore, the relation $\varphi_\infty(x)=\varphi_\infty(y)$ cannot hold, *i.e.*, the mapping φ_∞ is one-to-one.

The mapping φ_∞ is surjective. By the arguments of the construction of the mapping φ_∞, every point of $\varphi_\infty(M)$ is an interior point of $\varphi_\infty(M)$, that is, $\varphi_\infty(M)$ is open. Moreover, $\varphi_\infty(M)$ is closed from the continuity of φ_∞. It is clear that $\varphi_\infty(M)\neq\varnothing$ and M' is connected. Therefore $\varphi_\infty(M)=M'$.

As the mapping φ_∞ is one-to-one, onto and continuous, it is clear that it is a homeomorphism. $\square$

We have completed the proof of Theorem B from Lemma 3.1–3.4.

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