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Structure of the C^*-Algebras of Nilpotent Lie Groups

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Abstract. We show that the algebraic structure of the group C^*-algebra $C^*(G)$ of a simply connected, connected nilpotent Lie group G is described as repeating finitely the extension of C^*-algebras with T_2-spectrums by themselves and one more extension by a commutative C^*-algebra on the fixed point space $(\mathfrak{G}^*)^G$ of $\mathfrak{G}^*$ under the coadjoint action of G. Using this result, we show that $C^*(G)$ has no non-trivial projections.

1. Introduction.

It is generally a difficult problem to determine the algebraic structure of its C^*-algebra $C^*(G)$ when a connected Lie group G is given. In the representation theory, it is hard to study the spectrum $\hat{G}$ of G if G is a connected solvable Lie group of non-type I. However, if G is a simply connected, connected nilpotent Lie group, then it is known that $\hat{G}$ is homeomorphic to the quotient space $\mathfrak{G}^*/G$ of $\mathfrak{G}^*$ under the coadjoint action of G. This is called the Kirillov-Bernat (K-B) correspondence. Therefore, the study of the representation theory of G in this case is equivalent to the analysis of $\mathfrak{G}^*/G$.

In this paper, we first study $\mathfrak{G}^*/G$ more precisely. We next describe the structure of the C^*-algebra $C^*(G)$ of a simply connected, connected nilpotent Lie group G as repeating finitely the extension of C^*-algebras with T_2-spectrums by themselves and one more extension by a commutative C^*-algebra on the fixed point space $(\mathfrak{G}^*)^G$ under the coadjoint action of G. Secondly, using this result, we prove that $C^*(G)$ has no non-trivial projections. Lastly, we comment about non-trivial projections of $C^*(G)$ in case that G is an exponential Lie group.

2. Preliminaries.

Let G be an n-dimensional simply connected, connected nilpotent Lie group, and $\mathfrak{G}$ its Lie algebra, and $\mathfrak{G}^*$ the real dual space of $\mathfrak{G}$. Let $\{\mathfrak{G}_i\}_{i=0}^{m+1}$ be the descending central sequence of $\mathfrak{G}$, where $\mathfrak{G}_i=[\mathfrak{G}, \mathfrak{G}_{i-1}]$ $(1 \leq i \leq m+1)$, $\mathfrak{G}_0=\mathfrak{G}$, $\mathfrak{G}_{m+1}=0$.

Let $\mathfrak{G}_i^*$ be the real dual space of $\mathfrak{G}_i$, and $\mathfrak{G}_i^\perp$ be the subspace of $\mathfrak{G}^*$ annihilating

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on $\mathfrak{G}_i$. Then we have $\mathfrak{G}^* = \mathfrak{G}_i^* \oplus \mathfrak{G}_i^\perp$ as a vector space. Every element ϕ in $\mathfrak{G}_i^*$ can be identified with $\phi \oplus 0$ in $\mathfrak{G}^*$. Let $X_{01}^*, X_{02}^*, \cdots, X_{0a_0}^*$ be a basis of $\mathfrak{G}_1^\perp$. Similarly, let $X_{i1}^*, X_{i2}^*, \cdots, X_{ia_i}^*$ be a basis of $\mathfrak{G}_i^* \cap \mathfrak{G}_{i+1}^\perp$ $(1 \leq i \leq m)$ and U_i $(0 \leq i \leq m)$ be the subspaces of $\mathfrak{G}^*$ spanned by them. They are naturally identified with a_i-dimensional Euclidean spaces $\mathbf{R}^{a_i}$ $(0 \leq i \leq m)$. Every element ϕ in $\mathfrak{G}^*$ can be parameterized with $\phi = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m)$, $\alpha_i \in \mathbf{R}^{a_i}$ $(0 \leq i \leq m)$. This parameterization is essential to our Theorem 4.

Let Ad be the adjoint representation of G in Aut($\mathfrak{G}$), and Ad* the coadjoint action of G in $\mathfrak{G}^*$ defined by $\mathrm{Ad}^*(g)\phi(X) = \phi(\mathrm{Ad}(g^{-1})X)$, $(X \in \mathfrak{G}, \phi \in \mathfrak{G}^*, g \in G)$. Let $(\mathfrak{G}^*)^G$ be the fixed point space of $\mathfrak{G}^*$ under Ad*. Using the above parameterization, put

$$V_0 = \{\phi = (\alpha_0, 0, \cdots, 0) \in \mathfrak{G}^* \mid \alpha_0 \in \mathbf{R}^{a_0}\}\,.$$

Then, we can see that:

LEMMA 1. $V_0 = (\mathfrak{G}^*)^G$.

PROOF. Let ϕ be an element of V_0. By definition, ϕ is in $\mathfrak{G}_1^\perp$. Then we have

$$\begin{aligned}\mathrm{Ad}^*(g)(\phi)(Y) &= \phi(\mathrm{Ad}(g^{-1})Y) = \phi(\mathrm{Ad}(\exp(-X))Y)\,, \qquad \text{where } g = \exp(X)\\ &= \phi(\exp(\mathrm{ad}(-X))Y)\\ &= \phi\left(Y - [X, Y] + \frac{1}{2!}\mathrm{ad}(X)^2 Y - \cdots + \frac{(-1)^m}{m!}\mathrm{ad}(X)^m Y\right) = \phi(Y)\end{aligned}$$

for every g in G and Y in $\mathfrak{G}^*$. So ϕ is in $(\mathfrak{G}^*)^G$.

On the contrary, let ϕ be an element of $(\mathfrak{G}^*)^G$. By the same calculation, we have

$$\phi(Y) = \phi\left(Y - [X, Y] + \frac{1}{2!}\mathrm{ad}(X)^2 Y - \cdots + \frac{(-1)^m}{m!}\mathrm{ad}(X)^m Y\right)$$

for every X, Y in $\mathfrak{G}^*$. It implies that

$$\phi\left(-[X, Y] + \frac{1}{2!}\mathrm{ad}(X)^2 Y - \cdots + \frac{(-1)^m}{m!}\mathrm{ad}(X)^m Y\right) = 0\,.$$

Then, replacing Y with $\mathrm{ad}(X)^{m-1}Y$, we have that $\phi(\mathrm{ad}(X)^m Y) = 0$. Moreover, replacing Y with $\mathrm{ad}(X)^k Y$ $(1 \leq k \leq m-2)$, we have that $\phi(\mathrm{ad}(X)^{k+1}Y) = 0$ $(1 \leq k \leq m-2)$. Therefore, we conclude that $\phi([X, Y]) = 0$ for every X, Y in $\mathfrak{G}^*$. So ϕ is in V_0. □

Next, put

$$V_k = \{\phi = (\alpha_0, \alpha_1, \cdots, \alpha_k, 0, \cdots, 0) \in \mathfrak{G}^* \mid \alpha_j \in \mathbf{R}^{a_j} (0 \leq j \leq k-1), \alpha_k \in \mathbf{R}^{a_k} \setminus \{0\}\}\,,$$

$(1 \leq k \leq m)$. Then we can decompose $\mathfrak{G}^*$ into

$$V_0 \cup V_1 \cup V_2 \cup \cdots \cup V_k \cup \cdots \cup V_m$$

consisting of $m+1$ pieces of subsets of $\mathfrak{G}^*$.

Next we can see explicitly the coadjoint orbit for every element in V_k of $\mathfrak{G}^*$. In the following we denote by ϕ_{α_k} the functional corresponding to $\phi=(0, \cdots, 0, \alpha_k, 0, \cdots, 0)$. For example, we have that:

LEMMA 2. *The orbit* $\mathrm{Ad}^*(G)\phi$ *for an element* $\phi=(\alpha_0, \alpha_1, 0, \cdots, 0)$ *in* V_1 *of* $\mathfrak{G}^*$ *is given by the subset*

$$\{(\alpha_0-\mathrm{ad}^*(X)\alpha_1, \alpha_1, 0, \cdots, 0) \mid X\in\mathfrak{G}\},$$

where $(\mathrm{ad}^*(X)\phi_{\alpha_1})(Y)=\phi_{\alpha_1}([X, Y])$, $Y\in\mathfrak{G}$ *when* α_1 *in* $\mathbf{R}^{a_1}\backslash\{0\}$ *is identified with* ϕ_{α_1} *in* $\mathfrak{G}^*$.

PROOF. The functional corresponding to $\phi=(\alpha_0, \alpha_1, 0, \cdots, 0)$ in V_1 is given by $\phi_{\alpha_0}+\phi_{\alpha_1}$. By the direct computation, we have

$$\begin{aligned}
\mathrm{Ad}^*(g)(\phi_{\alpha_0}+\phi_{\alpha_1})(Y) &= \mathrm{Ad}^*(g)(\phi_{\alpha_0})(Y)+\mathrm{Ad}^*(g)(\phi_{\alpha_1})(Y)\\
&= \phi_{\alpha_0}(Y)+\phi_{\alpha_1}(\mathrm{Ad}(g^{-1})Y)\\
&= \phi_{\alpha_0}(Y)+\phi_{\alpha_1}\Big(Y-[X, Y]+\frac{1}{2!}\mathrm{ad}(X)^2Y\\
&\qquad -\cdots+\frac{(-1)^m}{m!}\mathrm{ad}(X)^mY\Big), \qquad \text{where } g=\exp(X)\\
&= \phi_{\alpha_0}(Y)+\phi_{\alpha_1}(Y-[X, Y])\\
&= \phi_{\alpha_0}(Y)-(\mathrm{ad}^*(X)\phi_{\alpha_1})(Y)+\phi_{\alpha_1}(Y).
\end{aligned}$$

We next show that $\mathrm{ad}^*(X)\phi_{\alpha_1}$ is in V_0. By the direct computation, we have

$$\begin{aligned}
&\mathrm{Ad}^*(h)(\mathrm{ad}^*(X)\phi_{\alpha_1})(Y)=(\mathrm{ad}^*(X)\phi_{\alpha_1})(\exp(\mathrm{ad}(-Z))Y), \qquad \text{where } h=\exp(Z)\\
&=\Big(\mathrm{ad}^*(X)\phi_{\alpha_1}\Big)\Big(Y-[Z, Y]+\frac{1}{2!}\mathrm{ad}(Z)^2Y-\cdots+\frac{(-1)^m}{m!}\mathrm{ad}(Z)^mY\Big)\\
&=\phi_{\alpha_1}\Big([X, Y]-[X, [Z, Y]]+\frac{1}{2!}\mathrm{ad}(X)\mathrm{ad}(Z)^2Y-\cdots+\frac{(-1)^m}{m!}\mathrm{ad}(X)\mathrm{ad}(Z)^mY\Big)\\
&=\phi_{\alpha_1}([X, Y])=(\mathrm{ad}^*(X)\phi_{\alpha_1})(Y).
\end{aligned}$$

It then follows that $\mathrm{Ad}^*(G)(\mathrm{ad}^*(X)\phi_{\alpha_1})=\mathrm{ad}^*(X)\phi_{\alpha_1}$, so that $\mathrm{ad}^*(X)\phi_{\alpha_1}$ is in V_0. □

In general, the orbit $\mathrm{Ad}^*(G)\phi$ for an element $\phi=(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_k, 0, \cdots, 0)$ in V_k of $\mathfrak{G}^*$ is given by the subset

$$\begin{aligned}
\{(&\alpha_0-\mathrm{ad}^*(X)\alpha_1+(2!)^{-1}\mathrm{ad}^*(X)^2\alpha_2+\cdots+(-1)^k(k!)^{-1}\mathrm{ad}^*(X)^k\alpha_k,\\
&\alpha_1-\mathrm{ad}^*(X)\alpha_2+\cdots+(-1)^{k-1}((k-1)!)^{-1}\mathrm{ad}^*(X)^{k-1}\alpha_k,\\
&\alpha_2-\mathrm{ad}^*(X)\alpha_3+\cdots+(-1)^{k-2}((k-2)!)^{-1}\mathrm{ad}^*(X)^{k-2}\alpha_k,\\
&\cdots\cdots, \quad \alpha_{k-1}-\mathrm{ad}^*(X)\alpha_k, \alpha_k, 0, \cdots, 0) \mid X\in\mathfrak{G}\}
\end{aligned}$$

where α_i is identified with ϕ_{α_i} in $\mathfrak{G}^*$ $(i=0, 1, \cdots, k)$.

In the subsets $V_0, V_1, \cdots, V_m$ of $\mathfrak{G}^*$, the coadjoint action of G effects to parameters on the left side of a non-zero parameter on the right end. Furthermore, we decompose V_1 into the subsets $\{V_{1i}\}_{i=1}^{3^{a_1}-1}$ of $\mathfrak{G}^*$, which are combinationally defined by whether each of the parameters $\{l_{1i}\}_{i=1}^{a_1}$ about $\{X_{1i}^*\}_{i=1}^{a_1}$ is zero, greater than zero or less than zero. For example, V_{11} is given by the subset

$$\{(\alpha_0, \alpha_1, 0, \cdots, 0) \mid \alpha_0 \in \mathbf{R}^{a_0}, \alpha_1 = (l_{11}, 0, \cdots, 0), l_{11} > 0\} ,$$

and V_{12} is given by the subset

$$\{(\alpha_0, \alpha_1, 0, \cdots, 0) \mid \alpha_0 \in \mathbf{R}^{a_0}, \alpha_1 = (l_{11}, 0, \cdots, 0), l_{11} < 0\} ,$$

and V_{13} is given by the subset

$$\{(\alpha_0, \alpha_1, 0, \cdots, 0) \mid \alpha_0 \in \mathbf{R}^{a_0}, \alpha_1 = (0, l_{12}, 0, \cdots, 0), l_{12} > 0\} .$$

More generally, V_{1i} for some i is given by the subset

$$\{(\alpha_0, \alpha_1, 0, \cdots, 0) \mid \alpha_0 \in \mathbf{R}^{a_0}, \alpha_1 = (l_{11}, l_{12}, l_{13}, \cdots, l_{1j}, 0, \cdots, 0) ,$$
$$l_{11} > 0, l_{12} = 0, l_{13} < 0, \cdots, l_{1j} > 0\} .$$

Furthermore, we decompose V_k $(k=2, \cdots, m)$ into the subsets $\{V_{ki}\}_{i=1}^{3^{a_1}3^{a_2}\cdots 3^{a_{k-1}}(3^{a_k}-1)}$ of $\mathfrak{G}^*$, which are combinationally defined by whether each of the parameters $\{l_{ji}\}_{i=1}^{a_j}$ $(1 \le j \le k)$ about $\{X_{ji}^*\}_{i=1}^{a_j}$ $(1 \le j \le k)$ is zero, greater than zero or less than zero. Therefore we can decompose $\mathfrak{G}^*$ into $1 + (3^{a_1}-1) + 3^{a_1}(3^{a_2}-1) + \cdots + 3^{a_1}3^{a_2} \cdots 3^{a_{k-1}}(3^{a_k}-1) + \cdots + 3^{a_1}3^{a_2} \cdots 3^{a_{m-1}}(3^{a_m}-1)$ (say l) pieces of subsets of $\mathfrak{G}^*$.

Then, letting q be the quotient map from $\mathfrak{G}^*$ to $\mathfrak{G}^*/G$, we consider the subsets $q(V_0)$ and $\{q(V_{ki})\}_{i=1}^{3^{a_1}3^{a_2}\cdots 3^{a_{k-1}}(3^{a_k}-1)}$ $(1 \le k \le m)$ of $\mathfrak{G}^*/G$. And let $\Omega_0, \Omega_1, \cdots, \Omega_l$ be those subsets of $\mathfrak{G}^*/G$. Note that it happens that $\Omega_i = \Omega_j$ for $i<j$. In this case let $\Omega_i = \{\emptyset\}$. Under this setup, using Lemma 3.1 in [4] and Theorem 10.5.4 in [2], which are stated later as Theorems 1 and 2 respectively, we prove our main theorems in the next section. Before further study, we give an example here for the convenience of understanding.

EXAMPLE 1. Let G be the simply connected, connected nilpotent Lie group defined by all 4×4 upper triangular real matrices with 1 on the diagonal. Then the Lie algebra $\mathfrak{G}$ of G is defined by all 4×4 upper triangular matrices with 0 on the diagonal. Then the real dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ is defined by all 4×4 lower triangular matrices with 0 on the diagonal. In our setting, every element $\phi = (l_{ij})_{1 \le i,j \le 4}$ in $\mathfrak{G}^*$ is parameterized with $\phi = (\alpha_0, \alpha_1, \alpha_2)$ where $\alpha_0 = (l_{21}, l_{32}, l_{43})$, $\alpha_1 = (l_{31}, l_{42})$, $\alpha_2 = l_{41}$. The coadjoint action of G on $\mathfrak{G}^*$ is defined by $\mathrm{Ad}^*(g)\phi(X) = \phi(\mathrm{Ad}(g^{-1})X) = \mathrm{Tr}(\mathrm{Ad}(g^{-1})X\phi)$ where $g \in G$, $X \in \mathfrak{G}$, and Tr is the natural trace on $M_4(\mathbf{R})$. Then computing this, we have

$$\mathrm{Ad}^*(g)\phi = (\beta_0, \beta_1, \beta_2) ,$$

where

$$\beta_0=(l_{21}-x_{23}l_{31}+(-x_{24}+(2!)^{-1}x_{23}x_{34})l_{41},\ x_{12}l_{31}-x_{12}x_{34}l_{41}+l_{32}-x_{34}l_{42}\,,$$
$$(x_{13}+(2!)^{-1}x_{12}x_{23})l_{41}+x_{23}l_{42}+l_{43})\,,\quad \beta_1=(l_{31}-x_{34}l_{41},\ x_{12}l_{41}+l_{42})\,,\quad \beta_2=l_{41}\,,$$

for $g^{-1}=\exp(X)$ and $X=(x_{ij})_{1\le i,j\le 4}$ in $\mathfrak{G}$. Then Ω_0 is identified with $\mathbf{R}^3$, and Ω_k $(1\le k\le 4)$ are identified with $\mathbf{R}\times(0,\infty)$, where representatives of Ω_k have the form $(\alpha_0, l_{31}, l_{42}, 0)$ with either $l_{31}=0$ or $l_{42}=0$, and the closures $\overline{\Omega_k}$ $(1\le k\le 4)$ are equal to $\Omega_0\cup\Omega_k$. The sets Ω_k $(5\le k\le 8)$ are identified with $\mathbf{R}\times(0,\infty)\times(0,\infty)$, where representatives of Ω_k have the form $(\alpha_0, l_{31}, l_{42}, 0)$ with $l_{31}\ne 0$ and $l_{42}\ne 0$, and the closure $\overline{\bigcup_{k=5}^{8}\Omega_k}$ contains $\bigcup_{i=0}^{4}\Omega_i$. The sets Ω_9, Ω_{10} are identified with $\mathbf{R}\times(0,\infty)$, where representatives of Ω_k $(9\le k\le 10)$ have the form $(\alpha_0, \alpha_1, l_{41})$ with $l_{41}\ne 0$, and the closure $\overline{\Omega_9\cup\Omega_{10}}$ are equal to $\mathfrak{G}^*/G$.

3. Main theorems.

In this section we prove that the C^*-algebra $C^*(G)$ of a simply connected, connected nilpotent Lie group G is obtained by repeating finitely the extension of C^*-algebras with T_2-spectrum by themselves and one more extension by a commutative C^*-algebra on a Euclidean space. Using this result, we prove that $C^*(G)$ has no non-trivial projections.

First of all, we prove the following lemma which is stated in [4]:

Lemma 3 [4]. *The image Ω_0 of the fixed point space $(\mathfrak{G}^*)^G$ is a locally compact T_2-space in the relative topology of Ω_0 and closed in $\mathfrak{G}^*/G$.*

Proof. First, it is known that $\hat{G}$ is locally compact, which can be found in [1]. Using K-B correspondence we have that $\mathfrak{G}^*/G$ is locally compact. So Ω_0 is locally compact with its relative topology.

Next, let $[\phi_1]$, $[\phi_2]$ be two distinct points in Ω_0. Then $q^{-1}([\phi_1])=\{\phi_1\}$, $q^{-1}([\phi_2])=\{\phi_2\}$ are also two distinct points in $\mathfrak{G}^*$. Since $\mathfrak{G}^*$ is a T_2-space, there exist two open neighborhoods U_1, U_2 of ϕ_1, ϕ_2 respectively such that $U_1\cap U_2=\varnothing$. Since $q(U_1)$, $q(U_2)$ are open in $\mathfrak{G}^*/G$, $q(U_1)\cap\Omega_0$, $q(U_2)\cap\Omega_0$ are two disjoint open neighborhoods of $[\phi_1]$, $[\phi_2]$ respectively in Ω_0.

Lastly, let $\{[\phi_n]\}$ be a sequence of Ω_0. Suppose that $[\phi]$ is in $\mathfrak{G}^*/G$ and $[\phi_n]$ converges to $[\phi]$. If $[\phi]$ is not in Ω_0, then $q^{-1}([\phi])\cap(\mathfrak{G}^*)^G=\varnothing$. Since $\hat{G}$ is a T_1-space, $\{[\phi]\}$ is closed in $\mathfrak{G}^*/G$ so that $q^{-1}([\phi])$ is closed in $\mathfrak{G}^*$. By normality of $\mathfrak{G}^*$, there exists an open set O of $\mathfrak{G}^*$ such that $q^{-1}([\phi])\subset O$ and $O\cap(\mathfrak{G}^*)^G=\varnothing$. It follows that $q(O)$ is an open neighborhood of $[\phi]$ in $\mathfrak{G}^*/G$ and $q(O)\cap\Omega_0=\varnothing$, which contradicts our assumption. □

From this result, we can consider the C^*-algebra $C_0(\Omega_0)$ consisting of all complex valued continuous functions on Ω_0 vanishing at infinity.

We proved the following theorem in [4], which was considered as the first key lemma for our main theorems. We prepare the notation for this theorem.

Now, let Φ be the Kirillov-Bernat mapping from the coadjoint orbit space $\mathfrak{G}^*/G$ to the spectrum $\hat{G}$ of G. Put $\Phi([\phi])=\chi_\phi$ for every element $[\phi]$ in Ω_0, where $[\phi]$ is identified with ϕ in $\mathfrak{G}^*$, and χ_ϕ is defined by $\chi_\phi(\exp(X))=e^{i\phi(X)}$ for every X in $\mathfrak{G}$. Let $\tilde{\chi}_\phi$ be the element in spectrum $\widehat{C^*(G)}$ of $C^*(G)$ corresponding to χ_ϕ. Let $\ker(\tilde{\chi}_\phi)$ be the kernel of $\tilde{\chi}_\phi$. Let $\mathfrak{J}_0=\bigcap_{[\phi]\in\Omega_0}\ker(\tilde{\chi}_\phi)$ be the intersection of those kernels for every element $[\phi]$ in Ω_0. Then it is clear that $\mathfrak{J}_0$ is a two-sided closed ideal of $C^*(G)$. Then, the following theorem holds:

THEOREM 1 [4]. *The quotient C^*-algebra $C^*(G)/\mathfrak{J}_0$ of $C^*(G)$ by the ideal $\mathfrak{J}_0$ is isomorphic to $C_0(\Omega_0)$.*

Next we investigate the difference space $(\mathfrak{G}^*/G)\backslash\Omega_0$ corresponding to the spectrum $\hat{\mathfrak{J}}_0$ of $\mathfrak{J}_0$. Then the following lemma holds:

LEMMA 4. *The subsets Ω_i $(1\leq i\leq l)$ of $\mathfrak{G}^*/G$ are all non compact connected T_2-spaces in the relative topology of Ω_i, and closed in $(\mathfrak{G}^*/G)\backslash(\bigcup_{j=0}^{i-1}\Omega_j)$.*

PROOF. We can take the subset V_{kj} of $\mathfrak{G}^*$ for some k, j such that $q(V_{kj})=\Omega_i$. Each element of V_{kj} can be parameterized with $(\alpha_0, l_{11}, \cdots, l_{1a_1}, \cdots, l_{k1}, \cdots, l_{ka_k}, 0, \cdots, 0)$. Put $W_i=V_{kj}$. Since W_i is connected, Ω_i which is the continuous image of W_i by q is also connected.

We next show that Ω_i is non compact. Let $U(r_1, \cdots, r_k)$ be the open subsets of V_k defined by the product spaces

$$\mathbf{R}^{a_0}\times U(r_1)\times\cdots\times U(r_{k-1})\times(U(r_k)\backslash\{0\})\times\{0\}\times\cdots\times\{0\}$$

where $U(r_j)$ is the open ball in $\mathbf{R}^{a_j}$ with the radius r_j in $\mathbf{N}$ and center 0 $(j=1, \cdots, k)$. Since q is an open map and $q(U(r_1, \cdots, r_k)\cap W_i)=q(U(r_1, \cdots, r_k))\cap\Omega_i$, the family $\{q(U(r_1, \cdots, r_k)\cap W_i)\}_{(r_1,\cdots,r_k)\in\mathbf{N}^k}$ is an open covering of Ω_i with respect to the relative topology. It is clear that every finite subcovering does not contain Ω_i. Therefore, Ω_i is non compact.

We next show that Ω_i is closed in $(\mathfrak{G}^*/G)\backslash(\bigcup_{j=0}^{i-1}\Omega_j)$. Suppose that a sequence $([\phi_n])_{n\in\mathbf{N}}$ of Ω_i converges to $[\phi]$ in $(\mathfrak{G}^*/G)\backslash(\bigcup_{j=0}^{i-1}\Omega_j)$. We show that $[\phi]$ is in Ω_i. If not so, say $[\phi]\in\Omega_j$ $(j>i)$, there exists $\psi\in W_j$ such that $q(\psi)=[\phi]$, where $q(W_j)=\Omega_j$ as before. We can take a small open neighborhood U of ψ such that $U\cap W_i=\varnothing$ since ψ has a nonzero G-invariant parameter l_{st} such that l_{st} is zero for every element in W_i, or ψ has $l_{st}>0$ (<0) such that $l_{st}<0$ (>0) for every element in W_i respectively. Then consider G-invariant open subset $\mathrm{Ad}^*(G)(U)$ of $\mathfrak{G}^*$, where $\mathrm{Ad}^*(G)(U)$ means the union $\bigcup_{g\in G}\mathrm{Ad}^*(g)(U)$ of open subsets $\mathrm{Ad}^*(g)(U)$ in $\mathfrak{G}^*$. Then we have $\mathrm{Ad}^*(G)(U)\cap W_i=\varnothing$ since every element in U has a G-invariant parameter l_{uv} such that l_{uv} is zero for every element in W_i, or $l_{uv}>0$ (<0) such that $l_{uv}<0$ (>0) for every element in W_i respectively. It then follows that $q(\mathrm{Ad}^*(G)(U))\cap\Omega_i=\varnothing$, which is a contradiction.

We next show that Ω_i is a T_2-space with respect to the relative topology. Let $[\phi]$

and $[\psi]$ be two distinct points in Ω_i. Then the preimages $q^{-1}([\phi])$ and $q^{-1}([\psi])$ are disjoint. Let $\phi=(\alpha_0, 0, \cdots, \alpha_{i_1}, 0, \cdots, \alpha_{i_k}, 0, \cdots, 0)$ and $\psi=(\beta_0, 0, \cdots, \beta_{i_1}, 0, \cdots, \beta_{i_k}, 0, \cdots, 0)$ be two arbitrary points of $q^{-1}([\phi]) \cap W_i$ and $q^{-1}([\psi]) \cap W_i$ respectively. Since $[\phi]$ and $[\psi]$ are two distinct points in Ω_i, there exists a term such that $\alpha_{i_j} \neq \beta_{i_j}$ where $\alpha_{i_j}=(l_{i_j1}, \cdots, l_{i_ja_{i_j}})$, $\beta_{i_j}=(m_{i_j1}, \cdots, m_{i_ja_{i_j}})$ with respect to $X^*_{i_j1}, \cdots, X^*_{i_ja_{i_j}}$, such that $l_{i_ts}=m_{i_ts}$ for G-invariant parameters l_{i_ts} and m_{i_ts} of α_{i_t} and β_{i_t} respectively, if necessary, replacing the basis $\{X^*_{i_ts}\}_{s=1}^{a_{i_t}}$ $(j<t\leq k)$, and there exists a G-invariant parameter $l_{i_ju} \neq m_{i_ju}$ of α_{i_j} and β_{i_j}, if necessary, replacing the basis $\{X^*_{i_jt}\}_{t=1}^{a_{i_j}}$. Since $\mathfrak{G}^*$ is of course T_2-space, let U_ϕ and U_ψ be two disjoint open neighborhoods of ϕ and ψ respectively, separating l_{i_ju} and m_{i_ju}. Now put $S_i=q^{-1}((\mathfrak{G}^*/G)\setminus(\bigcup_{j=0}^{i-1}\Omega_j))$. Then we can consider two G-invariant open subsets $\mathrm{Ad}^*(G)(U_\phi) \cap S_i$ and $\mathrm{Ad}^*(G)(U_\psi) \cap S_i$ of $\mathfrak{G}^*$. Then put $T_\phi=\mathrm{Ad}^*(G)(U_\phi) \cap S_i$ and $T_\psi=\mathrm{Ad}^*(G)(U_\psi) \cap S_i$. Then $q(T_\phi)$ and $q(T_\psi)$ are two open neighborhoods in $\mathfrak{G}^*/G$ since q is an open map. They are also open in $(\mathfrak{G}^*/G)\setminus(\bigcup_{j=0}^{i-1}\Omega_j)$. Then $q(T_\phi) \cap \Omega_i$ and $q(T_\psi) \cap \Omega_i$ are disjoint and open in Ω_i. Therefore, Ω_i is a T_2-space, as desired. □

Using this lemma, we can consider the decreasing sequence $\{\mathfrak{I}_j\}_{j=0}^{l}$ $(\mathfrak{I}_j \supset \mathfrak{I}_{j+1})$, $\mathfrak{I}_l=\{0\}$ of C^*-subalgebras of $C^*(G)$ corresponding to subsets $(\mathfrak{G}^*/G)\setminus(\bigcup_{i=0}^{j}\Omega_i)$ $(0\leq j\leq l)$ of $\mathfrak{G}^*/G$. Since $C^*(G)$ is liminal, so are its C^*-subalgebras $\{\mathfrak{I}_j\}_{j=0}^{l-1}$. Let $\{\mathscr{C}_j\}_{j=0}^{l-2}$ be the quotient C^*-algebras $\mathfrak{I}_j/\mathfrak{I}_{j+1}$ of $\mathfrak{I}_j$ by $\mathfrak{I}_{j+1}$, which are also liminal. Then the spectrum $\hat{\mathscr{C}}_i$ of $\mathscr{C}_i$ is equal to Ω_{i+1}.

In general, the following holds. We need this result to prove our theorems:

THEOREM 2 [2]. *Let $\mathfrak{A}$ be a liminal C^*-algebra with the T_2-spectrum $\hat{\mathfrak{A}}$. Let $\mathfrak{F}=((\mathfrak{A}/\ker(\pi))_{[\pi]\in\hat{\mathfrak{A}}}, \Theta)$ be a continuous field of elementary C^*-algebras over $\hat{\mathfrak{A}}$ defined by $\mathfrak{A}$. Let $\tilde{\mathfrak{A}}$ be the C^*-algebra defined by $\mathfrak{F}$. Then the correspondence from a in $\mathfrak{A}$ to $\tilde{a}$ in $\tilde{\mathfrak{A}}$ gives an isomorphism from $\mathfrak{A}$ to $\tilde{\mathfrak{A}}$, where $\tilde{a}$ is an element in Θ defined by* $\tilde{a}([\pi])=a+\ker(\pi)$.

Applying this to the quotients $\{\mathscr{C}_j\}_{j=0}^{l-2}$ in exact sequences, and using the above results, we have the following theorem:

THEOREM 3. *The C^*-algebra $C^*(G)$ for every simply connected, connected nilpotent Lie group G can be obtained by repeating finitely the extension of the C^*-algebras defined by a continuous field of elementary C^*-algebras over Ω_i $(1\leq i\leq l)$ by themselves, and one more extension by $C_0(\Omega_0)$ with spectrum homeomorphic to $\mathbf{R}^{a_0}$. Moreover, Ω_i is homotopic to Euclidean space $\mathbf{R}^{k_i}$ for some k_i $(1\leq i\leq l)$.*

PROOF. Using Theorem 1 and Lemma 4, we have the following exact sequences:

$$0 \longrightarrow \mathfrak{I}_0 \xrightarrow{\iota_0} C^*(G) \xrightarrow{q_0} C_0(\Omega_0) \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{I}_j \xrightarrow{\iota_j} \mathfrak{I}_{j-1} \xrightarrow{q_j} \mathfrak{I}_{j-1}/\mathfrak{I}_j\,(=\mathscr{C}_{j-1}) \longrightarrow 0\,,$$

where $1 \leq j \leq l-1$. The quotient $\mathscr{C}_j$ $(0 \leq j \leq l-2)$ in this exact sequence has the spectrum which is identified with Ω_{j+1}. Also, the ideal $\mathfrak{J}_{l-1}$ in the last exact sequence has the spectrum which is identified with Ω_l. Since $\mathscr{C}_j$ $(0 \leq j \leq l-2)$ and $\mathfrak{J}_{l-1}$ are liminal C^*-algebras with T_2-spectrums, we apply Theorem 2 to them. Hence, those can be considered as the C^*-algebras of continuous vector fields of continuous fields. This shows that $C^*(G)$ is obtained by the extension of $\mathfrak{J}_{l-1}$ with this property by $\mathscr{C}_{l-2}$ with this one and repeating the extension by $\mathscr{C}_j$ $(0 \leq j \leq l-3)$ with this one, and one more extension by $C_0(\Omega_0)$.

We next show that Ω_i is homotopic to $\mathbf{R}^j$ for some j. Let W_i be the subset of $\mathfrak{G}^*$ corresponding to Ω_i as considered in Lemma 4. Suppose that W_i is a subset of V_k. We can pick up G-invariant non-zero parameters for all elements in W_i. Let $(\beta_0, \beta_1, \cdots, \beta_k, 0, \cdots, 0)$ be the parametrization of them. We denote by S_i the set of all elements of this form. Then, we can consider the strong retraction r from W_i to the subset S_i of $\mathfrak{G}^*$ defined by r ; $W_i \times I \to \mathfrak{G}^*$,

$$r((\alpha_0, \alpha_1, \cdots, \alpha_k, 0, \cdots, 0), t) = (t\alpha_0, t\alpha_1, \cdots, t\alpha_k, 0, \cdots, 0) + ((1-t)\beta_0, (1-t)\beta_1, \cdots, (1-t)\beta_k, 0, \cdots, 0),$$

where I means the interval $[0, 1]$ and $t \in I$, and $t\alpha_0$ means the pointwise multiplication. Then, it is clear that r induces the strong retraction from Ω_i to $q(S_i)$. We can also show that $q(S_i)$ is homeomorphic to $\mathbf{R}^j$ for some j. Therefore, Ω_i is homotopic to $\mathbf{R}^j$ for some j. □

REMARK 1. $\mathscr{C}_j$ $(0 \leq j \leq l-2)$ and $\mathfrak{J}_{l-1}$ are written as

$$\{\tilde{a} : \Omega_{j+1} \to \bigcup_{[\phi] \in \Omega_{j+1}} \mathscr{C}_j / \ker(\pi_\phi),\ \|\tilde{a}(\cdot)\| \in C_0(\Omega_{j+1})\} \qquad (0 \leq j \leq l-2)$$

and

$$\{\tilde{a} : \Omega_l \to \bigcup_{[\phi] \in \Omega_l} \mathfrak{J}_{l-1} / \ker(\pi_\phi),\ \|\tilde{a}(\cdot)\| \in C_0(\Omega_l)\},$$

where π_ϕ is an irreducible representation corresponding to $[\phi]$, and $\|\tilde{a}(\cdot)\|$ maps $[\phi]$ to $\|\tilde{a}([\phi])\|$. Since $\mathscr{C}_j$ $(0 \leq j \leq l-2)$ and $\mathfrak{J}_{l-1}$ are liminal, $\mathscr{C}_j / \ker(\pi_\phi)$ and $\mathfrak{J}_{l-1} / \ker(\pi_\psi)$ are isomorphic to $\mathbf{K}(H_{\pi_\phi})$ and $\mathbf{K}(H_{\pi_\psi})$ respectively. It is unclear whether or not those continuous fields satisfies Fell's condition. If so, the above continuous fields may be written as $C_0(\Omega_{j+1}) \otimes \mathbf{K}(H)$, $(0 \leq j \leq l-1)$ for a Hilbert space H. Then, homotopy equivalence of Ω_j to $\mathbf{R}^n$ for some n may be useful to the calculation in K-theory for the above exact sequences.

REMARK 2. $\mathscr{C}_j$ $(0 \leq j \leq l-2)$ and $\mathfrak{J}_{l-1}$ have no non-trivial projections. Let $\mathfrak{A}$ be one of them. Suppose that p is a non-trivial projection in $\mathfrak{A}$. Let $\tilde{p}$ be the continuous vector field corresponding to p. Then, $\tilde{p}([\pi])$ is a non-trivial projection in $\mathfrak{A}/\ker(\pi)$ for some $[\pi]$ in $\hat{\mathfrak{A}}$ so that the norm of $\tilde{p}([\pi])$ is one. If the inverse image $\|\tilde{p}(\cdot)\|^{-1}(0)$ of 0 is non-empty, then $\|\tilde{p}(\cdot)\|^{-1}(0)$, $\|\tilde{p}(\cdot)\|^{-1}(1)$ are non-empty clopen sets, and $\Omega_j = \|\tilde{p}(\cdot)\|^{-1}(0) \cup \|\tilde{p}(\cdot)\|^{-1}(1)$, which is impossible by the connectivity of Ω_j. So $\Omega_j =$

$\|\tilde{p}(\cdot)\|^{-1}(1)$. Hence, $\tilde{p}$ does not vanish at infinity, which is a contradiction. Therefore, $\mathfrak{A}$ has no non-trivial projections.

As a consequence of Theorem 3, the following theorem is verified:

THEOREM 4. *The C^*-algebra $C^*(G)$ of every simply connected, connected nilpotent Lie group G has no non-trivial projections.*

PROOF. Suppose that $C^*(G)$ has a non-trivial projection p. We use the structure theorem of $C^*(G)$. Remember exact sequences in the proof of Theorem 3. Now, if p is not in $\mathfrak{I}_0$, then $q_0(p)$ is a non-trivial projection in $C_0(\Omega_0)$, but $C_0(\Omega_0)$ has no non-trivial projections, which is a contradiction. So p is in $\mathfrak{I}_0$. Similarly, if p is not in $\mathfrak{I}_1$, then we have a contradiction. So p is in $\mathfrak{I}_1$. Repeating this process finitely, we have that p is in $\mathfrak{I}_{l-1}$, but $\mathfrak{I}_{l-1}$ has no non-trivial projections, which is a contradiction. Therefore, we conclude that $C^*(G)$ has no non-trivial projections. □

REMARK 3. In Theorem 4, if G is commutative, this result is evident since $C^*(G)$ is isomorphic to $C_0(\hat{G})$ and $\hat{G}$ is homeomorphic to the Euclidean space $\mathbf{R}^n$ where n is the dimension of G. Also, if G is an exponential Lie group, this result is false in general. For example, if G is a real $ax+b$ group, then $C^*(G)$ has the direct sum $\mathbf{K}\oplus\mathbf{K}$ as a closed ideal where $\mathbf{K}$ is the C^*-algebra consisting of all compact operators on a countably infinite dimensional Hilbert space. Therefore, $C^*(G)$ has a non-trivial projection. On the other hand, let E be an exponential Lie group and N a simply-connected, connected nilpotent Lie group and $G=N\times E$. Then $C^*(G)$ is isomorphic to $C^*(N)\otimes C^*(E)$. From the above structure theorem of $C^*(N)$, we have that $C^*(G)$ has no non-trivial projections.

REMARK 4. As an example of connected solvable Lie groups of non-type I, let G be the 5-dimensional Mautner group. It is of the form $\mathbf{C}^2\rtimes_\alpha\mathbf{R}$ where α is defined by $\alpha_t(z_1, z_2)=(e^{it}z_1, e^{it\theta}z_2)$, $t\in\mathbf{R}$, $z_1, z_2\in\mathbf{C}$, $\theta\in\mathbf{R}\backslash\mathbf{Q}$. Then it is known that $C^*(G)$ has a non-trivial projection.

Now, let G be a semi-simple Lie group and Ad(G) the adjoint group defined by the quotient of G by its center Z. We can consider the existence problem of non-trivial projections of $C^*(G)$. Then the following result is known:

THEOREM 5 [5]. *Let G be a real connected semisimple Lie group with finite center Z. Then the following statements are equivalent*:

(1) *The tensor product $C^*(G)\otimes\mathbf{K}$ has no non-trivial projections.*
(2) *$C^*(G)$ has no non-trivial minimal projections.*
(3) Ad(G) *has at least one simple factor which is isomorphic to the Lorentz group $SO_0(2n+1, 1)$ for some $n\geq 1$.*

From Remarks 3, 4 and Theorem 5, the next problems may be of independent interest:

PROBLEM. Let G be an exponential Lie group. Then describe the necessary and sufficient condition that $C^*(G)$ has no non-trivial projections in terms of the inner structure of G, and study the same thing in the case of type I Lie groups.

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