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Singular Limit of Some Quasilinear Wave Equations with Damping and Restoring Terms

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Abstract. A mixed problem for some hyperbolic equation with small parameter ε under the presence of a restoring term $|u|^{\alpha}u$ and a reduced problem for a parabolic type are considered. Several ε weighted energy estimates can be obtained by the method of difference quotients. It is shown that the solution u_{ε} of the mixed problem converges, uniformly on any finite time interval, to the solution u of the problem for the parabolic equation in an appropriate Hilbert space as $\varepsilon\to 0$.

Introduction.

Let $\Omega\subset\mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We now consider the mixed problem

$$\varepsilon^2 u_{tt}-M(\|\nabla u\|_2^2)\Delta u+\delta u_t+\mu|u|^{\alpha}u=0\,, \qquad t>0\,, \quad x\in\Omega\,, \tag{0.1}$$

$$u(0,x)=u_{0\varepsilon}(x)\,, \quad u_t(0,x)=u_{1\varepsilon}(x)\,, \qquad x\in\Omega\,, \tag{0.2}$$

$$u(t,x)\big|_{\partial\Omega}=0\,, \qquad t\geq 0\,. \tag{0.3}$$

Here ε is a positive parameter with $0<\varepsilon\leq 1$. $\delta>0$, $\mu>0$ and $\alpha>0$ are given constants. $M(s)$ is a positive C^1-function on $[0,\infty)$. $u=u(t,x)$ is an unknown real valued function on $[0,\infty)\times\Omega$. Δ is the Laplace operator in $\mathbf{R}^n$.

If ε is a fixed positive number, then Hosoya and Yamada [5] obtained the global solutions and decay properties for the equation of this kind. Their method depends on a kind of monotonicity of the so called restoring term $\mu|u|^{\alpha}u$. For the related results, see [4], [6], [7], [8] and the references therein.

Our purpose of this article is to study the behavior of the solution u_{ε} for the mixed problem (0.1)–(0.3) as $\varepsilon\to 0$. For the equation (0.1) with an inhomogeneous term instead of $\mu|u|^{\alpha}u$ in one space dimension, the singular limit was proved by Esham and Weinacht [2] in a classical sense on some local time interval. They prescribed initial data $u_{1\varepsilon}$ which was not uniform with respect to ε and required "boundary initial layer correctors" to obtain the uniform convergence in space and time. On the other hand, with regard

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to the ε-uniformity of the initial data $u_{1\varepsilon}$, the singular limit on any compact interval of time is known in Matsuyama [10]. In [10] we discussed a singular perturbation problem of "hyperbolic-parabolic type" for a class of damped nonlinear wave equations with a special type of quasilinear term as well as a cubic nonlinearity by constructing the stable set in three space dimensions. That is to say, we treated the equation (0.1) with $M(s)=a+2bs$ for $s\geq 0$ and a cubic blowing-up term μu^3. Here $a>0$, $b\geq 0$ and $\mu<0$ are given constants. However, it seems that the singular perturbation problem for a wider class of quasilinear wave equations is difficult to discuss globally in time by the method of [10] in general space dimensions, while, for the equation (0.1) with $\delta>0$ and $\mu>0$, we can use a kind of monotonicity of the term $\mu|u|^\alpha u$ and expect the global existence of the solution u_ε. Therefore, by virtue of the presence of such a restoring term $\mu|u|^\alpha u$ we will see more easily the ε dependency of the solution u_ε to the problem (0.1)–(0.3) than that of the blowing-up case [10]. If $M(s)\equiv 1$ and the initial data $\{u_{0\varepsilon}, u_{1\varepsilon}\}$ satisfies $u_{0\varepsilon}\in H_0^1(\Omega)$, $u_{1\varepsilon}\in L^2(\Omega)$, then there is a work of Benaouda and Madonne-Tort [1]. On the other hand, in the case of $M(s)\not\equiv 1$, we cannot expect such a regularity of initial data, since, in general, the problem (0.1)–(0.3) is not possibly globally solvable. Therefore, it is necessary for the present problem to enhance the regularity of the initial data to some extent. In the same way as [10], we derive ε weighted energy estimates of the equation (0.1) by using the method of difference quotients and evaluate the difference of the solutions in an appropriate Hilbert space.

We formulate the singular limit problem for the equation (0.1) precisely. Formally, letting $\varepsilon\to 0$, we can consider the reduced problem

$$\delta u_t - M(\|\nabla u\|_2^2)\Delta u+\mu|u|^\alpha u=0\,, \qquad t>0\,, \quad x\in\Omega\,, \tag{0.4}$$

$$u(0,x)=u_0(x)\,, \qquad x\in\Omega\,, \tag{0.5}$$

$$u(t,x)\,|_{\partial\Omega}=0\,, \qquad t\geq 0\,. \tag{0.6}$$

This suggests that the solution u_ε of the mixed problem (0.1)–(0.3) converges to the solution u of the mixed problem (0.4)–(0.6) as $\varepsilon\to 0$. Let n be an integer with $1\leq n\leq 3$. Then it is known in [10] that the mixed problem (0.4)–(0.6) has a unique global strong solution under the additional assumptions. However, since we shall discuss in general space dimensions, we cannot apply the result of [10] to the problem (0.4)–(0.6). So we shall prove the global solvability of the problem (0.4)–(0.6) by the Galerkin method.

Throughout this paper we impose the following assumptions on M, α, $u_{0\varepsilon}$, $u_{1\varepsilon}$ and u_0:

(A.1) $M\in C^1[0,\infty)$ and $M(s)\geq m_0>0$ for $s\geq 0$,

(A.2) $0<\alpha\leq 2/(n-4)$ if $n\geq 5$ and $0<\alpha<\infty$ if $n=1, 2, 3, 4$,

(A.3) $u_{0\varepsilon}\in H_0^1(\Omega)\cap H^2(\Omega)$ and $u_{1\varepsilon}\in H_0^1(\Omega)$,

(A.4) $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$.

Then we can state the singular limit of the mixed problem (0.1)–(0.3).

THEOREM. *Assume that* (A.1)–(A.4), $\delta>0$ *and* $\mu>0$. *Suppose that there exist* $\eta_0>0$

(*depending on* $\sup_{0<\varepsilon\le 1}\|\nabla u_{0\varepsilon}\|_2$, $\sup_{0<\varepsilon\le 1}\|u_{1\varepsilon}\|_2$, *and* $\sup_{0<\varepsilon\le 1}\|u_{0\varepsilon}\|_{\alpha+2}$) *and* $\eta_1>0$ (*depending on* $\|\nabla u_0\|_2$ *and* $\|u_0\|_{\alpha+2}$), *independent of* ε, *such that*

$$\sup_{0<\varepsilon\le 1}(\|\Delta u_{0\varepsilon}\|_2+\|\nabla u_{1\varepsilon}\|_2)<\eta_0\,, \tag{0.7}$$

$$\|\Delta u_0\|_2<\eta_1 \qquad \text{and} \tag{0.8}$$

$$u_{0\varepsilon}\to u_0 \qquad \text{strongly in } H_0^1(\Omega)\cap H^2(\Omega) \tag{0.9}$$

as $\varepsilon\to 0$. *Then, for any finite time interval* $[0, T]$ *the solution* u_ε *of the mixed problem* (0.1)–(0.3) *converges to the solution* u *of the mixed problem* (0.4)–(0.6) *strongly in* $L^\infty(0, T; H_0^1(\Omega))$ *as* $\varepsilon\to 0$. *Furthermore* u_ε' *converges to* u' *strongly in* $L^2((0, T)\times\Omega)$ *as* $\varepsilon\to 0$.

Our plan in this paper is as follows. In section 1 we derive ε weighted energy estimates following the results of Hosoya and Yamada [5]. In section 2 we give the global solvability of the mixed problem (0.4)–(0.5). In section 3 we show some energy estimates playing an important role in our argument and prove Theorem.

We conclude this section by stating several notations. Throughout this paper the functions considered are all real valued and the notations for their norms are adopted as usual ones (see, e.g., Lions [9]). Let $(\cdot,\cdot)$ denote the scalar product in $L^2(\Omega)$. $\|u\|_p$ stands for the usual $L^p(\Omega)$ norm of $u\in L^p(\Omega)$ and $\|u\|_{2,2}$ means $H^2(\Omega)$ norm of $u\in H^2(\Omega)$. $\max\{M(k)\,;\,0\le k\le s^2\}$ and $\max\{|M'(k)|\,;\,0\le k\le s^2\}$ are denoted by M_0 and M_1, respectively. We often suppress the space variable x when no confusion arises. Also we often abbreviate $(d/dt)u_\varepsilon(t)$, $(d^2/dt^2)u_\varepsilon(t)$ and $(d/dt)u(t)$ to $u_\varepsilon'(t)$, $u_\varepsilon''(t)$ and $u_t(t)$ (or $u'(t)$), respectively.

1. The hyperbolic problem.

In this section we shall derive the ε weighted energy estimates of the solution for the mixed problem (0.1)–(0.3) following Hosoya et al. [5].

First we formulate our problem precisely. In [5] it was shown that the mixed problem (0.1)–(0.3) has a unique global solution.

PROPOSITION 1.1. *Assume that* (A.1), (A.2), (A.3), $\delta>0$ *and* $\mu>0$. *Let* ε *be a fixed number with* $0<\varepsilon\le 1$. *Then there exists a number* $\eta_0>0$ (*depending on* $\sup_{0<\varepsilon\le 1}\|\nabla u_{0\varepsilon}\|_2$, $\sup_{0<\varepsilon\le 1}\|u_{1\varepsilon}\|_2$ *and* $\sup_{0<\varepsilon\le 1}\|u_{0\varepsilon}\|_{\alpha+2}$) *such that, if*

$$\sup_{0<\varepsilon\le 1}\{\|\Delta u_{0\varepsilon}\|_2+\|\nabla u_{1\varepsilon}\|_2\}<\eta_0\,,$$

then the problem (0.1)–(0.3) *has a unique solution* $u_\varepsilon(t, x)$ *which satisfies*

$$u_\varepsilon\in C([0,\infty)\,;\,H_0^1(\Omega))\cap C_w([0,\infty)\,;\,H^2(\Omega))\,, \tag{1.1}$$

$$u_\varepsilon'\in C([0,\infty)\,;\,L^2(\Omega))\cap C_w([0,\infty)\,;\,H_0^1(\Omega))\,, \tag{1.2}$$

(1.3) $$u_\varepsilon'' \in C_w([0, \infty)\,;\, L^2(\Omega))\,,$$

where the subscript "w" means the weak continuity with respect to t.

LEMMA 1.2. *Let $u_\varepsilon(t, x)$ be as in Proposition 1.1. Then there exists a positive constant C_1 (depending on $\sup_{0<\varepsilon\leq 1}\|\nabla u_{0\varepsilon}\|_2$, $\sup_{0<\varepsilon\leq 1}\|u_{1\varepsilon}\|_2$ and $\sup_{0<\varepsilon\leq 1}\|u_{0\varepsilon}\|_{\alpha+2}$) independent of ε such that*

(1.4) $$\varepsilon^2\|u_\varepsilon'(t)\|_2^2+\|\nabla u_\varepsilon(t)\|_2^2+\|u_\varepsilon(t)\|_{\alpha+2}^{\alpha+2}+\int_0^t\|u_\varepsilon'(t)\|_2^2 ds\leq C_1^2$$

for all $t\geq 0$.

PROOF. Multiplying Eq. (0.1) by $u_\varepsilon'(t)$, we have

(1.5) $$\begin{aligned}\varepsilon^2(u_\varepsilon''(t), u_\varepsilon'(t))+M(\|\nabla u_\varepsilon(t)\|_2^2)(\nabla u_\varepsilon(t), \nabla u_\varepsilon'(t))\\ +\mu(|u_\varepsilon(t)|^\alpha u_\varepsilon(t), u_\varepsilon'(t))+\delta\|u_\varepsilon'(t)\|_2^2=0\,.\end{aligned}$$

We integrate (1.5) over $[0, T]$ to obtain

(1.6) $$\begin{aligned}&\frac{\varepsilon^2}{2}\|u_\varepsilon'(t)\|_2^2+\frac{1}{2}\bar{M}(\|\nabla u_\varepsilon(t)\|_2^2)+\frac{\mu}{\alpha+2}\|u_\varepsilon(t)\|_{\alpha+2}^{\alpha+2}+\delta\int_0^t\|u_\varepsilon'(s)\|_2^2 ds\\ &=\frac{\varepsilon^2}{2}\|u_{1\varepsilon}\|_2^2+\frac{1}{2}\bar{M}(\|\nabla u_{0\varepsilon}\|_2^2)+\frac{\mu}{\alpha+2}\|u_{0\varepsilon}\|_{\alpha+2}^{\alpha+2}\,,\qquad t\in[0, T]\,,\end{aligned}$$

where $\bar{M}(s)=\int_0^s M(r)dr$. By (A.2) and Sobolev's lemma, $H^2(\Omega)$ is imbedded in $L^{\alpha+2}(\Omega)$. Then the right hand side of (1.6) is bounded by a positive constant independent of ε. Since we have the assumption (A.1), (1.4) easily follows. This ends the proof of Lemma 1.2. □

We next derive the ε weighted higher order derivatives of u_ε.

LEMMA 1.3. *Let $u_\varepsilon(t, x)$ be as in Lemma 1.2. Then there exists a positive constant η_0 depending on $\sup_{0<\varepsilon\leq 1}\|\nabla u_{0\varepsilon}\|_2$, $\sup_{0<\varepsilon\leq 1}\|u_{1\varepsilon}\|_2$ and $\sup_{0<\varepsilon\leq 1}\|u_{0\varepsilon}\|_{\alpha+2}$ such that, if*

$$\sup_{0<\varepsilon\leq 1}\{\|\Delta u_{0\varepsilon}\|_2+\|\nabla u_{1\varepsilon}\|_2\}<\eta_0\,,$$

then

(1.7) $$\|\Delta u_\varepsilon(t)\|_2+\varepsilon\|\nabla u_\varepsilon'(t)\|_2\leq C_2 \qquad \text{for all} \quad t\geq 0\,,$$

(1.8) $$\int_0^t \nabla u_\varepsilon'(s)\|_2^2 ds\leq C_3 \qquad \text{for all} \quad t\geq 0$$

with some positive constants C_2 and C_3 independent of ε.

PROOF. Let

$$Z_\varepsilon(t) \equiv \varepsilon^2 \|\nabla u'_\varepsilon(t)\|_2^2 + M(\|\nabla u_\varepsilon(t)\|_2^2)\|\Delta u_\varepsilon(t)\|_2^2 .$$

Then, using the idea of Yosida approximations, we can apply the same derivation as in [5, Lemma 3.3] to Eq. (0.1) to obtain

$$(1.9)\qquad Z_\varepsilon(t) + \frac{3\delta}{2}\int_0^t \|\nabla u'_\varepsilon(s)\|_2^2 ds$$
$$\le Z_\varepsilon(0) + \frac{C_1^2}{2\delta}\int_0^t \|\Delta u_\varepsilon(s)\|_2^{2\alpha+2} ds + 2C_2 M_1 \int_0^t \|\Delta u_\varepsilon(s)\|_2^2 \|\nabla u'_\varepsilon(s)\|_2 ds$$

and

$$(1.10)\qquad \varepsilon^2\delta(\nabla u'_\varepsilon(t), \nabla u_\varepsilon(t)) + \frac{\delta^2}{2}\|\nabla u_\varepsilon(t)\|_2^2 + m_0\delta\int_0^t \|\Delta u_\varepsilon(s)\|_2^2 ds$$
$$\le H_\varepsilon + \varepsilon^2\delta\int_0^t \|\nabla u'_\varepsilon(s)\|_2^2 ds ,$$

where H_ε is defined by

$$(1.11)\qquad H_\varepsilon = \varepsilon^2\delta(\nabla u_{1\varepsilon}, \nabla u_{0\varepsilon}) + \frac{\delta^2}{2}\|\nabla u_{0\varepsilon}\|_2^2 .$$

Now we set

$$W_\varepsilon(t) \equiv Z_\varepsilon(t) + \varepsilon^2\delta(\nabla u'_\varepsilon(t), \nabla u_\varepsilon(t)) + \frac{\delta^2}{2}\|\nabla u_\varepsilon(t)\|_2^2 .$$

Then we will derive the following estimates:

$$(1.12)\qquad W_\varepsilon(t) + \frac{\delta}{4}\int_0^t \|\nabla u'_\varepsilon(s)\|_2^2 ds \le W_\varepsilon(0) + \int_0^t E_\varepsilon(s)\|\Delta u_\varepsilon(s)\|_2^2 ds \qquad \text{and}$$

$$(1.13)\qquad C_0\{\varepsilon^2\|\nabla u'_\varepsilon(t)\|_2^2 + \|\Delta u_\varepsilon(t)\|_2^2\} \le W_\varepsilon(t)$$

for some positive constant C_0, where $E_\varepsilon(s)$ is defined by

$$(1.14)\qquad E_\varepsilon(s) = \frac{C_1^2}{2\delta}\|\Delta u_\varepsilon(s)\|_2^{2\alpha} + \frac{4C_2M_1}{\delta}\|\Delta u_\varepsilon(s)\|_2^2 - m_0\delta .$$

For the derivation of (1.12) and (1.13), noting the definition of $W_\varepsilon(t)$ we add (1.9) to (1.10) to obtain

$$(1.15)\qquad W_\varepsilon(t)+\frac{3\delta}{2}\int_0^t \|\nabla u_\varepsilon'(s)\|_2^2 ds+m_0\delta\int_0^t \|\Delta u_\varepsilon(s)\|_2^2 ds$$
$$\leq W_\varepsilon(0)+\frac{C_1^2}{2\delta}\int_0^t \|\Delta u_\varepsilon(s)\|_2^{2\alpha+2}ds+2C_2M_1\int_0^t \|\Delta u_\varepsilon(t)\|_2^2\|\nabla u_\varepsilon'(s)\|_2 ds$$
$$+\varepsilon^2\delta\int_0^t \|\nabla u_\varepsilon'(s)\|_2^2 ds$$
$$\leq W_\varepsilon(0)+\frac{C_1^2}{2\delta}\int_0^t \|\Delta u_\varepsilon(s)\|_2^{2\alpha+2}ds+\frac{4C_2M_1}{\delta}\int_0^t \|\Delta u_\varepsilon(s)\|_2^4 ds$$
$$+\frac{\delta}{4}\int_0^t \|\nabla u_\varepsilon'(s)\|_2^2 ds+\delta\int_0^t \|\nabla u_\varepsilon'(s)\|_2^2 ds\,.$$

Hence the desired estimate (1.12) is equivalent to (1.15).

We note that $M(\|\nabla u_\varepsilon(t)\|_2^2)\geq m_0$. Then it is easy to see that

$$(1.16)\qquad Z_\varepsilon(t)\geq\varepsilon^2\|\nabla u_\varepsilon'(t)\|_2^2+m_0\|\Delta u_\varepsilon(t)\|_2^2\,.$$

We observe here that

$$(1.17)\qquad \varepsilon^2\delta(\nabla u_\varepsilon'(t),\nabla u_\varepsilon(t))\geq-\varepsilon^2\delta\|\nabla u_\varepsilon'(t)\|_2\|\nabla u_\varepsilon(t)\|_2$$
$$\geq-\frac{\varepsilon^4}{2}\|\nabla u_\varepsilon'(t)\|_2^2-\frac{\delta^2}{2}\|\nabla u_\varepsilon(t)\|_2^2$$
$$\geq-\frac{\varepsilon^2}{2}\|\nabla u_\varepsilon'(t)\|_2^2-\frac{\delta^2}{2}\|\nabla u_\varepsilon(t)\|_2^2\,.$$

Hence from (1.16) and (1.17) we deduce that

$$(1.18)\qquad W_\varepsilon(t)\geq\frac{\varepsilon^2}{2}\|\nabla u_\varepsilon'(t)\|_2^2+m_0\|\Delta u_\varepsilon(t)\|_2^2\,.$$

If we choose C_0 so that $C_0=\min\{1/2,m_0\}$, then (1.13) can be obtained.

We are ready to deduce ε weighted energy estimates. Take the initial data $\{u_{0\varepsilon},u_{1\varepsilon}\}$ satisfying

$$(1.19)\qquad \sup_{0<\varepsilon\leq1}\left\{\frac{C_1^2}{2\delta}\|\Delta u_{0\varepsilon}\|_2^{2\alpha}+\frac{4C_2M_1}{\delta}\|\Delta u_{0\varepsilon}\|_2^2\right\}\leq\frac{m_0\delta}{4}\qquad\text{and}$$

$$(1.20)\qquad \sup_{0<\varepsilon\leq1}\left\{\frac{C_1^2}{2\delta}\left(\frac{W_\varepsilon(0)}{C_0}\right)^\alpha+\frac{4C_2M_1}{\delta}\frac{W_\varepsilon(0)}{C_0}\right\}<\frac{m_0\delta}{2}\,.$$

We will show

$$\sup_{0<\varepsilon\leq 1}\left\{\frac{C_1^2}{2\delta}\|\Delta u_\varepsilon(t)\|_2^{2\alpha}+\frac{4C_2M_1}{\delta}\|\Delta u_\varepsilon(t)\|_2^2\right\}<\frac{m_0\delta}{2} \tag{1.21}$$

for all $t\geq 0$. Suppose that there exists some positive number t^* such that

$$\frac{C_1^2}{2\delta}\|\Delta u_\varepsilon(t)\|_2^{2\alpha}+\frac{4C_2M_1}{\delta}\|\Delta u_\varepsilon(t)\|_2^2<\frac{m_0\delta}{2} \quad \text{on } [0, t^*) \qquad \text{and} \tag{1.22}$$

$$\frac{C_1^2}{2\delta}\|\Delta u_\varepsilon(t^*)\|_2^{2\alpha}+\frac{4C_2M_1}{\delta}\|\Delta u_\varepsilon(t^*)\|_2^2=\frac{m_0\delta}{2}. \tag{1.23}$$

Then it follows from (1.12) with $t=t^*$ and (1.22) that

$$W_\varepsilon(t^*)+\frac{\delta}{4}\int_0^{t^*}\|\nabla u_\varepsilon'(s)\|_2^2 ds\leq W_\varepsilon(0). \tag{1.24}$$

Furthermore using (1.13) we have

$$\|\Delta u_\varepsilon(t^*)\|_2^2\leq\frac{W_\varepsilon(t^*)}{C_0}\leq\frac{W_\varepsilon(0)}{C_0}. \tag{1.25}$$

Therefore, combining (1.24) with (1.25) we get

$$\frac{C_1^2}{2\delta}\|\Delta u_\varepsilon(t^*)\|_2^{2\alpha}+\frac{4C_2M_1}{\delta}\|\Delta u_\varepsilon(t^*)\|_2^2\leq\frac{C_1^2}{2\delta}\left(\frac{W_\varepsilon(0)}{C_0}\right)^\alpha+\frac{4C_2M_1}{\delta}\frac{W_\varepsilon(0)}{C_0}<\frac{m_0\delta}{2}, \tag{1.26}$$

which contradicts (1.23).

If $\|\Delta u_{0\varepsilon}\|_2$ and $\|\nabla u_{1\varepsilon}\|_2$ are sufficiently small, one can see that (1.19) and (1.20) are valid. Hence (1.7) follows from (1.21). If we use (1.12) and (1.21), then (1.8) is also valid. This completes the proof of Lemma 1.3. □

2. The parabolic problem.

In this section we describe the unique global solvability of the parabolic equation by using the Galerkin method. The proof of the global existence is the routine work of Hosoya et al. [5]. So we shall only describe the key estimates.

Let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of eigenvalues for

$$-\Delta w=\lambda w \quad \text{in } \Omega \qquad \text{and} \qquad w=0 \quad \text{on } \partial\Omega.$$

Let $w_j\in H_0^1(\Omega)\cap H^2(\Omega)$ be the corresponding eigenfunction to λ_j and take $\{w_j\}_{j=1}^\infty$ as a completely orthonormal system in $L^2(\Omega)$. We construct approximate solution u_m $(m=1, 2, \cdots)$ in the form

$$u_m(t)=\sum_{j=1}^m g_{jm}(t)w_j,$$

where g_{jm} $(j=1, 2, \cdots, m)$ are determined by

$$(2.1)\qquad \delta(u_m'(t), w_j)+M(\|\nabla u_m(t)\|^2)(\nabla u_m(t), \nabla w_j) + \mu(|u_m(t)|^\alpha u_m(t), w_j)=0\,, \qquad (j=1, 2, \cdots, m)\,,$$

with the initial condition

$$(2.2)\qquad u_m(0)=u_{0m}\equiv \sum_{j=1}^{m}(u_0, w_j)w_j \to u_0 \quad \text{in} \quad H_0^1(\Omega)\cap H^2(\Omega) \qquad \text{as} \quad m\to\infty\,.$$

Then the system (2.1) has a unique C^0-class solution $u_m(t)$ on some interval $[0, T_m)$, since the second term in the left hand side of (2.1) is locally Lipschitz continuous. Note that $u_m'(t)$ is absolutely continuous. So (2.1) holds a.e. in $[0, T_m)$. On the other hand, $u_m(t)$ can be extended to $[0, \infty)$. In fact we have the following lemmas.

First we have

LEMMA 2.1. *Let $u_m(t)$ be a solution of* (2.1), (2.2). *Then there exists a positive constant C_1 depending on $\|\nabla u_0\|_2$ and $\|u_0\|_{\alpha+2}$ such that*

$$(2.3)\qquad \int_0^t \|u'(s)\|_2^2 ds+\|\nabla u(t)\|_2^2+\|u(t)\|_{\alpha+2}^{\alpha+2}\le C_1^2$$

for all $t\ge 0$.

Now we set $W(t)$ as

$$(2.4)\qquad W(t)\equiv M(\|\nabla u(t)\|_2^2)\|\Delta u(t)\|_2^2+\frac{\delta}{2}\|\nabla u(t)\|_2^2\,.$$

Then we have the following.

LEMMA 2.2. *Let $u_m(t)$ be as in Lemma* 2.1. *Then we have the following estimates*

$$(2.5)\qquad \frac{d}{dt}W(t)\le E(t)\|\Delta u_m(t)\|_2^2 \qquad \textit{and}$$

$$(2.6)\qquad m_0\|\Delta u_m(t)\|_2^2\le W(t)\le C_3\|\Delta u_m(t)\|_2^2$$

for some constant $C_3>0$, where $E(s)$ is defined by

$$(2.7)\qquad E(t)=\frac{2M_0M_1}{\delta}\|\Delta u_m(t)\|_2^2+\frac{2\mu C_1M_1}{\delta}\|\Delta u_m(t)\|_2^{\alpha+1}+\frac{2C_2\mu^2}{\delta}\|\Delta u_m(t)\|_2^{2\alpha}-m_0\,.$$

We now have the smallness of initial data u_0 as follows. Let $u_m(t)$ be as in Lemma 3.1 with initial data u_0 satisfying

$$(2.8)\qquad \frac{2M_0M_1}{\delta}\|\Delta u_0\|_2^2+\frac{2\mu C_1M_1}{\delta}\|\Delta u_0\|_2^{\alpha+1}+\frac{2C_2\mu^2}{\delta}\|\Delta u_0\|_2^{2\alpha}<\frac{m_0}{4}\,,$$

(2.9) $$\frac{2M_0M_1}{\delta}\frac{W(0)}{m_0}+\frac{2\mu C_1M_1}{\delta}\left(\frac{W(0)}{m_0}\right)^{(\alpha+1)/2}+\frac{2C_2\mu^2}{\delta}\left(\frac{W(0)}{m_0}\right)^{\alpha}<\frac{m_0}{2}.$$

Then we can show that

(2.10) $$\frac{2M_0M_1}{\delta}\|\Delta u_m(t)\|_2^2+\frac{2\mu C_1M_1}{\delta}\|\Delta u_m(t)\|_2^{\alpha+1}+\frac{2C_2\mu^2}{\delta}\|\Delta u_m(t)\|_2^{2\alpha}<\frac{m_0}{2}$$

for all $t\geq 0$. Hence we can pass to the limit: $m\to\infty$. Therefore we get the following theorem concerning the global existence of the problem (0.4)–(0.6).

THEOREM 2.3. *Suppose that* (A.1), (A.2), (A.4), $\delta>0$ *and* $\mu>0$. *Then there exists a number* $\eta_1>0$ (*depending on* $\|\nabla u_0\|_2$ *and* $\|u_0\|_{\alpha+2}$) *such that, if* u_0 *satisfies* $\|\Delta u_0\|_2<\eta_1$, *then the problem* (0.4)–(0.6) *has a unique solution* $u(t, x)$ *such that*

(2.11) $$u\in L^\infty(0,\infty\,;H_0^1(\Omega)\cap H^2(\Omega)),$$

(2.12) $$u'\in L^\infty(0,\infty\,;L^2(\Omega))\cap L^2((0,T)\times\Omega).$$

3. Proof of Theorem.

In this section we derive several energy estimates of the equations (0.1) and (0.4) needed for our argument and prove Theorem. In the course of calculations below, various constants are simply denoted by C and change from line to line.

First we have the following estimates for (0.4).

LEMMA 3.1. *Let* $u(t, x)$ *be a solution of the problem* (0.4)–(0.6) *with initial data* u_0 *which satisfy the same assumption as in Theorem* 2.3. *Then, for any* $T>0$ *we have the following estimates*

(3.1) $$\|\nabla u\|_{L^\infty(0,T;L^2(\Omega))}+\|\Delta u\|_{L^\infty(0,T;L^2(\Omega))}\leq C,\qquad t\in[0,T],$$

(3.2) $$\left\|\frac{d}{dt}u(t)\right\|_{L^2((0,T)\times\Omega)}+\left\|\frac{d}{dt}u(t)\right\|_{L^\infty(0,T;L^2(\Omega))}\leq C,\qquad t\in[0,T].$$

PROOF. This is an immediate consequence of Theorem 2.3. □

We next derive a series of energy estimates for (0.1).

LEMMA 3.2. *Let* $u_\varepsilon(t, x)$ *be as in Lemma* 1.3. *Then, for any* $T>0$ *we have the following estimates*

(3.3) $$\sup_{0<\varepsilon\leq 1}\|\nabla u_\varepsilon\|_{C([0,T];L^2(\Omega))}+\sup_{0<\varepsilon\leq 1}\|\Delta u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\leq C,$$

(3.4) $$\sup_{0<\varepsilon\leq 1}\|\nabla u_\varepsilon'\|_{L^2((0,T)\times\Omega)}\leq C,$$

(3.5) $$\sup_{0<\varepsilon\le 1} \varepsilon\|\nabla u_\varepsilon'\|_{L^\infty(0,T;L^2(\Omega))}\le C\,,$$

(3.6) $$\sup_{0<\varepsilon\le 1} \|u_\varepsilon'\|_{L^2((0,T)\times\Omega)}\le C\,.$$

PROOF. This is an immediate consequence of Lemma 1.3. □

We set the nonlinear operator f by $f(u)\equiv|u|^\alpha u$. Then the following estimates are essential to the proof of main theorem.

LEMMA 3.3. *Let $u_\varepsilon(t, x)$ be as in Lemma 1.3. Then we have the following estimates*

(3.7) $$\sup_{0<\varepsilon\le 1}\left\|\frac{d}{dt}u_\varepsilon\right\|_{L^\infty(0,T;L^2(\Omega))}\le C\,,$$

(3.8) $$\sup_{0<\varepsilon\le 1}\varepsilon^2\left\|\frac{d^2}{dt^2}u_\varepsilon\right\|_{L^\infty(0,T;L^2(\Omega))}\le C\,,$$

(3.9) $$\sup_{0<\varepsilon\le 1}\varepsilon\left\|\frac{d^2}{dt^2}u_\varepsilon\right\|_{L^2((0,T)\times\Omega)}\le C$$

for any T with $T>0$.

PROOF. In the sequel we suppress ε of u_ε. In view of Eq. (0.1) and using the imbedding $H^2(\Omega)\subset L^{N\alpha}(\Omega)$ we have

(3.10) $$\begin{aligned}\delta\|u_t(t)\|_2&\le\varepsilon^2\|u_{tt}(t)\|_2+M(\|\nabla u(t)\|_2^2)\|\Delta u(t)\|_2+\mu\,\||u(t)|^\alpha u(t)\|_2\\&\le\varepsilon^2\|u_{tt}(t)\|_2+M(\|\nabla u(t)\|_2^2)\|\Delta u(t)\|_2+C\|u(t)\|_{N\alpha}^\alpha\|u(t)\|_{2N/(N-2)}\\&\le\varepsilon^2\|u_{tt}(t)\|_2+C\|\Delta u(t)\|_2^\alpha\|\nabla u(t)\|_2\,,\end{aligned}$$

from which it is sufficient to prove (3.8).

Now we shall use a method of difference quotients because u_ε is not smooth enough. Let h be a positive real number and let t be a number with $0<t<T-h$. For each function $w:[0, T]\times\Omega\to\mathbf{R}$ we denote by $\tau_h w(t)$ the difference quotient $(1/h)(w(t+h)-w(t))$. Then we subtract Eq. (0.1) at $t+h$ from Eq. (0.1) at t and take the $L^2(\Omega)$ inner product of them with $2\tau_h u_t(t)$ to obtain

(3.11) $$\begin{aligned}&\varepsilon^2\frac{d}{dt}\|\tau_h u_t(t)\|_2^2+\frac{d}{dt}[M(\|\nabla u(t+h)\|_2^2)\|\tau_h\nabla u(t)\|_2^2]+2\delta\|\tau_h u_t(t)\|_2^2\\&=M'(\|\nabla u(t+h)\|_2^2)(\nabla u_t(t+h),\,\nabla u(t+h))\|\tau_h\nabla u(t)\|_2^2\\&\quad+2[\tau_h M(\|\nabla u(t)\|_2^2)](\Delta u(t+h),\,\tau_h u_t(t))-2\mu(\tau_h f(u(t)),\,\tau_h u_t(t))\,.\end{aligned}$$

Hence from (3.3) we get

$$\varepsilon^2 \frac{d}{dt}\|\tau_h u_t(t)\|_2^2 + \frac{d}{dt}[M(\|\nabla u(t+h)\|_2^2)\|\tau_h \nabla u(t)\|_2^2] + 2\delta\|\tau_h u_t(t)\|_2^2 \tag{3.12}$$

$$\leq CM_1\|\nabla u_t(t+h)\|_2\|\tau_h\nabla u(t)\|_2^2 + 2CM_1\|\tau_h\nabla u(t)\|_2\|\Delta u(t+h)\|_2\|\tau_h u_t(t)\|_2$$

$$+2\mu|(\tau_h f(u(t)), \tau_h u_t(t))|.$$

We note here that $\tau_h f(u(t))$ is bounded by

$$|\tau_h f(u(t))| \leq C(|u(t+h)|+|u(t)|)^\alpha |\tau_h u(t)| \tag{3.13}$$

because we have

$$|f(u)-f(v)| \leq C(|u|+|v|)^\alpha |u-v| \tag{3.14}$$

for any $u, v \in \mathbf{R}$. Then the last term in the right hand side of (3.12) is estimated by

$$2\mu|(\tau_h f(u(t)), \tau_h u_t(t))| \leq 2\mu \int_\Omega |\tau_h f(u(t))||\tau_h u_t(t)|\,dx \tag{3.15}$$

$$\leq C\int_\Omega (|u(t+h)|+|u(t)|)^\alpha |\tau_h u(t)||\tau_h u_t(t)|\,dx$$

$$\leq C(\|u(t+h)\|_{q\alpha}^\alpha + \|u(t)\|_{q\alpha}^\alpha)\|\tau_h u(t)\|_r \|\tau_h u_t(t)\|_2,$$

with $2/q+2/r=1$, where we have taken q and r such that

$$\frac{1}{q\alpha} \geq \frac{1}{2}-\frac{2}{n}, \qquad \frac{1}{r} \geq \frac{1}{2}-\frac{1}{n}, \tag{3.16}$$

which is possible on account of (A.2). From Sobolev's inequality, the regularity theory of elliptic equations and (3.3) it follows that

$$\|u(t)\|_{q\alpha} \leq C\|u(t)\|_{2,2} \leq C\|\Delta u(t)\|_2 \leq C, \tag{3.17}$$

$$\|\tau_h u(t)\|_r \leq C\|\tau_h \nabla u(t)\|_2. \tag{3.18}$$

Hence we get

$$2\mu|(\tau_h f(u(t)), \tau_h u_t(t))| \leq C(\|\Delta u(t+h)\|_2^\alpha + \|\Delta u(t)\|_2^\alpha)\|\tau_h \nabla u(t)\|_2 \|\tau_h u_t(t)\|_2 \tag{3.19}$$

$$\leq C\|\tau_h\nabla u(t)\|_2\|\tau_h u_t(t)\|_2$$

$$\leq \frac{C}{\varepsilon^2}\|\tau_h \nabla u(t)\|_2^2 + \frac{\varepsilon^2}{2}\|\tau_h u_t(t)\|_2^2.$$

Also the second term in the right hand side of (3.12) is estimated by

$$\frac{C}{\varepsilon^2}\|\tau_h \nabla u(t)\|_2^2 + \frac{\varepsilon^2}{2}\|\tau_h u_t(t)\|_2^2.$$

Hence from (3.12) and (3.19) we have

$$\varepsilon^2 \frac{d}{dt} \|\tau_h u_t(t)\|_2^2 + \frac{d}{dt} [M(\|\nabla u(t+h)\|_2^2)\|\tau_h \nabla u(t)\|_2^2] + 2\delta \|\tau_h u_t(t)\|_2^2 \tag{3.20}$$

$$\leq C\|\nabla u_t(t+h)\|_2 \|\tau_h \nabla u(t)\|_2^2 + \frac{C}{\varepsilon^2} \|\tau_h \nabla u(t)\|_2^2 + \varepsilon^2 \|\tau_h u_t(t)\|_2^2 .$$

Multiplying ε^2 to the both hand sides of (3.20) and integrating over $[0, T]$, we deduce from the property of the difference quotients, (3.4), (3.5) and (3.6) that

$$\varepsilon^4 \|\tau_h u_t(t)\|_2^2 + \varepsilon^2 M(\|\nabla u(t+h)\|_2^2)\|\tau_h \nabla u(t)\|_2^2 + 2\varepsilon^2 \delta \int_0^t \|\tau_h u_t(s)\|_2^2 ds \tag{3.21}$$

$$\leq C(h) + C\|\tau_h \nabla u\|_{L^2(0,T;L^2(\Omega))}^2 + C\varepsilon^2 \int_0^t \|\nabla u_t(s+h)\|_2 \|\tau_h \nabla u(s)\|_2^2 ds$$

$$+ \varepsilon^4 \int_0^t \|\tau_h u_t(s)\|_2^2 ds , \qquad t \in (0, T-h) ,$$

where $C(h) = \sup_{0<\varepsilon\leq 1} [\varepsilon^4 \|\tau_h u_t(0)\|_2^2 + \varepsilon^2 M(\|\nabla u(h)\|_2^2)\|\tau_h \nabla u(0)\|_2^2]$. From our hypothesis of the smallness of the initial data it is easy to see that $C(h)$ is bounded by a positive constant C independent of ε and h. Furthermore we note that $\|\tau_h \nabla u\|_{L^2((0,T)\times\Omega)}$ are bounded by a positive constant C independent of ε and h since we have (3.4) and (3.6). Then the third term in the right hand side of (3.21) is estimated by

$$C\varepsilon^2 \|\nabla u_t\|_{L^\infty(0,T;L^2(\Omega))} \|\tau_h \nabla u\|_{L^2((0,T)\times\Omega)}^2 \leq C\|\tau_h \nabla u\|_{L^2((0,T)\times\Omega)}^2 \leq C \tag{3.22}$$

because of (3.5). Thus combining (3.21) with (3.22) and using (3.4), we have

$$\varepsilon^4 \|\tau_h u_t(t)\|_2^2 + 2\varepsilon^2 \delta \int_0^t \|\tau_h u_t(s)\|_2^2 ds \leq C + \varepsilon^4 \int_0^t \|\tau_h u_t(s)\|_2^2 ds , \qquad t \in [0, T] . \tag{3.23}$$

Therefore, applying Gronwall's inequality to (3.23) and letting $h \to 0$, we deduce that

$$\varepsilon^4 \|u_{tt}(t)\|_2^2 \leq C \qquad \text{on} \quad [0, T] ,$$

$$\varepsilon^2 \int_0^t \|u_{tt}(s)\|^2 ds \leq C \qquad \text{on} \quad [0, T] .$$

This ends the proof of Lemma 3.3. □

PROOF OF THEOREM COMPLETED. We are now in a position to prove our main theorem. At first we subtract Eq. (0.4) from Eq. (0.1) to obtain

$$\varepsilon^2 \frac{d^2 u_\varepsilon}{dt^2} - M(\|\nabla u_\varepsilon\|_2^2)(\Delta u_\varepsilon - \Delta u) + \delta\left(\frac{du_\varepsilon}{dt} - \frac{du}{dt}\right) \tag{3.24}$$

$$= [M(\|\nabla u_\varepsilon\|_2^2) - M(\|\nabla u\|_2^2)]\Delta u - \mu(f(u_\varepsilon) - f(u)) .$$

We take the $L^2(\Omega)$ inner product of (3.24) with $2(u_\varepsilon'-u')$ to obtain

$$(3.25)\quad \frac{d}{dt}[M(\|\nabla u_\varepsilon\|_2^2)\|\nabla u_\varepsilon-\nabla u\|_2^2]+2\delta\|u_\varepsilon'-u'\|_2^2$$

$$\leq\left|\frac{d}{dt}M(\|\nabla u_\varepsilon\|_2^2)\right|\|\nabla u_\varepsilon-\nabla u\|_2^2+2M_1(\|\nabla u_\varepsilon\|_2+\|\nabla u\|_2)\|\Delta u\|_2\|\nabla u_\varepsilon-\nabla u\|_2\|u_\varepsilon'-u'\|_2$$

$$+2\mu\|f(u_\varepsilon)-f(u)\|_2\|u_\varepsilon'-u'\|_2+2\varepsilon^2\|u_\varepsilon''\|_2\|u_\varepsilon'-u'\|_2 .$$

By (3.13) and Hölder's inequality we have

$$(3.26)\qquad \|f(u_\varepsilon)-f(u)\|_2\leq C(\|u_\varepsilon\|_{q\alpha}^\alpha+\|u\|_{q\alpha}^\alpha)\|u_\varepsilon-u\|_r$$

with $2/q+2/r=1$. Here we have chosen q and r as in (3.16). By (3.17), (3.18), (3.3) and (3.26) we have

$$(3.27)\qquad \begin{aligned}\|f(u_\varepsilon)-f(u)\|_2&\leq C(\|\Delta u_\varepsilon\|_2^\alpha+\|\Delta u\|_2^\alpha)\|\nabla u_\varepsilon-\nabla u\|_2\\ &\leq C\|\nabla u_\varepsilon-\nabla u\|_2 .\end{aligned}$$

Here we note that

$$(3.28)\qquad \left|\frac{d}{dt}M(\|\nabla u_\varepsilon\|_2^2)\right|\leq 2M_1|(\nabla u_\varepsilon,\nabla u_\varepsilon')|=2M_1|(\Delta u_\varepsilon,u_\varepsilon')|\leq 2M_1\|\Delta u_\varepsilon\|_2\|u_\varepsilon'\|_2\leq C$$

because of (3.3) and (3.7). Then from (3.27), (3.28) and Poincaré's inequality it follows that

$$(3.29)\qquad \frac{d}{dt}[M(\|\nabla u_\varepsilon\|_2^2)\|\nabla u_\varepsilon-\nabla u\|_2^2]+2\delta\|u_\varepsilon'-u'\|_2^2$$

$$\leq C\|\nabla u_\varepsilon-\nabla u\|_2^2+C\|\nabla u_\varepsilon-\nabla u\|_2\|u_\varepsilon'-u'\|_2+\frac{2\varepsilon^4}{\delta}\|u_\varepsilon''\|_2^2+\frac{\delta}{2}\|u_\varepsilon'-u'\|_2^2$$

$$\leq C\|\nabla u_\varepsilon-\nabla u\|_2^2+\delta\|u_\varepsilon'-u'\|_2^2+\frac{2\varepsilon^4}{\delta}\|u_\varepsilon''\|_2^2 .$$

Hence using $M(\|\nabla u_\varepsilon\|_2^2)\geq m_0$ and (3.9) we integrate over $[0, T]$ to get

$$(3.30)\quad m_0\|\nabla u_\varepsilon-\nabla u\|_2^2+\delta\int_0^t\|u_\varepsilon'(s)-u'(s)\|_2^2ds$$

$$\leq C\varepsilon^2+M(\|\nabla u_{0\varepsilon}\|_2^2)\|\nabla u_{0\varepsilon}-\nabla u_0\|_2^2+C\int_0^t\|\nabla u_\varepsilon(s)-\nabla u(s)\|_2^2ds ,\qquad t\in[0, T] .$$

Therefore we deduce from Gronwall's inequality that

$$(3.31)\qquad \|\nabla u_\varepsilon(t)-\nabla u(t)\|_2^2\leq C\varepsilon^2+C\|\nabla u_{0\varepsilon}-\nabla u_0\|_2^2 ,$$

$$\int_0^t \|u_\varepsilon'(s)-u'(s)\|_2^2 ds \le C\varepsilon^2 + C\|\nabla u_{0\varepsilon}-\nabla u_0\|_2^2 \tag{3.32}$$

on $[0, T]$. We thus conclude from (3.31) and (3.32) that

$$u_\varepsilon \to u \quad \text{strongly in } L^\infty(0, T; H_0^1(\Omega)),$$
$$u_\varepsilon' \to u' \quad \text{strongly in } L^2((0, T)\times\Omega)$$

as $\varepsilon\to 0$. This completes the proof of Theorem. □

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