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On the Sum of Four Cubes and a Product of k Factors

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1. Introduction.

Page [10] [11] and Hooley [6] investigated the asymptotic behaviour of the number of representations of a natural number as a sum of squares and products of two positive factors, and established an asymptotic formula for the number of representations in each case where it can exist.

We consider here similar problems for cubes instead of squares. As is mentioned in Hooley [7, p. 180], an asymptotic formula for each case with at least five cubes and a product, or with at least two products and a cube is obtained by standard application of the circle method of Hardy and Littlewood. Indeed, let $k \geq 2$ be an integer and let $\nu(N)$ denote the number of representations of N as the sum of t products of k positive factors and s cubes. The number $\nu(N)$ is investigated in Waring's problem when $t=0$, and in the additive divisor problem when $s=0$. Here we consider the case $t, s \geq 1$. We introduce the functions

$$F(\alpha)=\sum_{m \leq N^{1/3}} e(m^3\alpha)\,, \qquad D_k(\alpha)=\sum_{n \leq N} d_k(n)e(n\alpha)\,, \tag{1.1}$$

where $e(\alpha)=\exp(2\pi i\alpha)$ and $d_k(n)$ denotes the number of ways of expressing n as a product of k positive factors. We have

$$\nu(N)=\int_0^1 D_k(\alpha)^t F(\alpha)^s e(-N\alpha)d\alpha\,.$$

We divide the unit interval [0, 1] into "major" and "minor" arcs. We don't give here definitions of major and minor arcs exactly, but we only indicate that major arcs, say $\mathfrak{M}$, is a union of narrow neighbourhoods of rational numbers with smaller denominators, and that $\mathfrak{M}$ is defined so that the integral

$$\int_{\mathfrak{M}} D_k(\alpha)^t F(\alpha)^s e(-N\alpha)d\alpha$$

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can be estimated asymptotically. And minor arcs, say $\mathfrak{m}$, is defined as the complement of $\mathfrak{M}$ to the unit interval. Therefore, establishing an asymptotic formula for $\nu(N)$ is reduced to giving a sufficient estimation for the integral on minor arcs

$$\int_{\mathfrak{m}} D_k(\alpha)^t F(\alpha)^s e(-N\alpha)d\alpha = \nu(N; \mathfrak{m}), \quad \text{say}.$$

For example, the estimate

$$\nu(N; \mathfrak{m}) = o(N^{t+s/3-1})$$

is sufficient.

When $t \geq 2$, we have

$$\nu(N; \mathfrak{m}) \ll \left(\max_{\alpha \in \mathfrak{m}} |F(\alpha)|\right)^s \left(\max_{\alpha \in \mathfrak{m}} |D_k(\alpha)|\right)^{t-2} \left(\int_0^1 |D_k(\alpha)|^2 d\alpha\right),$$

and then we can derive that $\nu(N; \mathfrak{m})$ is small enough by Weyl's inequality (see [14, Lemma 2.4]) and the well-known fact

$$\int_0^1 |D_k(\alpha)|^2 d\alpha = \sum_{n \leq N} d_k(n)^2 \ll N(\log N)^{k^2-1}. \tag{1.2}$$

Henceforth we consider only the case $t=1$. We have by Cauchy-Schwartz inequality

$$\nu(N; \mathfrak{m}) \ll \left(\int_0^1 |D_k(\alpha)|^2 d\alpha\right)^{1/2} \left(\int_{\mathfrak{m}} |F(\alpha)|^{2s} d\alpha\right)^{1/2} \tag{1.3}$$

and, in view of (1.2), the required bound for $\nu(N; \mathfrak{m})$ follows from

$$\int_{\mathfrak{m}} |F(\alpha)|^{2s} d\alpha \ll N^{2s/3-1}(\log N)^{-B} \tag{1.4}$$

with arbitrary fixed constant $B > 0$. Roughly speaking, to prove (1.4) is the same as to establish an asymptotic formula for the number of representations of a number as the sum of $2s$ cubes. When $2s > 8$, that is, $s \geq 5$, the bound (1.4) follows at once from Weyl's and Hua's inequality (see [14, Theorem 2.1]).

Next we set $s=4$. Suppose that

$$|\alpha - a/q| \leq q^{-2}, \qquad (q, a) = 1. \tag{1.5}$$

Then we have by a familiar argument (see [14, Lemma 2.2])

$$\begin{aligned} D_k(\alpha) &\ll \sum_{1 \leq N^{1-1/k}} d_{k-1}(l) \left| \sum_{m \leq N/l} e(lm\alpha) \right| \ll \sum_{l \leq N^{1-1/k}} \max\left(\frac{N}{l}, \|l\alpha\|^{-1}\right) N^{\varepsilon} \\ &\ll \left(\frac{N}{q} + N^{1-1/k} + q\right) N^{2\varepsilon} \end{aligned} \tag{1.6}$$

for any fixed $\varepsilon>0$, where $\|\beta\|$ denotes the distance between β and the nearest integer of it. When $s=4$, we can define our minor arcs $\mathfrak{m}$ so that one can find integers a and q satisfying (1.5) with

$$N^{4/9-\varepsilon}\leq q\leq N^{5/9+\varepsilon},$$

for any $\alpha\in\mathfrak{m}$. Thus it follows from (1.6) and Hua's inequality that

$$\nu(N;\mathfrak{m})\ll\max_{\alpha\in\mathfrak{m}}|D_k(\alpha)|\int_0^1|F(\alpha)|^4d\alpha\ll(N^{1-1/k}+N^{5/9})N^{2/3+\varepsilon} \tag{1.7}$$

with any fixed $\varepsilon>0$. Through this way, we obtain a sufficient estimate $\nu(N;\mathfrak{m})\ll N^{4/3}$ providing $k<3$, namely, only when $k=2$.

We have now observed that the circle method can establish an asymptotic formula for $\nu(N)$ when $s\geq 5$ or when $s=4$ and $k=2$. This limit was first got over by Hooley [7] in 1981. He published a new approach to problems in additive number theory, different from the circle method. He succeeded in obtaining an asymptotic formula for $\nu(N)$ when $s=4$, $k=3$ by his method, which is based on his decomposition [7, (33)] for the sum

$$\sum_{\substack{m_1^3+m_2^3+m_3^3+m_4^3\equiv N \pmod q\\ m_1^3+m_2^3+m_3^3+m_4^3<N}} 1\,.$$

By the method, he further showed in [8] that it is possible to prove an asymptotic formula for $\nu(N)$ when $s=3$, $k=2$.

In 1986, Vaughan [15] established an asymptotic formula for the number of representations of a natural number as the sum of eight positive cubes, in a frame of the circle method. He proved (1.4) for $s=4$, $B<2-4/\pi$ [15, Theorem B]. Here the constant B can not be large enough in (1.4), so we can not obtain directly a required bound for $\nu(N;\mathfrak{m})$ by (1.3) when $s=4$. We can, however, prove an asymptotic formula for $\nu(N)$ when $s=4$, $k\geq 3$, being based on Vaughan's ingenious treatment of the integral on minor arcs. To show this is the purpose of this paper.

Let $R_k(N)$ denote $\nu(N)$ for $s=4$, $k\geq 3$, or $R_k(N)$ be the number of representations of a natural number N as the sum of four positive cubes and a product of k positive factors;

$$N=l_1l_2\cdots l_k+m_1^3+m_2^3+m_3^3+m_4^3\,. \tag{1.8}$$

Hooley showed in [7]

$$R_3(N)=\frac{3}{8}\Gamma\left(\frac{4}{3}\right)^3\mathfrak{S}(N)N^{4/3}(\log N)^2+O(N^{4/3}(\log N)^{9/5}),$$

where Γ denotes the gamma function and $\mathfrak{S}(N)$ is what is called the singular series. (On the notation in section 5 below, $\mathfrak{S}(N)=2\mathfrak{S}_3^{(0)}(N)$.) Our result is

THEOREM 1. *For $k \geq 3$, we have*

$$R_k(N) = N^{4/3} \sum_{j=0}^{2} \xi_k^{(j)}(N)(\log N)^{k-1-j} + O(N^{4/3}(\log N)^{k-4}(\log\log N)^{C_k}),$$

where

$$C_k = \frac{1}{6} k(k-1)(k+4) + 3,$$

the implied constant depends only on k, and the coefficients $\xi_k^{(j)}(N)$ are defined explicitly in the section 5 (see (5.15) below). In particular, we have $\xi_k^{(j)}(N) \ll 1$ for all $0 \leq j \leq 2$, and $\xi_k^{(0)}(N) \gg 1$.

When we apply the circle method to our problem, we should first consider how to treat the integral for $R_k(N)$ on minor arcs. For $k \geq 3$, we can not give a satisfactory estimate for the integral on minor arcs directly, as yet. But if we pay attention to, instead of $R_k(N)$, the number of solutions of (1.8) such that $m_1 m_2 m_3$ has a prime divisor in the interval $[(\log N)^C, N^{1/21}]$ with an appropriate constant C, then we can estimate, by Vaughan's method, the corresponding integral on minor arcs satisfactorily through the way like (1.3). We should then handle the number, say $\widetilde{R_k}(N)$, of solutions of (1.8) such that $m_1 m_2 m_3$ has no prime divisor in $[(\log N)^C, N^{1/21}]$. It is inevitably hopeless to evaluate $\widetilde{R_k}(N)$ asymptotically, and it is all we can do with $\widetilde{R_k}(N)$ at present to give a sharp upper bound for $\widetilde{R_k}(N)$ (see (2.8) below) using Wolke's result [17].

This is the outline of our proof, and also shows the reason why our formula for $R_k(N)$ in Theorem 1 provides with only the first three main terms, though it is expected that

$$R_k(N) \sim N^{4/3} \sum_{j=0}^{k-1} \xi_k^{(j)}(N)(\log N)^{k-1-j}$$

with $\xi_k^{(j)}(N)$'s defined in the section 5 (see (5.14) below).

Here we relate briefly to a similar problem for h-th prowers with $h \geq 4$, instead of cubes. Let $\nu(N)$ be the number of representations of N as the sum of a product of k factors and s h-th powers. Using Hua's inequality, we can prove an asymptotic formula for $\nu(N)$

$$\text{for} \quad s \geq 2^{h-1} + 1, \qquad \text{and for} \quad s = 2^{h-1}, \ k < h,$$

by the way like (1.3), and (1.7), respectively. When $h \geq 6$, we can also prove an asymptotic formula for $\nu(N)$

$$\text{for} \quad s \geq 7 \cdot 2^{h-4} + 1,$$

through the way like (1.3), by using Heath-Brown's improvement of Hua's inequality [5, Theorem 2].

Our argument in this paper can show an asymptotic formula for $v(N)$

$$\text{for} \quad s=2^{h-1}, \qquad \text{and for} \quad h\geq 6,\ s=7\cdot 2^{h-4},$$

by combining with Vaughan's method [16] and Heath-Brown's method [5] (see also Boklan [1]).

When $h\geq 10$, we can improve the above restriction for s by Wooley's result [18] with the way like (1.3). In fact, an asymptotic formula for $v(N)$ is attainable for $s\geq s_0(h)$, where $s_0(h)$ is a certain function of h, and we only remark here $s_0(h)<7\cdot 2^{h-4}$ for $h\geq 10$, and $s_0(h)=(1+o(1))h^2\log h$ as $h\to\infty$, because the definition of $s_0(h)$ is pretty complicated (refer to Corollaries 1.2 and 1.2a in [18]).

We can apply our argument to another similar problem. As a direct consequence of the result due to Davenport [2], it follows that every sufficiently large number is representable as the sum of four positive cubes and a prime. We denote by $R_0(N)$ the number of such representations of a natural number N. In the same manner as for $R_k(N)$ with $k\geq 3$, we obtain an asymptotic formula for $R_0(N)$.

THEOREM 2. *We have*

$$R_0(N)=\Gamma\left(\frac{4}{3}\right)^3\mathfrak{S}_0(N)\int_2^N\frac{(N-t)^{1/3}}{\log t}dt+O(N^{4/3}(\log N)^{-4}(\log\log N)^4),$$

where $\mathfrak{S}_0(N)$ *is defined in the section* 5 (*see* (5.17) *below*). *In particular we have* $1\ll\mathfrak{S}_0(N)\ll 1$.

We note here that

$$\int_2^N\frac{(N-t)^{1/3}}{\log t}dt=\frac{3}{4}\frac{N^{4/3}}{\log N}+O\left(\frac{N^{4/3}}{(\log N)^2}\log\log N\right).$$

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2. Notation, definition and preliminaries.

Throughout this paper, N denotes a sufficiently large natural number, regarded as tending to infinity. The letter B is a fixed constant ≥ 5; k is a non-negative integer; and p stands for prime numbers. The letter ε is a fixed (small) positive constant. The constants implied by the symbols $\ll$ and O depend, at most, on k, B and ε.

We denote the Möbius function and Euler's totient function by $\mu(m)$ and $\varphi(m)$, respectively. We denote by $\rho_n(d)$ the number of solutions of the congruence

$$x^3\equiv n \pmod d,$$

with $1\leq x\leq d$. It is easily seen that

(2.1) $$\rho_n(p)=\begin{cases}1 & (p\not\equiv 1 \pmod 3, \text{ or } p\mid n),\\ 0 \text{ or } 3 & (p\equiv 1 \pmod 3) \text{ and } p\nmid n).\end{cases}$$

Let $\mathscr{I}$ be the set of all natural numbers $\le N^{1/3}$, and let

$$\mathscr{A}=\{m\in\mathscr{I};\ m \text{ has no prime factor } p \text{ such that } (\log N)^{4B_0}<p\le N^{1/21}\},$$

$$\bar{\mathscr{A}}=\mathscr{I}\backslash\mathscr{A},$$

where $B_0=B_0(k)=2B+k^2$. Making use of Selberg's upper bound sieve (see [4, Theorem 3.3]), we have

(2.2) $$\#\mathscr{A}\ll N^{1/3}\log\log N/\log N.$$

Here, the symbol # denotes the cardinality of the indicated set.

Besides the functions $F(\alpha)$ and $D_k(\alpha)$ introduced by (1.1), we use the function

$$F(\alpha;\mathscr{B})=\sum_{n\in\mathscr{B}} e(m^3\alpha),$$

for a subset $\mathscr{B}\subset\mathscr{I}$. In particular, $F(\alpha;\mathscr{I})=F(\alpha)$.

We put

$$r(m)=\sum_{\substack{m_1^3+m_2^3+m_3^3=m\\ m_1,m_2,m_3\in\mathscr{A}}} 1$$

so that

(2.3) $$F(\alpha;\mathscr{A})^3=\sum_{m\le 3N} r(m)e(m\alpha).$$

By (2.2), we have

(2.4) $$\sum_{m\le 3N} r(m)=F(0;\mathscr{A})^3=(\#\mathscr{A})^3\ll N(\log N)^{-3}(\log\log N)^3.$$

For convenience' sake, we define

$$d_0(n)=\begin{cases}1 & (\text{if } n \text{ is a prime}),\\ 0 & (\text{otherwise}),\end{cases}$$

$$D_0(\alpha)=\sum_{n\le N} d_0(n)e(n\alpha)=\sum_{p\le N} e(p\alpha).$$

Then, for $k\ge 0$, we see

$$R_k(N)=\sum_{n+m_1^3+m_2^3+m_3^3+m_4^3=N} d_k(n)=\int_0^1 D_k(\alpha)F(\alpha)^4e(-N\alpha)d\alpha.$$

Next, we put $B_1=B_1(k)=6B+3k^2$ and

$$Q_1=Q_1(k)=(\log N)^{2B_1}, \qquad Q_2=Q_2(k)=N(\log N)^{-B_1}.$$

For $1\le a\le q\le Q_1$, $(a,q)=1$, let

$$\mathfrak{M}_0(q,a)=\{\alpha;\ |\alpha-a/q|\le 1/Q_2\},$$

$$\mathfrak{M}_1(q,a)=\{\alpha;\ 1/Q_2<|\alpha-a/q|\le 1/(qN^{3/4})\},$$

and, for $Q_1<q\le N^{1/4}$, $1\le a\le q$, $(a,q)=1$, let

$$\mathfrak{M}_2(q,a)=\{\alpha;\ |\alpha-a/q|\le 1/(qN^{3/4})\}.$$

We should note that these families of arcs $\mathfrak{M}_j(q,a)$'s are pairwise disjoint, because if

$$q,q'\le N^{1/4}, \quad (q,a)=(q',a')=1, \quad a/q\ne a'/q',$$

then we see

$$\left|\frac{a}{q}-\frac{a'}{q'}\right|=\frac{|aq'-a'q|}{qq'}\ge\frac{1}{qq'}>\frac{q+q'}{qq'N^{3/4}}=\frac{1}{qN^{3/4}}+\frac{1}{q'N^{3/4}}.$$

Further, we set

$$\mathfrak{M}_j=\bigcup_{q\le Q_1}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}_j(q,a) \qquad (\text{for } j=0,1),$$

$$\mathfrak{M}_2=\bigcup_{Q_1<q\le N^{1/4}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}_2(q,a),$$

$$\mathfrak{m}=[N^{-3/4},1+N^{-3/4}]\backslash(\mathfrak{M}_0\cup\mathfrak{M}_1\cup\mathfrak{M}_2).$$

Then we have, for $k\ge 0$,

(2.5)
$$\begin{aligned}R_k(N)&=\int_{N^{-3/4}}^{1+N^{-3/4}}D_k(\alpha)F(\alpha)^4e(-N\alpha)d\alpha\\&=\int_{N^{-3/4}}^{1+N^{-3/4}}D_k(\alpha)F(\alpha)\{F(\alpha)^3-F(\alpha;\mathscr{A})^3\}e(-N\alpha)d\alpha\\&\quad+\int_{N^{-3/4}}^{1+N^{-3/4}}D_k(\alpha)F(\alpha)F(\alpha;\mathscr{A})^3e(-N\alpha)d\alpha\\&=I_k(\mathfrak{M}_0)+I_k(\mathfrak{M}_1)+I_k(\mathfrak{M}_2)+I_k(\mathfrak{m})+\widetilde{R_k}(N),\end{aligned}$$

where

$$\widetilde{R_k}(N)=\sum_{\substack{m_1,m_2,m_3\in\mathscr{A}\\ m_4<(N-m_1^3-m_2^3-m_3^3)^{1/3}}}d_k(N-m_1^3-m_2^3-m_3^3-m_4^3),$$

and

$$I_k(\mathfrak{N})=\int_{\mathfrak{N}} D_k(\alpha)F(\alpha)\{F(\alpha)^3-F(\alpha;\mathscr{A})^3\}e(-N\alpha)d\alpha$$

for $\mathfrak{N}=\mathfrak{M}_0$, $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{m}$.

We require

$$I_k(\mathfrak{m})\ll N^{4/3}(\log N)^{-B}, \tag{2.6}$$

$$I_k(\mathfrak{M}_j)\ll N^{4/3}(\log N)^{-B} \quad \text{for} \quad j=1,2, \tag{2.7}$$

$$\widetilde{R}_k(N)\ll N^{4/3}(\log N)^{k-4}(\log\log N)^{C'_k} \tag{2.8}$$

with $C'_0=7/2$ and $C'_k=k+2$ for $k\geq 3$. We also require that $I_k(\mathfrak{M}_0)$ equals the right-hand side of the asymptotic formula for $R_k(N)$ in our Theorems 1 and 2, namely,

$$I_k(\mathfrak{M}_0)=N^{4/3}\sum_{j=0}^{2}\xi_k^{(j)}(N)(\log N)^{k-1-j}+O(N^{4/3}(\log N)^{k-4}(\log\log N)^{C_k}) \tag{2.9}$$

for $k\geq 3$, and

$$I_0(\mathfrak{M}_0)=\Gamma\left(\frac{4}{3}\right)\mathfrak{S}_0(N)\int_2^N\frac{(N-t)^{1/3}}{\log t}dt+O(N^{4/3}(\log N)^{-4}(\log\log N)^4). \tag{2.10}$$

We shall prove (2.6) in the section 3, (2.7) in the section 4, (2.8) in the section 6, (2.9) and (2.10) in the section 5.

3. Estimation of $I_k(\mathfrak{m})$.

Our estimate for $I_k(\mathfrak{m})$ is entirely based on Vaughan's work [15] on Waring's problem for cubes. We have, for any $\mathscr{B}\subset\mathscr{I}$,

$$\int_{\mathfrak{m}}|F(\alpha)|^2|F(\alpha;\mathscr{A})|^2|F(\alpha;\mathscr{B})|^4d\alpha\ll N^{5/3}(\log N)^{-B_0} \tag{3.1}$$

by his method (see the estimation for "$I(\mathscr{C})$" and "$I(\mathscr{D})$" in [15, pp. 137–138]). We observe the outline of a proof for (3.1) here.

We put

$$G_d(\alpha)=\sum_{\substack{m_1,m_2\in\mathscr{A}\\(m_1,m_2)=d}}e((m_1^3-m_2^3)\alpha), \qquad G(\alpha)=\sum_{\substack{m_1,m_2\in\mathscr{A}\\(m_1,m_2)\leq Q_0}}e((m_1^3-m_2^3)\alpha),$$

so that

$$|F(\alpha;\mathscr{A})|^2=\sum_{Q_0<d\leq N^{1/3}}G_d(\alpha)+G(\alpha).$$

Then

$$\int_{\mathfrak{m}} |F(\alpha)|^2 |F(\alpha; \bar{\mathscr{A}})|^2 |F(\alpha; \mathscr{B})|^4 d\alpha = I_1 + I_2 , \tag{3.2}$$

where

$$I_1 = \sum_{Q_0 < d \le N^{1/3}} \int_{\mathfrak{m}} |F(\alpha)|^2 G_d(\alpha) |F(\alpha; \mathscr{B})|^4 d\alpha ,$$

$$I_2 = \int_{\mathfrak{m}} |F(\alpha)|^2 G(\alpha) |F(\alpha; \mathscr{B})|^4 d\alpha .$$

Let $R(P)$ denote the number of solutions of

$$m_1^3 + m_2^3 + m_3^3 + m_4^3 = m_5^3 + m_6^3 + m_7^3 + m_8^3$$

with $m_j \le P$. Vaughan [15, Theorem 2] proved $R(P) \ll P^5$, which yields

$$\int_0^1 |G_d(\alpha)|^4 d\alpha \le R(N^{1/3}/d) \ll N^{5/3} d^{-5} , \qquad \text{and}$$

$$\int_0^1 |F(\alpha; \mathscr{B})|^8 d\alpha \le R(N^{1/3}) \ll N^{5/3} \qquad \text{for any} \quad \mathscr{B} \subset \mathscr{I} . \tag{3.3}$$

Thus we have

$$I_1 \ll \sum_{Q_0 < d \le N^{1/3}} \left(\int_0^1 |G_d(\alpha)|^4 d\alpha \right)^{1/4} \left(\int_0^1 |F(\alpha)|^8 d\alpha \right)^{1/4} \left(\int_0^1 |F(\alpha; \mathscr{B})|^8 d\alpha \right)^{1/2} \tag{3.4}$$

$$\ll N^{5/3} \sum_{Q_0 < d \le N^{1/3}} d^{-5/4} \ll N^{5/3} Q_0^{-1/4} .$$

Next, by [15, Lemma 1], we have $F(\alpha) \ll N^{1/4} (\log N)$ uniformly for $\alpha \in \mathfrak{m}$, and

$$I_2 \ll N^{1/4} (\log N) \left(\int_0^1 |G(\alpha)|^2 |F(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_0^1 |F(\alpha; \mathscr{B})|^8 d\alpha \right)^{1/2} \tag{3.5}$$

$$\ll N^{1/4} (\log N) V_0^{1/2} R(N^{1/3})^{1/2} ,$$

where V_0 denotes the number of solutions of $m_1^3 - m_2^3 + n_1^3 = m_3^3 - m_4^3 + n_2^3$ with

$$m_j \in \bar{\mathscr{A}}, \quad (m_1, m_2) \le Q_0 , \quad (m_3, m_4) \le Q_0 , \quad n_j \le N^{1/3} .$$

For these m_j's there exist primes p_1, p_2 such that

$$p_1 \nmid m_1 , \quad p_1 | m_2 , \quad p_2 \nmid m_3 , \quad p_2 | m_4 , \quad Q_0 < p_j \le N^{1/21}$$

by the definition of $\bar{\mathscr{A}}$. Therefore,

$$V_0 \le V \tag{3.6}$$

where V denotes the number of solutions of

$$x_1^3-p_1^3y_1^3+z_1^3=x_2^3-p_2^3y_2^3+z_2^3$$

with

$$x_j\le N^{1/3}\,,\quad Q_0<p_j\le N^{1/21}\,,\quad (x_j, p_j)=1\,,\quad y_j\le N^{1/3}/p_j\,,\quad z_j\le N^{1/3}\,.$$

Vaughan showed, in [15, p. 138],

$$V\ll N^{7/6}(\log N)^4 Q_0^{-3/4}\,. \tag{3.7}$$

So we have

$$I_2\ll N^{5/3}(\log N)^3 Q_0^{-3/8} \tag{3.8}$$

by (3.5), (3.6) and (3.7). Hence the bound (3.1) follows from (3.2), (3.4) and (3.8).

Now we can estimate $I_k(\mathfrak{m})$ easily. Since

$$\begin{aligned} F(\alpha)^3-F(\alpha;\mathscr{A})^3&=(F(\alpha)-F(\alpha;\mathscr{A}))(F(\alpha)^2+F(\alpha)F(\alpha;\mathscr{A})+F(\alpha;\mathscr{A})^2)\\ &\ll|F(\alpha;\bar{\mathscr{A}})|(|F(\alpha)|^2+|F(\alpha;\mathscr{A})|^2)\,, \end{aligned}$$

we have, using the Cauchy-Schwartz inequality, (3.1) and (1.2),

$$\begin{aligned} I_k(\mathfrak{m})&\ll\left(\int_0^1|D_k(\alpha)|^2d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}}|F(\alpha)|^2|F(\alpha;\bar{\mathscr{A}})|^2(|F(\alpha)|^4+|F(\alpha;\mathscr{A})|^4)d\alpha\right)^{1/2}\\ &\ll N^{4/3}(\log N)^{-B}\,. \end{aligned}$$

We have obtained (2.6) for $k\ge0$. (We should note that (1.2) is still true for $k=0$.)

4. Estimations of $I_k(\mathfrak{M}_1)$ and $I_k(\mathfrak{M}_2)$.

Essentially, it is not so difficult to treat $I_k(\mathfrak{M}_0)$, $I_k(\mathfrak{M}_1)$ and $I_k(\mathfrak{M}_2)$. In this section, we shall estimate $I_k(\mathfrak{M}_1)$ and $I_k(\mathfrak{M}_2)$. We start with summarizing known results on the function $F(\alpha)$.

Let

$$V(q,a)=\sum_{r=1}^{q}e\left(\frac{a}{q}r^3\right),\qquad V^*(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\frac{V(q,a)}{q}\right|^6,$$

$$\tilde{V}_j(q,h)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\frac{V(q,a)}{q}\right)^j e\left(-\frac{a}{q}h\right),$$

and let

$$v(\beta)=\frac{1}{3}\sum_{m\le N}m^{-2/3}e(m\beta)\,.$$

We note that $V^*(q)$ and $\tilde{V}_j(q, h)$ are multiplicative functions of q.

LEMMA 1. (i) *Assume that* $\alpha=(a/q)+\beta$, $(a, q)=1$ *and* $|\beta|\leq(6qN^{2/3})^{-1}$. *Then we have*

$$F(\alpha)=\frac{V(q, a)}{q}v(\beta)+O(q^{1/2+\varepsilon}).$$

(ii) *For* $|\beta|\leq 1/2$, *we have*

$$v(\beta)\ll\min(N^{1/3}, |\beta|^{-1/3}).$$

(iii) *For* $Q\geq 1$, *we have*

$$\sum_{q\leq Q} q^{1/2}V^*(q)\ll 1.$$

(iv) *For* $Q\geq 2$, $k\geq 1$ *and for any integer* h, *we have*

$$\sum_{q\leq Q}\frac{d_k(q)}{q}|\tilde{V}_1(q, h)|\ll(\log Q)^{C''_k},$$

where $C''_k=k(k+1)(k+5)/6$.

(v) *For* $Q\geq 2$ *and for any integer* h, *we have*

$$\sum_{q\leq Q}\frac{\mu(q)^2}{\varphi(q)}|\tilde{V}_1(q, h)|\ll\log Q.$$

(vi) *Suppose* $\delta>\varepsilon>0$ *and* $\Theta(q)\ll q^{-\delta}$. *Then the series* $\sum_{q=1}^{\infty}\Theta(q)\tilde{V}_4(q, N)$ *converges absolutely, and we have, for* $Q\geq 1$,

$$\sum_{q>Q}|\Theta(q)\tilde{V}_4(q, N)|\ll Q^{-\delta+\varepsilon}.$$

PROOF. As for (i) and (ii), see Theorem 4.1 and Lemma 2.8 in Vaughan's book [14].

We turn to (iii). For $l\geq 1$, we write $l=3u+v$ with integers u and $1\leq v\leq 3$. Then we have, as in [14, p. 50],

$$V^*(p^l)\ll\begin{cases}p^{-6u-3+l} & (v=1),\\ p^{-6u-6+l} & (v=2, 3),\end{cases}$$

$$\sum_{q\leq Q}q^{1/2}V^*(q)\leq\prod_{p\leq Q}\left(1+\sum_{l=1}^{\infty}p^{l/2}V^*(p^l)\right)\ll\prod_{p\leq Q}(1+O(p^{-3/2}))\ll 1.$$

We next prove (iv). According to Lemma 4.7 of [14], we have

$$\tilde{V}_4(p^l, h)\ll\begin{cases}p^{-u} & (\text{where } v=1 \text{ and } p\nmid h),\\ p^{-u-1} & (\text{when } v\neq 1 \text{ and } p\nmid h),\\ p^{-u-1/2+l} & (\text{when } v=1 \text{ and } p\mid h),\\ p^{-u-1+l} & (\text{when } v\neq 1 \text{ and } p\mid h),\end{cases}$$

for $l=3u+v$, $1\le v\le 3$. By simple calculation, we have also $\tilde{V}_1(p^l,h)=\rho_h(p^l)-\rho_h(p^{l-1})$, and

$$\tilde{V}_1(p,h)=\begin{cases}0 & (p\not\equiv 1 \ (\mathrm{mod}\,3), \text{ or } p\,|\,h)\,,\\ 2 \text{ or } -1 & (p\equiv 1 \ (\mathrm{mod}\,3) \text{ and } p\nmid h)\,,\end{cases}$$

and if $p\,|\,h$ then $\tilde{V}_1(p^v,h)\le p^{v-1}$ for $v=2, 3$.

Taking account of these results, we obtain

$$\begin{aligned}\sum_{q\le Q}\frac{d_k(q)}{q}|\tilde{V}_1(q,h)|&\le\prod_{p\le Q}\left(1+\sum_{l=1}^{\infty}\frac{d_k(p^l)}{p^l}|\tilde{V}_1(p^l,h)|\right)\\&\le\prod_{\substack{p\le Q\\ p\nmid h}}\left(1+\frac{2k}{p}+O(p^{-3+\varepsilon})\right)\prod_{\substack{p\le Q\\ p|h}}\left(1+\frac{C''_k}{p}+O(p^{-3/2+\varepsilon})\right)\\&\ll\prod_{p\le Q}\left(1+\frac{C''_k}{p}\right)\ll(\log Q)^{C''_k}\,.\end{aligned}$$

Similarly, we obtain (v) and

$$\sum_{q=1}^{\infty}q^{-\varepsilon}|\tilde{V}_4(q,N)|\ll\prod_{p\nmid N}(1+O(p^{-3/2}))\prod_{p|N}(1+O(p^{-1-\varepsilon}))\ll 1\,,$$

which gives (vi).

Now we proceed to estimate $I_k(\mathfrak{M}_1)$ and $I_k(\mathfrak{M}_2)$. Using Lemma 1 (i), (ii), (iii), we obtain

$$\begin{aligned}(4.1)\qquad\int_{\mathfrak{M}_1}|F(\alpha)|^6d\alpha&\ll\sum_{q\le Q_1}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{1/Q_2}^{q^{-1}N^{-3/4}}\left(\left|\frac{V(q,a)}{q}\right|^6|v(\beta)|^6+q^{3+6\varepsilon}\right)d\beta\\&\ll\sum_{q\le Q_1}V^*(q)\int_{1/Q_2}^{1/2}|\beta|^{-2}d\beta+N^{-3/4+\varepsilon}\ll Q_2=N(\log N)^{-B_1}\,,\end{aligned}$$

$$\begin{aligned}(4.2)\qquad\int_{\mathfrak{M}_2}|F(\alpha)|^6d\alpha&\ll\sum_{Q_1<q\le N^{1/4}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{0}^{q^{-1}N^{-3/4}}\left(\left|\frac{V(q,a)}{q}\right|^6|v(\beta)|^6+q^{3+6\varepsilon}\right)d\beta\\&\ll\frac{1}{\sqrt{Q_1}}\sum_{q\le N^{1/4}}\sqrt{q}\,V^*(q)\int_0^{1/2}\min(N^2,|\beta|^{-2})d\beta+N^{1/4+2\varepsilon}\\&\ll\frac{N}{\sqrt{Q_1}}=N(\log N)^{-B_1}\,.\end{aligned}$$

Combining (1.2), (3.3), (4.1), (4.2) with a trivial bound $|F(\alpha;\mathscr{B})|\ll N^{1/3}$ for $\mathscr{B}\subset\mathscr{I}$, we have for $j=1, 2$,

$$I_k(\mathfrak{M}_j) \ll \max_{\alpha\in[0,1]} (|F(\alpha)|+|F(\alpha;\mathscr{A})|)^{1/3}\left(\int_0^1 |D_k(\alpha)|^2 d\alpha\right)^{1/2}$$
$$\times\left(\int_{\mathfrak{M}_j} |F(\alpha)|^6 d\alpha\right)^{1/6}\left(\int_0^1 (|F(\alpha)|^8+|F(\alpha;\mathscr{A})|^8) d\alpha\right)^{1/3}$$
$$\ll N^{4/3}(\log N)^{-B}.$$

We have showed (2.7).

5. On the integral $I_k(\mathfrak{M}_0)$.

In order to evaluate $I_k(\mathfrak{M}_0)$, we need an appropriate approximation of $D_k(\alpha)$ for $\alpha\in\mathfrak{M}_0$. As for $k=0$, we know the following result.

LEMMA 2. *Let*

$$T_0(\beta;q)=\frac{\mu(q)}{\varphi(q)}\sum_{2\le n\le N}\frac{e(n\beta)}{\log n}.$$

(i) *Suppose that* $1\le a\le q\le Q_1$, $(a,q)=1$, $\alpha\in\mathfrak{M}_0(q,a)$ *and* $\alpha=(a/q)+\beta$. *Then we have*

$$D_0(\alpha)=T_0(\beta;q)+O(N\exp(-c_1\sqrt{\log N})),$$

where c_1 *is a positive constant depending only on* B.

(ii) *For* $|\beta|\le 1/2$, *we have*

$$T_0(\beta;q)\ll\frac{\mu(q)^2}{\varphi(q)}\min\left(\frac{N}{\log N},|\beta|^{-1}\right).$$

And for $|\beta|\le 1/\sqrt{N}$, *we have*

$$T_0(\beta;q)\ll\frac{\mu(q)^2}{\varphi(q)\log N}\min(N,|\beta|^{-1}).$$

For the proof of (i), see Prachar [13, VI, Satz 3.2, p. 180]. The inequalities in (ii) follow easily from the well-known estimate

$$\sum_{n\le x} e(n\beta)\ll\min(x,|\beta|^{-1}) \tag{5.1}$$

for $|\beta|\le 1/2$, by partial summation.

As for the case $k\ge 2$, Motohashi [9] showed a result adequate to our aim. Though he confined his attention within square free q's, his argument [9, Lemmas 5, 6 and p. 60] still works for all q's with slight differences, and establishes the following Lemma 3. Here we follow his way.

Before writing down our Lemma 3, we need some definitions. Let $s=\sigma+it$ be a complex variable with real σ and t, and $\zeta(s)$ be the Riemann zeta function, as usual. We define the numbers $\eta_k(h)$ by

$$(s-1)^k\zeta(s)^k s^{-1}=\sum_{h=0}^{\infty}\eta_k(h)(s-1)^h \qquad (\text{as } s\to 1)\,.$$

We set

$$\Phi_k(s;q)=\prod_{p\mid q}\left\{1-(1-p^{-s})^k\left(\sum_{\nu=0}^{g_p-2}\frac{d_k(p^\nu)}{p^{\nu s}}\right)-(1-p^{-1})^{-1}(1-p^{-s})^k\frac{d_k(p^{g_p-1})}{p^{(g_p-1)s}}\right\},$$

where $g_p=g_p(q)$ is the number such that p^{g_p} is the highest power of p dividing q. We denote, by $\Phi_k^{(h)}(q)$, the value of the h-th derivative of $\Phi_k(s;q)$ at $s=1$. For $k\geq 2$, $0\leq h\leq k-1$, we put

$$\Theta_k^{(h)}(q)=\frac{1}{(k-1-h)!}\sum_{j=0}^{h}\frac{1}{j!}\eta_k(h-j)\Phi_k^{(j)}(q)\,,$$

$$T_k(\beta;q)=\sum_{h=0}^{k-1}\Theta_k^{(h)}(q)\sum_{n\leq N}\{(\log n)^{k-1-h}+(k-1-h)(\log n)^{k-2-h}\}e(n\beta)\,.$$

Now we can write down our Lemma 3.

LEMMA 3. *Let* $k\geq 3$.

(i) *Suppose that* $1\leq a\leq q\leq Q_1$, $(a,q)=1$, $\alpha\in\mathfrak{M}_0(q,a)$ *and* $\alpha=(a/q)+\beta$. *Then we have*

$$D_k(\alpha)=T_k(\beta;q)+O(N^{1-(10k)^{-1}})\,.$$

(ii) *For* $|\beta|\leq 1/2$ *and for* $q\leq Q_1$, *we have*

$$T_k(\beta;q)\ll\frac{d_{k-1}(q)}{q}(\log N)^{k-1}\min(N,|\beta|^{-1})\,.$$

PROOF. For $k\geq 1$, $\sigma>1$, we introduce the functions

$$\Delta_k\left(s;\frac{a}{q}\right)=\sum_{n=1}^{\infty}e\left(\frac{a}{q}n\right)d_k(n)n^{-s}\,,$$

$$\Psi_k(s;q)=\frac{1}{\varphi(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}\Delta_k\left(s;\frac{a}{q}\right)=\frac{1}{\varphi(q)}\sum_{n=1}^{\infty}c_q(n)d_k(n)n^{-s}\,,$$

where $c_q(n)$ is the Ramanujan sum

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(\frac{a}{q}n\right).$$

We can see

$$\Psi_k(s;q)=\zeta(s)^k\Phi_k(s;q)\,, \tag{5.2}$$

by some calculation.

If $k\geq 2$ and $(a,q)=1$ then we have

$$\begin{aligned}\Delta_k\left(s;\frac{a}{q}\right)&=\sum_{b=1}^{q}\sum_{l=1}^{\infty}\sum_{\substack{m=1\\ am\equiv b\pmod q}}^{\infty}e\left(\frac{a}{q}lm\right)d_{k-1}(m)(lm)^{-s}\\&=\zeta(s)^k\prod_{p\mid q}\left(1-(1-p^{-s})^{k-1}\left(\sum_{\nu=0}^{g_p-1}\frac{d_k(p^\nu)}{p^{\nu s}}\right)\right)\\&\quad+q^{-1}\sum_{b=1}^{q-1}\Delta_1\left(s;\frac{a}{q}\right)\sum_{h\mid q}\sum_{\substack{l=1\\(l,h)=1}}^{h}e\left(-\frac{bl}{h}\right)\Delta_{k-1}\left(s;\frac{al}{q}\right).\end{aligned}$$

Then we see the following facts by induction on k with known results on the Lerch zeta functions $\Delta_1(s;a/q)$ and the Riemann zeta function $\zeta(s)$.

(i) $\Delta_k(s;a/q)$ can be analytically continued to a meromorphic function over the whole complex plane, which is holomorphic save a possible pole at $s=1$.

(ii) If $(a,q)=1$ then the meromorphic part of $\Delta_k(s;a/q)$ at $s=1$ does not depend on a, therefore, $\Delta_k(s;a/q)$ has the same meromorphic part as $\Psi_k(s;q)$ at $s=1$.

(iii) For any fixed $\varepsilon>0$, we have

$$\Delta_k\left(s;\frac{a}{q}\right)\ll q^{k-1}(1+|t|^{k(1-\sigma+\varepsilon)/2})\,,$$

uniformly for $|s-1|\geq 1/2$, $\sigma\geq 1/2$.

Now we suppose $q\leq Q_1$, $(a,q)=1$, and $x\geq\sqrt{N}$, and put $T=x^{4\delta}$ with $\delta=\delta(k)=(10k)^{-1}$. Applying Perron's formula with the facts listed above, we obtain

$$\begin{aligned}\sum_{n\leq x}d_k(n)e\left(\frac{a}{q}n\right)&=\frac{1}{2\pi i}\int_{1+\delta-iT}^{1+\delta+iT}\Delta_k\left(s;\frac{a}{q}\right)\frac{x^s}{s}ds+O(x^{1+\delta}T^{-1})\\&=\Upsilon_k(q)+O(x^{1-2\delta})\,,\end{aligned} \tag{5.3}$$

where $\Upsilon_k(q)$ is the residue of $\Psi_k(s;q)x^s/s$ at $s=1$. For $k\geq 2$, it follows from (5.2) that $\Psi_k(s;q)$ has a pole of order k at $s=1$. And by the definition of $\Theta_k^{(h)}(q)$, we have

$$\Upsilon_k(q)=x\sum_{h=0}^{k-1}\Theta_k^{(h)}(q)(\log x)^{k-1-h}\,.$$

Then, Lemma 3 (i) is derived from (5.3) by partial summation.

We turn to (ii). It follows from (5.1) that

$$\sum_{n\leq N}\{(\log n)^{k-1-h}+(k-1-h)(\log n)^{k-2-h}\}e(n\beta)\ll\min(N,|\beta|^{-1})\cdot(\log N)^{k-1-h}$$

for $0\le h\le k-1$ by partial summation, and we have

$$T_k(\beta;q)\ll\min(N,|\beta|^{-1})\cdot\sum_{h=0}^{k-1}|\Theta_k^{(h)}(q)|(\log N)^{k-1-h},\tag{5.4}$$

for $|\beta|\le 1/2$.

Next we estimate $\Theta_k^{(h)}(q)$. After some computation, we get

$$\Phi_k(s;q)=q^{-s}\prod_{p|q}\left\{\sum_{u=0}^{k-2}\binom{g_p+k-2}{u}(1-p^{-s})^u p^{-s(k-2-u)}\right.$$
$$\left.-\binom{g_p+k-2}{k-1}p^{s-1}(1-p^{-s})^{k-1}(1-p^{-1})^{-1}(1-p^{-s+1})\right\},$$

$$\Theta_k^{(0)}(q)=\frac{1}{(k-1)!}\Phi_k(1;q)\ll\frac{d_{k-1}(q)}{q}.$$

For $1\le h\le k-1$, we have

$$\Phi_k^{(h)}(q)=\frac{h!}{2\pi i}\int_{|s-1|=(\log 2q)^{-1}}\frac{\Phi_k(s;q)}{(s-1)^{h+1}}ds\ll q^{-1+\varepsilon},$$

and $\Theta_k^{(h)}(q)\ll q^{-1+\varepsilon}$. Combining these estimates with (5.4), we deduce Lemma 3 (ii), and we complete our proof of Lemma 3.

We now use our Lemmas 2 and 3 to evaluate $I_k(\mathfrak{M}_0)$. For simplicity, we write

$$S_k(\alpha)=D_k(\alpha)F(\alpha)F(\alpha;\mathscr{A})^3e(-N\alpha),\quad\text{and}$$

$$\begin{aligned}I_k(\mathfrak{M}_0)&=\int_{\mathfrak{M}_0}D_k(\alpha)F(\alpha)^4e(-N\alpha)d\alpha-\int_{\mathfrak{M}_0}S_k(\alpha)d\alpha\\&=I_k^{(0)}(\mathfrak{M}_0)-I_k^{(1)}(\mathfrak{M}_0),\quad\text{say}.\end{aligned}\tag{5.5}$$

At first we estmate $I_k^{(1)}(\mathfrak{M}_0)$ for $k\ge 3$. Suppose that $q\le Q_1$, and $|\beta|\le Q_2^{-1}$, then we have by Lemma 1 (i), (ii), Lemma 3 and (2.3)

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}S_k\left(\frac{a}{q}+\beta\right)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left\{T_k(\beta;q)\frac{V(q,a)}{q}v(\beta)\sum_{m\le 3N}r(m)e\left((m-N)\left(\frac{a}{q}+\beta\right)\right)\right.$$
$$\left.+O(N^{7/3-(10k)^{-1}}+q^{-1/2+\varepsilon}N^{2+\varepsilon})\right\}$$

$$=T_k(\beta;q)v(\beta)\sum_{m\le 3N}r(m)\tilde{V}_1(q,N-m)e((m-N)\beta)+O(qN^{7/3-(10k)^{-1}})$$

$$\ll\frac{d_{k-1}(q)}{q}(\log N)^{k-1}\min(N^{4/3},|\beta|^{-4/3})\sum_{m\le 3N}r(m)|\tilde{V}_1(q,N-m)|$$
$$+qN^{7/3-(10k)^{-1}}.$$

Therefore, by virtue of Lemma 1 (iv) and (2.4), we have

$$I_k^{(1)}(\mathfrak{M}_0)=\sum_{q\le Q_1}\int_{-Q_2^{-1}}^{Q_2^{-1}}\sum_{\substack{a=1\\(a,q)=1}}^{q}S_k\left(\frac{a}{q}+\beta\right)d\beta \tag{5.6}$$

$$\ll(\log N)^{k-1}\sum_{m\le 3N}r(m)\sum_{q\le Q_1}\frac{d_{k-1}(q)}{q}|\tilde{V}_1(q,N-m)|$$

$$\times\int_{-Q_2^{-1}}^{Q_2^{-1}}\min(N^{4/3},|\beta|^{-4/3})d\beta+N^{7/3-(10k)^{-1}}Q_1^2Q_2^{-1}$$

$$\ll N^{1/3}(\log N)^{k-1}(\log Q_1)^{C''_{k-1}}\sum_{m\le 3N}r(m)+N^{4/3-(20k)^{-1}}$$

$$\ll N^{4/3}(\log N)^{k-4}(\log\log N)^{(1/6)k(k-1)(k+4)+3}\,.$$

When $k=0$, we have similarly

$$I_0^{(1)}(\mathfrak{M}_0)\ll N^{4/3}(\log N)^{-4}(\log\log N)^4\,, \tag{5.7}$$

by Lemma 1 (i), (ii), (v), Lemma 2, (2.3) and (2.4).

We can also evaluate $I_k^{(0)}(\mathfrak{M}_0)$ straightforwardly. We consider the case $k\ge 3$ again. For $q\le Q_1$, $|\beta|\le Q_2^{-1}$, we have by Lemma 1 (i), (ii) and Lemma 3,

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}D_k\left(\frac{a}{q}+\beta\right)F\left(\frac{a}{q}+\beta\right)^4 e\left(-N\left(\frac{a}{q}+\beta\right)\right)$$

$$=T_k(\beta;q)v(\beta)^4e(-N\beta)\tilde{V}_4(q,N)+O(qN^{7/3-(10k)^{-1}})\,,$$

thus

$$I_k^{(0)}(\mathfrak{M}_0)=\sum_{q\le Q_1}\int_{-Q_2^{-1}}^{Q_2^{-1}}\sum_{\substack{a=1\\(a,q)=1}}^{q}D_k\left(\frac{a}{q}+\beta\right)F\left(\frac{a}{q}+\beta\right)^4 e\left(-N\left(\frac{a}{q}+\beta\right)\right)d\beta \tag{5.8}$$

$$=\sum_{q\le Q_1}\tilde{V}_4(q,N)\int_{-Q_2^{-1}}^{Q_2^{-1}}T_k(\beta;q)v(\beta)^4e(-N\beta)d\beta+O(N^{4/3-(20k)^{-1}})$$

$$=J_k+O(N^{4/3-(20k)^{-1}})\,,\quad\text{say}\,.$$

It follows at once from Lemma 1 (ii) and Lemma 3 (ii) that

$$\int_{Q_2^{-1}\le|\beta|\le 1/2}T_k(\beta;q)v(\beta)^4e(-N\beta)d\beta\ll\frac{d_{k-1}(q)}{q}(\log N)^{k-1}\int_{Q_2^{-1}}^{1/2}\beta^{7/3}d\beta \tag{5.9}$$

$$\ll\frac{d_{k-1}(q)}{q}Q_2^{4/3}(\log N)^{k-1}\,.$$

On the other hand, since we know

$$(5.10)\qquad \int_0^1 v(\beta)^4 e(-m\beta)d\beta=\Gamma\left(\frac{4}{3}\right)^3 m^{1/3}+O(N^{1/3-\delta})$$

for $m\le N$ with some constant $\delta>0$ (see [14, Theorem 2.3]), we have

$$(5.11)\qquad \begin{aligned}&\int_0^1 T_k(\beta;q)v(\beta)^4 e(-N\beta)d\beta\\ &=\sum_{h=0}^{k-1}\Theta_k^{(h)}(q)\sum_{n\le N}\{(\log n)^{k-1-h}+(k-1-h)(\log n)^{k-2-h}\}\\ &\qquad\times\int_0^1 v(\beta)^4 e(-(N-n)\beta)d\beta\\ &=\Gamma\left(\frac{4}{3}\right)^3\sum_{h=0}^{k-1}\Theta_k^{(h)}(q)S_{k,h}(N)+O(N^{4/3-\delta/2}),\end{aligned}$$

where

$$S_{k,h}(N)=\sum_{n<N}\{(\log n)^{k-1-h}+(k-1-h)(\log n)^{k-2-h}\}(N-n)^{1/3}.$$

Elementary calculation shows

$$(5.12)\qquad \begin{aligned}S_{k,h}(N)&=\int_1^N\{(\log x)^{k-1-h}+(k-1-h)(\log x)^{k-2-h}\}(N-x)^{1/3}dx\\ &\quad+O(N^{1/3}(\log N)^{k-1-h})\\ &=\frac{1}{3}\int_1^N x(\log x)^{k-1-h}(N-x)^{-2/3}dx+O(N^{1/3}(\log N)^{k-1-h})\\ &=\frac{1}{3}N^{4/3}\int_{1/N}^1 t(\log N+\log t)^{k-1-h}(1-t)^{-2/3}dt+O(N^{1/3}(\log N)^{k-1-h})\\ &=\frac{1}{3}N^{4/3}\sum_{i=0}^{k-1-h}\binom{k-1-h}{i}(\log N)^{k-1-h-i}\int_0^1 t(\log t)^i(1-t)^{-2/3}dt\\ &\quad+O(N^{1/3}(\log N)^{k-1-h}).\end{aligned}$$

Then, by (5.9), (5.11) and (5.12),

$$(5.13)\qquad \begin{aligned}J_k=&\frac{1}{3}\Gamma\left(\frac{4}{3}\right)^3 N^{4/3}\sum_{h=0}^{k-1}\left(\sum_{q\le Q_1}\Theta_k^{(h)}\tilde{V}_4(q,N)\right)\\ &\times\sum_{i=0}^{k-1-h}\binom{k-1-h}{i}(\log N)^{k-1-h-i}\int_0^1 t(\log t)^i(1-t)^{-2/3}dt\end{aligned}$$

$$+O\left(\sum_{q\le Q_1}\frac{d_{k-1}(q)}{q}|\tilde{V}_4(q,N)|Q_2^{4/3}(\log N)^{k-1}+N^{4/3-\delta/2}\right)$$

$$=\frac{1}{3}\Gamma\left(\frac{4}{3}\right)^3 N^{4/3}\sum_{h=0}^{k-1}\mathfrak{S}_k^{(h)}(N)$$

$$\times\sum_{i=0}^{k-1-h}\binom{k-1-h}{i}(\log N)^{k-1-h-i}\int_0^1 t(\log t)^i(1-t)^{-2/3}dt$$

$$+O(N^{4/3}(\log N)^{-B}),$$

where we have used Lemma 1 (vi), and put

$$\mathfrak{S}_k^{(h)}(N)=\sum_{q=1}^{\infty}\Theta_k^{(h)}\tilde{V}_4(q,N).$$

On writing $j=h+i$ in (5.13), and combining it with (5.8), we have

$$I_k^{(0)}(\mathfrak{M}_0)=\frac{1}{3}\Gamma\left(\frac{4}{3}\right)^3 N^{4/3}\sum_{j=0}^{k-1}(\log N)^{k-1-j}\sum_{h=0}^{j}\binom{k-1-h}{j-h}\mathfrak{S}_k^{(h)}(N) \tag{5.14}$$

$$\times\int_0^1 t(\log t)^{j-h}(1-t)^{-2/3}dt+O(N^{4/3}(\log N)^{-B})$$

$$=N^{4/3}\sum_{j=0}^{k-1}\xi_k^{(j)}(N)(\log N)^{k-1-j}+O(N^{4/3}(\log N)^{-B}),$$

where

$$\xi_k^{(j)}(N)=\frac{1}{3}\Gamma\left(\frac{4}{3}\right)^3\sum_{h=0}^{j}\binom{k-1-h}{j-h}\mathfrak{S}_k^{(h)}(N)\int_0^1 t(\log t)^{j-h}(1-t)^{-2/3}dt. \tag{5.15}$$

We note that

$$\xi_k^{(0)}(N)=\frac{1}{3}\Gamma\left(\frac{4}{3}\right)^3\mathfrak{S}_k^{(0)}(N)\int_0^1 t(1-t)^{-2/3}dt$$

$$=\frac{3}{4(k-1)!}\Gamma\left(\frac{4}{3}\right)^3\prod_p\left\{1+\sum_{i=1}^{\infty}\tilde{V}_4(p^l,N)\Phi_k^{(0)}(p^l)\right\},$$

with

$$\Phi_k^{(0)}(p^l)=\Phi_k(1;p^l)=\frac{1}{p^l}\sum_{u=0}^{k-2}\binom{l+k-2}{u}(1-p^{-1})^u p^{-(k-2-u)}.$$

In the case $k=0$, we obtain similarly (rather more simply)

$$I_0^{(0)}(\mathfrak{M}_0)=\Gamma\left(\frac{3}{4}\right)^3\mathfrak{S}_0(N)\int_2^N\frac{(N-t)^{1/3}}{\log t}dt+O(N^{4/3}(\log N)^{-B}),\tag{5.16}$$

by Lemma 1 (i), (ii), Lemma 2 and (5.10), where

$$\mathfrak{S}_0(N)=\sum_{q=1}^{\infty}\frac{\mu(q)}{\varphi(q)}\tilde{V}_4(q,N)=\prod_p\left(1-\frac{\tilde{V}_4(p,N)}{p-1}\right).\tag{5.17}$$

In view of (5.5), we have (2.9) by (5.6) and (5.14) for $k\geq 3$, and (2.10) by (5.7) and (5.16).

It remains to observe the properties of $\xi_k^{(j)}(N)$'s and $\mathfrak{S}_0(N)$ which are mentioned in our Theorems 1 and 2. By Lemma 1 (vi), we see at once $\xi_k^{(j)}(N)\ll 1$ for $0\leq j<k$, and $\mathfrak{S}_0(N)\ll 1$. In order to observe $\xi_k^{(0)}(N)\gg 1$ for $k\geq 3$ and $\mathfrak{S}_0(N)\gg 1$, it suffices to show

$$1+\sum_{l=1}^{\infty}\tilde{V}_4(p^l,N)\Phi_k^{(0)}(p^l)>0,\tag{5.18}$$

$$1-\frac{\tilde{V}_4(p,N)}{p-1}>0,\tag{5.19}$$

for all primes p. Since $|\tilde{V}_4(p,N)|\leq 2$ and $\tilde{V}_4(2,N)=\tilde{V}_4(3,N)=0$, we get (5.19). Next, for $k\geq 3$ we put $S_m=\sum_{l=0}^{m}\tilde{V}_4(p^l,N)$. Then, by [14, Lemma 2.12], we have $S_m\geq 0$ and

$$\begin{aligned}1+\sum_{l=1}^{m}\tilde{V}_4(p^l,N)\Phi_k^{(0)}(p^l)&\\ =1-\Phi_k^{(0)}(p)+\sum_{l=1}^{m-1}p^{-l}(1-p^{-1})^{k-1}\binom{l+k-2}{k-2}S_l+\Phi_k^{(0)}(p^m)S_m&\\ \geq(1-p^{-1})^{k-1},&\end{aligned}$$

which yields (5.18).

6. Treatment for $\widetilde{R_k}(N)$.

It remains to prove (2.8). We put, for $k\geq 0$,

$$W_k(m)=\sum_{n<m^{1/3}}d_k(m-n^3),$$

then

$$\widetilde{R_k}(N)=\sum_{m_1,m_2,m_3\in\mathscr{A}}W_k(N-m_1^3-m_2^3-m_3^3)=\sum_{m<N}r(N-m)W_k(m).\tag{6.1}$$

We are going to estimate $W_k(m)$. We first consider the case $k\geq 3$. By van der Corput's argument, one may show $W_k(m)\ll N^{1/3}(\log N)^C$ for $m\leq N$ with some constant C, but we need the best possible value of the exponent C of $\log N$ in the estimate for $W_k(m)$. To this end, we appeal to Wolke's result [17, Satz 1]. Applying Satz 1 of [17] with $f(n)=d_k(n)$ and $a_n=m-n^3$, we have

$$W_k(m) \ll N^{1/3} \exp\left((k-1) \sum_{p \le N} \frac{\rho_m(p)}{p}\right), \tag{6.2}$$

for $m \le N$. We should note that we can not show

$$\sum_{p \le N} \frac{\rho_m(p)}{p} = \log\log N + O(1) \tag{6.3}$$

uniformly for $m \le N$ as yet, though it may be expected when the polynomial x^3-m is irreducible over the rational number field. We can however show that (6.3) holds for almost all $m \le N$.

We put

$$j_0 = [\varepsilon \log N], \quad u = (\log N)^{1/\varepsilon}, \quad v = 2^{j_0} u,$$

and define the sets

$$\mathscr{E}(N, M) = \left\{ m < N; \left| \sum_{M < p \le 2M} \frac{\rho_m(p)-1}{p} \right| \ge \frac{1}{j_0} \right\},$$

$$\mathscr{E}(N) = \bigcup_{j=1}^{j_0} \mathscr{E}(N, 2^{j-1}u).$$

Since $N^{\varepsilon/2} < v < N^{\varepsilon}$, we see that if $m \notin \mathscr{E}(N)$ and $m < N$ then

$$\sum_{u < p \le v} \frac{\rho_m(p)}{p} = \sum_{j=1}^{j_0} \left(\sum_{2^{j-1}u < p \le 2^j u} \frac{1}{p} + O(j_0^{-1}) \right)$$
$$= \log\log N - \log\log\log N + O(1),$$

whence

$$\sum_{p \le N} \frac{\rho_m(p)}{p} = \sum_{p \le u} \frac{\rho_m(p)}{p} + \sum_{u < p \le v} \frac{\rho_m(p)}{p} + O(1) \tag{6.4}$$
$$\le \log\log N + \log\log\log N + O(1),$$

because of (2.1) and the well-known formula

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1).$$

Thus, by (2.4), (6.2) and (6.4), we have

$$\sum_{\substack{m < N \\ m \notin \mathscr{E}(N)}} r(N-m) W_k(m) \ll N^{4/3} (\log N)^{k-4} (\log\log N)^{k+2}. \tag{6.5}$$

Next, we estimate the cardinality of $\mathscr{E}(N)$ through the way indicated in Plaksin's paper [12]. For a natural number ν, we put

$$S(N, M, \nu)=\sum_{m\leq N}\left|\sum_{M<p\leq 2M}\frac{\rho_m(p)-1}{p}\right|^{2\nu}.$$

And let $\mathscr{C}_p$ be the set of non-principal characters χ modulo p for which χ^3 is principal. It is easily observed that $\#\mathscr{C}_p=2$ or 0 according as $p\equiv 1 \pmod 3$ or not, and that

$$\rho_m(p)-1=\sum_{\chi\in\mathscr{C}_p}\chi(m).$$

Making use of the Pólya-Vinogradov inequality, we get

$$(6.6)\qquad \begin{aligned} S(N, M, \nu)&=\sum_{\substack{M<p_i\leq 2M\\(1\leq i\leq 2\nu)}}\prod_{i=1}^{2\nu}\frac{1}{p_i}\sum_{\substack{\chi_i\in\mathscr{C}_{p_i}\\(1\leq i\leq 2\nu)}}\sum_{m\leq N}\prod_{i=0}^{2\nu}\chi_i(m)\\ &\ll NM^{-2\nu}S_1(M, \nu)+2^{5\nu}M^{\nu}\log((2M)^{2\nu}), \end{aligned}$$

where

$$S_1(M, \nu)=\sum_{\substack{M<p_i\leq 2M\\(1\leq i\leq 2\nu)}}\sum_{\substack{\chi_i\in\mathscr{C}_{p_i}\\(1\leq i\leq 2\nu)\\ \prod_{i=1}^{2\nu}\chi_i \text{ is principal}}}1.$$

We note that if $\chi_i\in\mathscr{C}_{p_i}$ $(1\leq i\leq 2\nu)$ and $\prod_{i=1}^{2\nu}\chi_i$ is principal, then $\prod_{i=1}^{2\nu}p_i$ is a powerful number. A natural number l is called "powerful" if $p^2|l$ for all prime factors p of l. The number of powerful numbers not exceeding x is $O(\sqrt{x})$. (See Golomb [3].) So we have

$$S_1(M, \nu)\leq 2^{2\nu}(2\nu)!\sum_{\substack{l\leq (2M)^{2\nu}\\ l \text{ is powerful}}}1\ll 2^{3\nu}(2\nu)!M^{\nu}.$$

Then, by (6.6) and the definition of $\mathscr{E}(N, M)$, we obtain

$$(6.7)\qquad \begin{aligned} \#\mathscr{E}(N, M)&\leq j_0^{2\nu}S(N, M, \nu)\\ &\ll(\log N)^{3\nu}((2\nu)!NM^{-\nu}+M^{\nu}\log((2M)^{2\nu})). \end{aligned}$$

We now suppose that $(\log N)^{1/\varepsilon}\leq M\leq N^{\varepsilon}$, and choose ν so as to satisfy $M^{\nu}\leq\sqrt{N}<M^{\nu+1}$. Then we see

$$N^{1/2-\varepsilon}<M^{\nu}\leq N^{1/2}\quad\text{and}\quad \nu\leq\frac{\varepsilon}{2}\frac{\log N}{\log\log N},$$

therefore we have $(2\nu)!\leq\exp(2\nu\log(2\nu))\leq N^{\varepsilon}$, and $(\log N)^{3\nu}\leq N^{2\varepsilon}$. So (6.7) gives

$$\#\mathscr{E}(N, M)\ll N^{1/2+4\varepsilon}.$$

Hence,

$$\#\mathscr{E}(N)\le \sum_{j=1}^{j_0} \#\mathscr{E}(N, 2^{j-1}u) \ll N^{1/2+5\varepsilon}. \tag{6.8}$$

On the other hand, we know

$$\int_0^1 |F(\alpha)|^{2^j} d\alpha \ll (N^{1/3})^{2^j-j+\varepsilon} \qquad \text{for} \quad 1\le j\le 3,$$

by Hua's inequality (see [14, Lemma 2.5]). Using this, we see

$$\begin{aligned}\sum_{m\le N} r(N-m)^2 &\le \int_0^1 |F(\alpha)|^6 d\alpha \\ &\ll \left(\int_0^1 |F(\alpha)|^4 d\alpha\right)^{1/2}\left(\int_0^1 |F(\alpha)|^8 d\alpha\right)^{1/2} \\ &\ll N^{7/6+\varepsilon}.\end{aligned} \tag{6.9}$$

By (6.8), (6.9) and a trivial bound $W_k(m) \ll N^{1/3+\varepsilon}$, we have

$$\begin{aligned}\sum_{m\in\mathscr{E}(N)} r(N-m)W_k(m) &\ll N^{1/3+\varepsilon}\left(\sum_{m\in\mathscr{E}(N)} 1\right)^{1/2}\left(\sum_{m\le N} r(m)^2\right)^{1/2} \\ &\ll N^{7/6+4\varepsilon}.\end{aligned} \tag{6.10}$$

In view of (6.1), the inequalities (6.5) and (6.10) yield (2.8) with $C_k'=k+2$ for $k\ge 3$, which completes our proof of Theorem 1.

We proceed to the case $k=0$. By (2.4), we have at once

$$\widetilde{R_0}(N) \ll N^{4/3}(\log N)^{-3}(\log\log N)^3,$$

and which is sufficient to obtain an asymptotic formula for $R_0(N)$. To show (2.8), we should improve this trivial bound slightly.

We use Selberg's upper bound sieve. Taking $z=N^{1/7}$ and $\kappa=3$ in Theorem 4.1 of [4], we obtain

$$W_0(m) \ll N^{1/3} \prod_{p\le N^{1/7}}\left(1-\frac{\rho_m(p)}{p}\right) \ll N^{1/3}\exp\left(-\sum_{p\le N^{1/7}}\frac{\rho_m(p)}{p}\right),$$

for $m\le N$. It follows from the definition of the set $\mathscr{E}(N)$ that if $m\notin\mathscr{E}(N)$ and $m\le N$ then

$$\begin{aligned}\sum_{p\le N^{1/7}}\frac{\rho_m(p)}{p} &\ge \sum_{\substack{p\le u \\ p\not\equiv 1 \ (\mathrm{mod}\ 3)}}\frac{1}{p} + \sum_{u<p\le v}\frac{\rho_m(p)}{p} \\ &= \log\log N - \frac{1}{2}\log\log\log\log N + O(1),\end{aligned}$$

whence

$$W_0(m) \ll N^{1/3}(\log N)^{-1}(\log\log N)^{1/2} .$$

Therefore, in the same manner as for the case $k \geq 3$, we conclude that the inequality (2.8) holds for $k=0$ as well, with $C'_0 = 7/2$. Now we complete our proof of Theorem 2.

References

[1] K. D. Boklan, The asymptotic formula in Waring's problem, Mathematika **41** (1994), 329–347.
[2] H. Davenport, On Waring's problem for cubes, Acta Math. **71** (1939), 123–143.
[3] S. W. Golomb, Powerful numbers, Amer. Math. Monthly **77** (1970), 848–852.
[4] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press (1974).
[5] D. R. Heath-Brown, Weyl's inequality, Hua's inequality and Waring's problem, J. London Math. Soc. (2) **38** (1988), 216–230.
[6] C. Hooley, On the representation of a number as the sum of a square and a product, Math. Z. **69** (1958), 211–227.
[7] C. Hooley, On a new approach to various problems of Waring's type, *Recent Progress in Analytic Number Theory, Vol. 1*, Academic Press (1981), 127–191.
[8] C. Hooley, Waring's problem for two squares and three cubes, J. Reine Angew. Math. **328** (1981), 161–207.
[9] Y. Motohashi, An asymptotic series for an additive divisor problem, Math. Z. **170** (1980), 43–63.
[10] A. Page, On the representation of a number as a sum of squares and products, I and II, Proc. London Math. Soc. (2) **36** (1933), 241–256; 471–484.
[11] A. Page, On the representation of a number as a sum of squares and products, III, Proc. London Math. Soc. (2) **37** (1934), 1–16.
[12] V. A. Plaksin, On a question of Hua Loo-Keng (Russian), Mat. Zametki **47** (1990), 78–90. English transl.: Math. Notes **47** (1990), 278–286.
[13] K. Prachar, *Primzahlverteilung*, Springer (1957 (reprint, 1978)).
[14] R. C. Vaughan, *The Hardy-Littlewood Method*, Cambridge Univ. Press (1981).
[15] R. C. Vaughan, On Waring's problem for cubes, J. Reine Angew. Math. **365** (1986), 122–170.
[16] R. C. Vaughan, On Waring's problem for smaller exponents, II, Mathematika **33** (1986), 6–22.
[17] D. Wolke, Multiplikative Funktionen auf schnell wachsenden Folgen, J. Reine Angew. Math. **251** (1971), 54–67.
[18] T. D. Wooley, On Vinogradov's mean value theorem, Mathematika **39** (1992), 379–399.

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