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Extrinsic Characterizations of Circles in a Complex Projective Space Imbedded in a Euclidean Space

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0. Introduction.

It is well-known that a curve on a sphere S^2 in R^3 is a geodesic (that is, a great circle) or a (small) circle if and only if it is a circle as a curve in R^3. This can be considered as an *extrinsic* characterization of circles on S^2 in R^3.

On the other hand, Adachi, Udagawa and the second author ([1]) investigate circles in a complex projective space $CP^n(c)$ of constant holomorphic sectional curvature c. Moreover it is known that $CP^n(c)$ can be imbedded in $R^{n(n+2)}$ by using the eigenfunctions associated with the first eigenvalue of the Laplacian. Note that the imbedding of S^2 in R^3 is nothing but the case where $n=1$.

The main purpose of this paper is to give some *extrinsic* characterizations of circles in $CP^n(c)$ imbedded in $R^{n(n+2)}$ (cf. Theorems 2, 5 and 6), which can be considered as generalizations of the above-mentioned well-known result. The notion of finite type submanifolds introduced by the first author ([2]) plays an important role.

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1. Preliminaries.

Let $(M, \langle\,,\,\rangle)$ be an n-dimensional Riemannian manifold. A curve $\gamma: I \to M$ is called a *helix* (parametrized by its arc length s) of order $d(\leq n)$ if there exist an orthonormal system $\{V_1=\dot{\gamma}, V_2, \cdots, V_d\}$ along γ and positive constants $\{k_1, \cdots, k_{d-1}\}$ which satisfy the system of ordinary differential equations

$$\nabla_{\dot{\gamma}} V_i = -k_{i-1} V_{i-1} + k_i V_{i+1}$$

for $1 \leq i \leq d$, where $V_0 = V_{d+1} = 0$ and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. When $d=2$, the curve γ is called

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a *circle*. The second author and Ohnita ([4]) study helixes γ in a non-flat complex space form $M(c)$, by using continuous functions $\tau_{ij}(s)=\langle V_i(s), JV_j(s)\rangle$ on γ for $1\leq i<j\leq d$, where $\{V_1,\cdots, V_d\}$ is a system of curvature vectors of γ and J is the complex structure of $M(c)$. The functions τ_{ij} are called the *complex torsions* of γ. In particular, τ_{12} and τ_{23} are called the first and the second complex torsions of γ, respectively. For simplisity we denote τ_{12} and τ_{23} by τ_1 and τ_2, respectively. When γ is a circle on a Kaehler manifold, we only have the first complex torsion τ_1. Moreover the complex torsion is constant along a circle on a Kaehler manifold. In fact, we have

$$\begin{aligned}\nabla_{\dot\gamma}\langle V_1, JV_2\rangle &=\langle \nabla_{\dot\gamma}V_1, JV_2\rangle+\langle V_1, J\nabla_{\dot\gamma}V_2\rangle \\ &=k_1\cdot\langle V_2, JV_2\rangle-k_1\cdot\langle V_1, JV_1\rangle=0\,.\end{aligned}$$

Using this fact and the fact that an n-dimensional complex projective space $CP^n(4)$ is a base manifold of the principal S^1-bundle $\pi: S^{2n+1}(1)\to CP^n(4)$, we can investigate the circles in a complex projective space.

In this paper, we apply two main tools to provide extrinsic characterizations of circles in $CP^n(4)$. One is the first standard (isometric) imbedding F of $CP^n(4)$ into Euclidean space $R^{n(n+2)}$. The map $F: CP^n(4)\to R^{n(n+2)}$ is defined as

$$F: CP^n(4)\xrightarrow{\text{minimal}} S^{n(n+2)-1}\left(\frac{2(n+1)}{n}\right)\xrightarrow{\text{totally umbilic}} R^{n(n+2)}\,.$$

The map F has various geometric properties. For instance, the second fundamental form of F is parallel and the image of a geodesic of $CP^n(4)$ under the map F is a circle (in the usual sense of Euclidean geometry) with curvature 2 in $R^{n(n+2)}$ (see, [6]).

On the other hand, consider the map $\tilde F: C^{n+1}\to C^{(n+1)^2}$ defined by

$$\tilde F(z)=z\otimes\bar z=(z_i\bar z_j)_{0\leq i,j\leq n}\,, \tag{1.1}$$

where $z=(z_0,\cdots, z_n)\in C^{n+1}$. Since it holds that $\tilde F(\kappa z)=\tilde F(z)$ for $\kappa\in C$ satisfying $|\kappa|=1$, we may regard $\tilde F$ as a mapping of $CP^n(4)$ into $C^{(n+1)^2}$, where $z_0,\cdots, z_n$ are regarded as homogeneous coordinates in $CP^n(4)$ satisfying $\sum_{i=0}^n z_i\bar z_i=1$. It is well-known that the map $\tilde F$ can be decomposed as

$$\begin{aligned}\tilde F: CP^n(4)&\xrightarrow{\text{minimal}} S^{n(n+2)-1}\left(\frac{2(n+1)}{n}\right)\xrightarrow{\text{totally umbilic}} R^{n(n+2)}\\ &\xrightarrow{\text{totally geodesic}} C^{(n+1)^2}\ (=R^{2(n+1)^2})\,.\end{aligned}$$

In the following, we mix the map $F: CP^n(4)\to R^{n(n+2)}$ and the map $\tilde F: CP^n(4)\to C^{(n+1)^2}$.

The other is the notion of finite type submanifolds introduced by the first author more than a decade ago. Here we review this notion briefly. A Riemannian submanifold M (not necessarily compact) of R^m is said to be of *finite type* if each component of its position vector $X: M\to R^m$ can be written as a finite sum of eigenfunctions of the

Laplacian Δ of M, that is,

$$X=X_0+\sum_{i=1}^{k} X_i\,, \tag{1.2}$$

where X_0 is a constant vector and $\Delta X_i=\lambda_i X_i$, $i=1, 2,\cdots, k$. Here we denote the isometric immersion of M into R^m by the same letter X. If, in particular, all eigenvalues $\{\lambda_1, \lambda_2,\cdots, \lambda_k\}$ are mutually different, then M is said to be of *k-type*. The decomposition (1.2) is called the *spectral decomposition* of the isometric immersion X (cf. [2], for details).

In terms of the theory of finite type submanifolds, a well-known result of Takahashi ([7]) can be restated as follows: A submanifold of R^m is of 1-type if and only if it is either a minimal submanifold of R^m or a minimal submanifold of a hypersphere of R^m.

Let M be a finite type submanifold whose spectral decomposition is given by (1.2). If we define a polynomial P by

$$P(t)=\prod_{i=1}^{k} (t-\lambda_i)\,, \tag{1.3}$$

then $P(\Delta)(X-X_0)=0$. The polynomial P is called the *minimal polynomial* of the finite type submanifold M. It is proved in [2, 3] that if M is compact and if there exists a constant vector X_0 and nontrivial polynomial P such that $P(\Delta)(X-X_0)=0$, then M is of finite type (see, [2]). By virtue of this characterization we can algebraically deal with finite type submanifolds. If M is non-compact, then the existence of a nontrivial polynomial P satisfying $P(\Delta)(X-X_0)=0$ does not imply that M is of finite type in general. However, if M is 1-dimensional, then the existence of the polynomial P satisfying the above condition guarantees that M is of finite type (see, [3]).

Finally we review the fundamental results about circles in CP^n. Let N be the outward unit normal on the unit sphere $S^{2n+1}(1)\subset R^{2n+2}=C^{n+1}$. We denote by J the natural complex structure on C^{n+1}. In the following we mix the complex structures of C^{n+1} and $CP^n(4)$. The relation between the Riemannian connection ∇ of $CP^n(4)$ and the Riemannian connection $\tilde{\nabla}$ of $S^{2n+1}(1)$ is given by (see, [5])

$$\tilde{\nabla}_X Y=\nabla_X Y+\langle X, JY\rangle JN \tag{1.4}$$

for any vector fields X and Y on $CP^n(4)$, where $\langle\ ,\ \rangle$ is the natural metric on C^{n+1}. For the sake of simplicity, we identify a vector field on $CP^n(4)$ with its horizontal lift on $S^{2n+1}(1)$.

In order to prove Theorem 1 in section 2, we recall the following results (for details, see [1]).

PROPOSITION 1. *Let γ be a circle with the complex torsion τ in $CP^n(4)$ satisfying $\nabla_{\dot\gamma}\dot\gamma=kY$ and $\nabla_{\dot\gamma}Y=-k\dot\gamma$. Then a horizontal lift $\tilde\gamma$ of γ in $S^{2n+1}(1)$ is a helix of order* 2, 3 *or* 5 *in $S^{2n+1}(1)$ according as $\tau=0$, $\tau=\pm 1$ or $\tau\neq 0, \pm 1$, respectively. Moreover $\tilde\gamma$*

satisfies the differential equations

$$(1.5)\qquad \begin{cases} \tilde{\nabla}_{\dot{\tilde\gamma}}\dot{\tilde\gamma}=kY\,, \\ \tilde{\nabla}_{\dot{\tilde\gamma}}Y=-k\dot{\tilde\gamma}+\tau JN\,, \\ \tilde{\nabla}_{\dot{\tilde\gamma}}JN=-\tau Y+\sqrt{1-\tau^2}\,Z\,, \\ \tilde{\nabla}_{\dot{\tilde\gamma}}Z=-\sqrt{1-\tau^2}\,JN+kW\,, \\ \tilde{\nabla}_{\dot{\tilde\gamma}}W=-kZ\,, \end{cases}$$

where $Z=1/\sqrt{1-\tau^2}\cdot(J\dot{\tilde\gamma}+\tau Y)$ *and* $W=1/\sqrt{1-\tau^2}\cdot(JY-\tau\dot{\tilde\gamma})$.

PROPOSITION 2. *Let* γ *be a circle with the complex torsion* τ *in* $CP^n(4)$ *satisfying* $\nabla_{\dot\gamma}\dot\gamma=kY$ *and* $\nabla_{\dot\gamma}Y=-k\dot\gamma$. *If a horizontal lift* $\tilde\gamma$ *of* γ *on* $S^{2n+1}(1)$ *satisfies the initial conditions* $\tilde\gamma(0)=x$, $\dot{\tilde\gamma}(0)=u$ *and* $\ddot{\tilde\gamma}(0)+\tilde\gamma(0)=kv$, *then* $\tilde\gamma$ *is expressed as follows*:

(1) *When* $\tau=0$,

$$(1.6)\qquad \tilde\gamma(s)=\frac{k}{k^2+1}(kx+v)+\frac{\cos(\sqrt{k^2+1}\,s)}{k^2+1}(x-kv)+\frac{\sin(\sqrt{k^2+1}\,s)}{\sqrt{k^2+1}}u\,.$$

(2) *When* $\tau=\pm1$,

$$(1.7)\qquad \tilde\gamma(s)=\frac{1}{1+\alpha^2}(e^{\alpha is}+\alpha^2e^{\beta is})x+\frac{\alpha}{1+\alpha^2}(-e^{\alpha is}+e^{\beta is})Ju\,,$$

where $\alpha+\beta=\mp k$ *and* $\alpha\beta=-1$.

(3) *When* $\tau\neq0,\ \pm1$,

$$(1.8)\qquad \tilde\gamma(s)=Ae^{ais}+Be^{bis}+Ce^{cis}\,,$$

where $a+b+c=0$, $ab+bc+ca=-k^2-1$, $abc=-\tau k$ *and*

$$A=\frac{1}{(a-b)(c-a)}\{-(1+bc)x+aJu+kv\}\,,$$

$$B=\frac{1}{(b-c)(a-b)}\{-(1+ca)x+bJu+kv\}\,,$$

$$C=\frac{1}{(c-a)(b-c)}\{-(1+ab)x+cJu+kv\}\,.$$

2. The image $F(\gamma)$ of a circle γ.

THEOREM 1. *Let* F *be the first standard imbedding of* $CP^n(4)$ *into* $R^{n(n+2)}$. *Then the image* $F(\gamma)$ *of a circle* γ *in* $CP^n(4)$ *with complex torsion* τ *is of* 1*-type,* 2*-type or* 3*-type in* $R^{n(n+2)}$ *according as* $\tau=\pm1$, $\tau=0$ *or* $\tau\neq\pm1,0$.

PROOF. Let γ be a circle with the complex torsion τ in $CP^n(4)$ and $\tilde{\gamma}$ a horizontal lift of γ on $S^{2n+1}(1)$. Our discussion is divided into three cases.

Case 1: $\tau=\pm 1$. It follows from (1.7) that $\tilde{\gamma}$ lies on the linear subspace C^2 spanned by $\{x, Jx, u, Ju\}$. Since $\langle x, u\rangle=\langle x, Ju\rangle=0$, without loss of generality we may regard x, u in C^2 as $x=(1, 0)$, $u=(0, 1)$. Then $\tilde{\gamma}$ is expressed as $\tilde{\gamma}=(z_0, z_1)$, where

$$z_0=\frac{1}{1+\alpha^2}(e^{\alpha is}+\alpha^2 e^{\beta is}), \qquad z_1=\frac{i\alpha}{1+\alpha^2}(-e^{\alpha is}+e^{\beta is}).$$

Hence

$$|z_0|^2=\frac{1}{(1+\alpha^2)^2}(1+\alpha^4+2\alpha^2\cos(\alpha-\beta)s),$$

$$z_0\overline{z_1}=-\frac{i\alpha}{(1+\alpha^2)^2}(-1+e^{(\alpha-\beta)is}-\alpha^2 e^{(\beta-\alpha)is}+\alpha^2),$$

$$|z_1|^2=\frac{2\alpha^2}{(1+\alpha^2)^2}(1-\cos(\alpha-\beta)s).$$

Since $F(\gamma)=\tilde{\gamma}\otimes\overline{\tilde{\gamma}}$, the above calculation shows that $F(\gamma)$ is of 1-type (with eigenfunction $e^{(\alpha-\beta)is}$) in $C^{(n+1)^2}$.

Case 2: $\tau=0$. Since $\langle u, Jv\rangle=\tau=0$, without loss of generality we may regard three vectors x, u and v as: $x=(1, 0, 0)$, $u=(0, 1, 0)$, $v=(0, 0, 1)$ in C^3. Then (1.6) implies that $\tilde{\gamma}=(1/l^2)(z_0, z_1, z_2)$, where $z_0=k^2+\cos ls$, $z_1=l\cdot\sin ls$, $z_2=k(1-\cos ls)$ and $l^2=k^2+1$. Hence a calculation yields that $F(\gamma)=\tilde{\gamma}\otimes\overline{\tilde{\gamma}}$ is of 2-type (with eigenfunctions e^{lis} and e^{2lis}) in $C^{(n+1)^2}$.

Case 3: $\tau\neq 0, \pm 1$. We set $\cos\beta=\langle Ju, v\rangle=-\tau$ so that $\beta\neq 0 \pmod{\pi/2}$. Then without loss of generality we may regard three vectors x, u and v as $x=(1, 0, 0)$, $u=(0, -i\cdot\sin\beta, -i\cdot\cos\beta)$ and $v=(0, 0, 1)$ in C^3. So (1.8) implies that

$$\tilde{\gamma}=\frac{1}{(a-b)(b-c)(c-a)}(z_0, z_1, z_2),$$

where

$$\begin{aligned}
z_0&=-(b-c)(1+bc)-(c-a)(1+ca)e^{bis}-(a-b)(1+ab)e^{cis},\\
z_1&=\sin\beta\{a(b-c)e^{ais}+b(c-a)e^{bis}+c(a-b)e^{cis}\},\\
z_2&=\cos\beta\{a(b-c)e^{ais}+b(c-a)e^{bis}+c(a-b)e^{cis}\}\\
&\quad+k\{(b-c)e^{ais}+(c-a)e^{bis}+(a-b)e^{cis}\}.
\end{aligned}$$

Thus by a direct calculation, we see that the curve $F(\gamma)=\tilde{\gamma}\otimes\overline{\tilde{\gamma}}$ in $C^{(n+1)^2}$ is of 3-type (with eigenfunctions $e^{(a-b)is}$, $e^{(b-c)is}$ and $e^{(c-a)is}$). □

REMARK. Proposition 2 and Theorem 1 imply the following: Let γ be a circle in

$CP^n(4)$ and $\tilde{\gamma}$ a horizontal lift of γ on $S^{2n+1}(1)$. Then

(1) $\tilde{\gamma}$ is of 1-type in C^{n+1} if and only if $\tilde{\gamma}\otimes\bar{\tilde{\gamma}}$ is of 2-type in $C^{(n+1)^2}$.

(2) $\tilde{\gamma}$ is of 2-type in C^{n+1} if and only if $\tilde{\gamma}\otimes\bar{\tilde{\gamma}}$ is of 1-type in $C^{(n+1)^2}$.

(3) $\tilde{\gamma}$ is of 3-type in C^{n+1} if and only if $\tilde{\gamma}\otimes\bar{\tilde{\gamma}}$ is of 3-type in $C^{(n+1)^2}$.

3. Circles with complex torsion $\tau=\pm 1$.

The purpose of this section is to prove the following.

THEOREM 2. *Let γ be a curve in $CP^n(4)$. Then γ is a geodesic or a circle with the complex torsion $\tau=1$ or -1 in $CP^n(4)$ if and only if $F(\gamma)$ is of* 1*-type in $R^{n(n+2)}$.*

PROOF. Let $\tilde{\gamma}$ be a horizontal lift of γ on $S^{2n+1}(1)$. We set $\tilde{\gamma}=z=(z_0,\cdots,z_n)\in C^{n+1}$. First of all we recall that $F(\gamma)$ is of k-type in $C^{(n+1)^2}$ if and only if there exist $a_1,\cdots,a_k\in R$ and $\eta\in C^{(n+1)^2}$ satisfying

$$\Delta^k(z\otimes\bar{z})+a_1\Delta^{k-1}(z\otimes\bar{z})+\cdots+a_{k-1}\Delta(z\otimes\bar{z})+a_k(z\otimes\bar{z})+\eta=0\,, \tag{3.1}$$

where $\Delta=-d/ds^2$ and s is the arc-length of γ. We denote by $\tilde{\nabla}$ and $\bar{\nabla}$ the Riemannian connections of $S^{2n+1}(1)$ and C^{n+1}, respectively. Let H be the mean curvature vector of $\tilde{\gamma}$ in C^{n+1} and h be the mean curvature vector of $\tilde{\gamma}$ in $S^{2n+1}(1)$. We denote by D the normal connection of $\tilde{\gamma}$ in C^{n+1} and we put $t=d/ds$. Then we have

$$H=h-z\,,\quad \bar{\nabla}_t z=t\,,\quad \bar{\nabla}_t t=H\quad\text{and}\quad D_tH=D_th\,. \tag{3.2}$$

It follows from (3.1) and (3.2) that $F(\gamma)$ is of 1-type if and only if there exist $\eta\in C^{(n+1)^2}$ and $a_1\in R$ satisfying

$$H\otimes\bar{z}+z\otimes\bar{H}+2t\otimes\bar{t}=\eta+a_1(z\otimes\bar{z})\,. \tag{3.3}$$

Differentiating (3.3) in the direction of t with respect to $\bar{\nabla}$ and using (3.2), we obtain

$$\begin{aligned}-A_Ht\otimes\bar{z}+D_th\otimes\bar{z}+H\otimes\bar{t}+t\otimes\bar{H}-z\otimes A_{\bar{H}}t\\ +z\otimes D_t\bar{h}+2H\otimes\bar{t}+2t\otimes\bar{H}=a_1(t\otimes\bar{z}+z\otimes\bar{t})\,,\end{aligned} \tag{3.4}$$

where A_H is the shape operator of $\tilde{\gamma}$ with respect to H in C^{n+1}.

On the other hand, from the Frenet formula for $\tilde{\gamma}$ in $S^{2n+1}(1)$ we can set $\tilde{\nabla}_t t=k_1v$ and $\tilde{\nabla}_t v=-k_1t+k_2w$, where k_1 and k_2 are functions on $\tilde{\gamma}$. Hence, by using $D_th=D_t(k_1v)$, we get

$$D_th=k_1'v+k_1k_2w\,. \tag{3.5}$$

Since $A_z=-\mathrm{Id}$, we obtain

$$A_Ht=(k_1^2+1)t\,. \tag{3.6}$$

We here remark that $A_{\bar{H}}t=\overline{A_Ht}$ and $D_t\bar{h}=\overline{D_th}$. From (3.4), (3.5) and (3.6) we obtain the following equation for $\tilde{\gamma}$:

$$(3.7)\qquad k_1'v\otimes\bar z+k_1k_2w\otimes\bar z+3H\otimes\bar t+3t\otimes\bar H \\ +k_1'z\otimes\bar v+k_1k_2z\otimes\bar w=(a_1+k_1^2+1)(t\otimes\bar z+z\otimes\bar t)\,.$$

For simplicity we choose orthonormal vectors e_1, e_2, e_3 and e_4 in C^{n+1} as follows: Put $p=\tilde\gamma(0)=z(0)=(1,0,\cdots,0)=e_1\in C^{n+1}$ and $u(=t(0))=(0,1,\cdots,0)=e_2$. Since $\tilde\nabla_t t=k_1v$, (1.4) implies that v is horizontal, that is, (e_1-component of v)$=0$. So we may choose e_3 in such a way that the vector v is expressed as

$$v=(i\cos\beta)e_2+(\sin\beta)e_3$$

at p, where $\beta=\langle Ju, v\rangle$. We set $w=(ai, bi, c+di, e, 0,\cdots,0)$. We choose e_4 such that $e\in R$. It follows from $\langle w, v\rangle=0$ that $b\cos\beta+c\sin\beta=0$. So at p, w is expressed as

$$w=aie_1+(i\mu\cdot\sin\beta)e_2+(-\mu\cdot\cos\beta+di)e_3+e\cdot e_4\,,$$

where $a, \mu, d, e\in R$. Moreover, we have

$$H=h-z=k_1v-e_1=(ik_1\cos\beta)e_2+(k_1\sin\beta)e_3-e_1$$

at p. Substituting these equalities into (3.7), we obtain

$$(3.8)\qquad k_1'\{i\cos\beta(e_2\otimes e_1-e_1\otimes e_2)+\sin\beta(e_1\otimes e_3+e_3\otimes e_1)\} \\ +k_1k_2\{i\mu\sin\beta(e_2\otimes e_1-e_1\otimes e_2)-\mu\cos\beta(e_3\otimes e_1+e_1\otimes e_3) \\ +di(e_3\otimes e_1-e_1\otimes e_3)+e(e_4\otimes e_1+e_1\otimes e_4)\} \\ +3k_1\sin\beta(e_2\otimes e_3+e_3\otimes e_2)=(a_1+k_1^2+4)(e_1\otimes e_2+e_2\otimes e_1)\,.$$

Taking the $(e_1\otimes e_2+e_2\otimes e_1)$-component of (3.8), we get $a_1+k_1^2+4=0$ at p. Since p can be chosen arbitrarily on $\tilde\gamma$, k_1 is constant on $\tilde\gamma$.

First we consider the case where $k_1\equiv 0$ on $\tilde\gamma$, that is, $\tilde\gamma$ is a horizontal great circle in $S^{2n+1}(1)$, so that γ is a geodesic in $CP^n(4)$. Hence $F(\gamma)$ is of 1-type in $C^{(n+1)^2}$ (cf. section 1).

Next we consider the case where $k_1\neq 0$ on $\tilde\gamma$. By taking the $(e_4\otimes e_1+e_1\otimes e_4)$-component of (3.8), we obtain $k_1k_2e=0$, that is, $k_2e=0$. If $k_2\equiv 0$ on $\tilde\gamma$, then $\tilde\gamma$ is a horizontal small circle of $S^{2n+1}(1)$, so that $\tilde\gamma$ is of 1-type in C^{n+1}. Thus $F(\gamma)$ is of 2-type in $C^{(n+1)^2}$ (see, Remark in Section 2). This is a contradiction. So, without loss of generality we may assume that $k_2\neq 0$ at p. (Hence the continuity of k_2 guarantees that there exists a positive number s_0 satisfying $k_2\neq 0$ on $I_0=\{s\,|-s_0<s<s_0\}$). Thus $e=0$. On the other hand, by taking the $(e_2\otimes e_3+e_3\otimes e_2)$-component of (3.8), we get $\sin\beta=0$. Also, by taking the $(e_1\otimes e_3+e_3\otimes e_1)$-component and the $(e_3\otimes e_1-e_1\otimes e_3)$-component of (3.8), we get $\mu=d=0$. Hence $w=aie_1$, so that $w=\pm Jz$ at p. Therefore $w=\pm Jz$ on I_0, that is, the curve $\tilde\gamma$ satisfies $\tilde\nabla_t t=k_1v$, $\tilde\nabla_t v=-k_1t\pm k_2Jz$ on I_0. Since Jz is a vertical vector, the curve γ in $CP^n(4)$ is a circle on I_0 with curvature k_1. By the assumption we can see that the complex torsion τ of γ is 1 or -1 (see, Theorem 1). So (1.5) implies that $k_2=|\tau|=1$ on I_0. Hence the continuity of k_2 tells us that $w(s)=Jz(s)$ for any s

$(-\infty < s < \infty)$. Therefore the above discussion asserts that the curve γ is a circle with the complex torsion $\tau=1$ or -1 in $CP^n(4)$. □

REMARK. We can restate Theorem 2 as follows:

THEOREM 2′. *Let γ be a curve in $CP^n(4)$. Then γ is a geodesic or a circle with the complex torsion $\tau=1$ or -1 in $CP^n(4)$ if and only if $F(\gamma)$ is a circle in $R^{n(n+2)}$.*

4. Circles with complex torsion $\tau=0$.

In this section we study the class of curves γ in $CP^n(4)$ satisfying that $F(\gamma)$ is of 2-type in $C^{(n+1)^2}$.

First we establish the following.

THEOREM 3. *Let γ be a curve in $CP^n(4)$. If $F(\gamma)$ is of 2-type in $R^{n(n+2)}$, then the first curvature k_1 of γ is constant along γ.*

PROOF. It follows from (3.1) and (3.2) that $F(\gamma)$ is of 2-type if and only if there exist $\eta \in C^{(n+1)^2}$ and $a_1, a_2 \in R$ satisfying

$$
\begin{aligned}
(4.1)\qquad & -\{\nabla_t(A_H t)\otimes \bar z + z\otimes \overline{\nabla_t(A_H t)} + \sigma(t, A_H t)\otimes \bar z + z\otimes \overline{\sigma(t, A_H t)}\} \\
& -4(A_H t\otimes \bar t + t\otimes \overline{A_H t}) + 6H\otimes \bar H - (A_{D_tH} t\otimes \bar z + z\otimes \overline{A_{D_tH} t}) \\
& +(D_t^2 H\otimes \bar z + z\otimes \overline{D_t^2 H}) + 4(D_t H\otimes \bar t + t\otimes \overline{D_t H}) \\
& + a_1(H\otimes \bar z + z\otimes \bar H + 2t\otimes \bar t) + a_2(z\otimes \bar z) + \eta = 0\,,
\end{aligned}
$$

where σ is the second fundamental form of $\tilde\gamma$ (which is a horizontal lift of γ) in C^{n+1}. By differentiating (4.1) in the direction of t with respect to the Riemannian connection $\bar\nabla$ of C^{n+1}, from (3.2) we get

$$
\begin{aligned}
(4.2)\qquad & -\{\nabla_t^2(A_H t)\otimes \bar z + z\otimes \overline{\nabla_t^2(A_H t)} + \sigma(t, \nabla_t(A_H t))\otimes \bar z + z\otimes \overline{\sigma(t, \nabla_t(A_H t))} \\
& + 5\nabla_t(A_H t)\otimes \bar t + 5t\otimes \overline{\nabla_t(A_H t)}\} + (A_{\sigma(t, A_H t)} t\otimes \bar z + z\otimes \overline{A_{\sigma(t,A_H t)} t}) \\
& -\{D_t(\sigma(t, A_H t))\otimes \bar z + z\otimes \overline{D_t(\sigma(t, A_H t))}\} - 5\{\sigma(t, A_H t)\otimes \bar t + t\otimes \overline{\sigma(t, A_H t)}\} \\
& -10(A_H t\otimes \bar H + H\otimes \overline{A_H t}) + 10(D_t H\otimes \bar H + H\otimes \overline{D_t H}) \\
& -\{\nabla_t(A_{D_tH} t)\otimes \bar z + z\otimes \overline{\nabla_t A_{D_tH} t)}\} - \{\sigma(t, A_{D_tH} t)\otimes \bar z + z\otimes \overline{\sigma(t, A_{D_tH} t)}\} \\
& -5(A_{D_tH} t\otimes \bar t + t\otimes \overline{A_{D_tH} t}) - (A_{D_t^2H} t\otimes \bar z + z\otimes \overline{A_{D_t^2H} t}) + (D_t^3 H\otimes \bar z + z\otimes \overline{D_t^3 H}) \\
& + 5(D_t^2 H\otimes \bar t + t\otimes \overline{D_t^2 H}) + a_1\{-(A_H t\otimes \bar z + z\otimes \overline{A_H t}) + (D_t H\otimes \bar z + z\otimes \overline{D_t H}) \\
& + 3(H\otimes \bar t + t\otimes \bar H)\} + a_2(t\otimes \bar z + z\otimes \bar t) = 0\,.
\end{aligned}
$$

As a matter of course the following vectors are scalar multiples of t:

$$\nabla_t^2(A_H t)\,,\ \nabla_t(A_H t)\,,\ A_{\sigma(t,A_H t)} t\,,\ A_H t\,,\ \nabla_t(A_{D_tH} t)\,,\ A_{D_tH} t\,,\ A_{D_t^2H} t\,.$$

Also, from (3.2) and (3.5) we get $D_t H \perp z$ so that $D_t^l H \perp z$ for $l=1, 2, \cdots$. Furthermore, it follows from (3.2), (3.5) and (3.6) that

$$\sigma(t, A_H t) = (k_1^2+1)H = (k_1^2+1)k_1 v - (k_1^2+1)z ,$$
$$D_t(\sigma(t, A_H t)) = 2k_1 k_1'(k_1 v - z) + (k_1^2+1)D_t H ,$$
$$\sigma(t, \nabla_t(A_H t)) = 2k_1 k_1' \sigma(t, t) = 2k_1 k_1'(k_1 v - z) ,$$
$$\sigma(t, A_{D_t H} t) = k_1 k_1' H = k_1 k_1'(k_1 v - z) .$$

Therefore by taking the $(z \otimes \bar{z} + \bar{z} \otimes z)$-component of (4.2) we have

$$-2k_1 k_1' + 2k_1 k_1' + k_1 k_1' = 0 ,$$

so that k_1 is constant along $\tilde{\gamma}$. This is equivalent to saying that the first curvature of $\gamma(=k_1)$ is constant along γ (cf. (1.4)). □

The main purpose of this section is to prove the following.

THEOREM 4. *Let γ be a curve in $CP^n(4)$. Then the first complex torsion τ_1 of γ is zero and $F(\gamma)$ is of 2-type in $R^{n(n+2)}$ if and only if γ is either*

(1) *a circle which lies on some totally geodesic $RP^2(1)$ in $CP^n(4)$, or*

(2) *a helix of order 4 which lies on some totally geodesic $CP^2(4)$ in $CP^n(4)$ and satisfies*

$$\nabla_t t = k_1 v , \qquad \nabla_t v = -k_1 t + k_2 J v ,$$
$$\nabla_t(Jv) = -k_2 v + k_1(-Jt) , \qquad \nabla_t(-Jt) = -k_1 J v , \tag{4.3}$$

where ∇ is the Riemannian connection of $CP^n(4)$, $t = \dot{\gamma}$, $\langle t, Jv\rangle = 0$ and $9k_1^2 + 2k_2^2 = 18$.

PROOF. Let $\tilde{\gamma}$ be a horizontal lift of γ on $S^{2n+1}(1)$. We denote by k_1 and k_2 the first and the second curvatures of $\tilde{\gamma}$, respectively.

First, we consider the case where $k_2 \equiv 0$ on $\tilde{\gamma}$. In this case, γ is of case (1) in our Theorem (cf. Theorem 1 and [1]). So, we may assume that $k_2 \neq 0$ at $p = z(0)$, so that $k_2(s) \neq 0$ $(-s_0 < s < s_0)$ for some $s_0 > 0$. Now we shall prove that k_2 is constant.

It follows from Theorem 3 that the first curvature k_1 is constant so that $\nabla_t(A_H t) = 0$. Therefore, by a direct calculation, (4.2) becomes

$$\begin{aligned} K(t \otimes \bar{z} + z \otimes \bar{t}) &+ k_1 k_2(a_1 - k_1^2 - 1)(w \otimes \bar{z} + z \otimes \bar{w}) \\ &+ k_1(3a_1 - 15k_1^2 - 15)(v \otimes \bar{t} + t \otimes \bar{v}) - (3a_1 - 15k_1^2 - 15)(z \otimes \bar{t} + t \otimes \bar{z}) \\ &+ 10k_1^2 k_2(w \otimes \bar{v} + v \otimes \bar{w}) - 10k_1 k_2(w \otimes \bar{z} + z \otimes \bar{w}) \\ &+ 5(D_t^2 H \otimes \bar{t} + t \otimes \overline{D_t^2 H}) + (D_t^3 H \otimes \bar{z} + z \otimes \overline{D_t^3 H}) = 0 , \end{aligned} \tag{4.4}$$

where $K = (k_1^2+1)^2 + k_1^2 k_2^2 - a_1(k_1^2+1) + a_2$.

By assumption, $\{z, t, v\}$ is a totally real orthonormal frame along $\tilde{\gamma}$. It follows from $\tilde{\nabla}_t v = -k_1 t + k_2 w$ that $D_t v = k_2 w$. Here D is the normal connection of $\tilde{\gamma}$ in C^{n+1}. Since $\tau = \langle t, Jv\rangle = 0$, without loss of generality we define canonical basis e_1, e_2, e_3 in C^{n+1}

as: $e_1=z(0)=\tilde{\gamma}(0)$, $e_2=t(0)$, $e_3=v(0)$. We set $w=aJz+\mu Jt+cJv+fe_4$, where $\{z, t, v, e_4\}$ is a totally real orthonormal frame along $\tilde{\gamma}$. Here a, μ and c are real-valued functions on $\tilde{\gamma}$, since w is perpendicular to z, t and v. Note that in general $f(=$the coefficient of $e_4)$ is a complex-valued function. But without loss of generality we may choose $e_4\in C^{n+1}$ in such a way that $f(0)\in R$. Hence at the point $p=\tilde{\gamma}(0)$ we get

$$\bar{w}=-aJz-\mu Jt-cJu+fe_4 \quad \text{and} \quad \bar{v}=v$$

so that

$$w\otimes\bar{v}+v\otimes\bar{w}=a(Jz\otimes v-v\otimes Jz)+\mu(Jt\otimes v-v\otimes Jt)+f(e_4\otimes v+v\otimes e_4)\,.$$

By taking the $(e_4\otimes v+v\otimes e_4)$-component of (4.4), we obtain $f(0)=0$, so that $f\equiv 0$, because p is an arbitrary point on $\tilde{\gamma}$. Therefore the vector w on $\tilde{\gamma}$ is expressed as

$$w=aJz+\mu Jt+cJv\,.$$

Consequently, we have

$$w\otimes\bar{v}+v\otimes\bar{w}=a(Jz\otimes v-v\otimes Jz)+\mu(Jt\otimes v-v\otimes Jt)\,,$$

$$w\otimes\bar{z}+z\otimes\bar{w}=\mu(Jt\otimes z-z\otimes Jt)+c(Jv\otimes z-z\otimes Jv)$$

at p. Moreover, D_t^2H, $D_t^3H\perp z, t$. Thus by taking the $(t\otimes z+z\otimes t)$-component of (4.4), we get

$$K-(3a_1-15k_1^2-15)=0\,.$$

Since k_1, a_1, a_2 are constant, we see that k_2 is constant along $\tilde{\gamma}$. Now from

$$\bar{\nabla}_t v=\tilde{\nabla}_t v=-k_1t+k_2w\,,$$

we get

$$J\bar{\nabla}_t v=-k_1Jt-ak_2z-\mu k_2t-ck_2v\,,$$

which implies

$$\bar{\nabla}_t w=a'Jz+\mu'Jt+c'Jv+aJt+\mu k_1Jv-\mu Jz-c^2k_2v-ck_1Jt-cak_2z-c\mu k_2t\,.$$

Hence

$$D_tw=(a'-\mu)Jz+(\mu'+a-ck_1)Jt-c^2k_2v+(c'+\mu k_1)Jv\,. \tag{4.5}$$

Since k_1 and k_2 are constant, Equation (3.5) shows

$$D_tH=k_1k_2w \quad \text{and} \quad D_t^2H=k_1k_2D_tw\,. \tag{4.6}$$

From (4.5) and (4.6) we see at the point $p=\tilde{\gamma}(0)$ that

$$\begin{aligned} D_t^2H\otimes\bar{t}+t\otimes\overline{D_t^2H}=k_1k_2\{&(a'-\mu)(Jz\otimes t-t\otimes Jz)\\ &-c^2k_2(v\otimes t+t\otimes v)+(c'+\mu k_1)(Jv\otimes t-t\otimes Jv)\}\,. \end{aligned} \tag{4.7}$$

Now, by taking the $(t\otimes v+v\otimes t)$-component of (4.4), from (4.7) we have at p that

$$k_1(3a_1-15k_1^2-15)-c^2k_1k_2^2=0\,. \tag{4.8}$$

Since p is an arbitrary fixed point and moreover k_1 and k_2 are nonzero constants, the function c is constant on $\tilde{\gamma}$. Similarly, by taking the $(Jt\otimes v-v\otimes Jt)$-component of (4.4) and using (4.7), we find $\mu\equiv 0$. Thus

$$w=aJz+cJv\,,$$

which implies that a is constant, because $a^2+c^2=1$. Hence

$$\bar{\nabla}_t w=aJt+cJ\nabla_t v\,,$$

which yields that

$$\begin{aligned}\tilde{\nabla}_t w=\bar{\nabla}_t w&=aJt+cJ\bar{\nabla}_t v\\&=aJt+c(-k_1Jt-ak_2z-ck_2v)\\&=(a-ck_1)Jt-c^2k_2v-cak_2z\,.\end{aligned}$$

Since $\tilde{\nabla}_t w\perp z$, $ca=0$. Moreover, from Frenet formula we may put

$$\tilde{\nabla}_t w=-k_2v+k_3w_2\,.$$

Then we obtain

$$k_3w_2=(a-ck_1)Jt+a^2k_2v\,. \tag{4.9}$$

Since $k_3w_2\perp v$, we know that $a=0$ and $k_3=k_1$. So (4.9) asserts that $w=Jv$. This implies

$$\tilde{\nabla}_t w=-k_2v+k_1(-Jt)\,.$$

The following is trivial:

$$\tilde{\nabla}_t(-Jt)=\bar{\nabla}_t(-Jt)=-JH=-k_1Jv+Jz\,.$$

Since Jz is a vertical vector, the above discussion shows that our curve γ satisfies the differential equations (4.3). Moreover the above computation yields

$$\begin{aligned}D_t^2H&=k_1k_2'w_2-k_1k_2^2v+k_1k_2k_3w_2\\&=-k_1k_2^2v-k_1^2k_2Jt\,.\end{aligned}$$

Hence

$$\begin{aligned}D_t^3H&=-k_1k_2^2(\bar{\nabla}_t v)^{\perp}-k_1^2k_2(JH)^{\perp}\\&=-k_1k_2^3w-k_1^2k_2(k_1Jv-Jz)\\&=-k_1k_2(k_1^2+k_2^2)Jv+k_1^2k_2Jz\,.\end{aligned}$$

Therefore by taking the $(Jv\otimes z-z\otimes Jv)$-component and the $(v\otimes t+t\otimes v)$-component of (4.4) at p, we obtain

$$a_1=11+2k_1^2+k_2^2 \quad \text{and} \quad k_1(3a_1-15k_1^2-15)-5k_1k_2^2=0\,,$$

respectively.

From these equations we conclude that $9k_1^2+2k_2^2=18$. □

Next, we investigate the solutions of (4.3).

PROPOSITION 3. *Let* $\tilde{\gamma}(s)=\tilde{\gamma}(s\,;\,k)=(Ae^{i\alpha s}, Be^{i\beta s}, Ce^{i\varepsilon s})$ *be a curve in* C^3, *where*

$$A=\sqrt{\frac{4-k^2-\sqrt{(2-k^2)(8-k^2)}}{2(8-k^2)}}\,,\qquad B=\frac{2}{\sqrt{8-k^2}}\,,$$

$$C=\sqrt{\frac{4-k^2+\sqrt{(2-k^2)(8-k^2)}}{2(8-k^2)}}\,,$$

$$\alpha=(\sqrt{2-k^2}+\sqrt{8-k^2})/\sqrt{2}\,,\qquad \beta=\sqrt{2-k^2}/\sqrt{2}\,,$$
$$\varepsilon=(\sqrt{2-k^2}-\sqrt{8-k^2})/\sqrt{2}$$

and $0<k<\sqrt{2}$. *Then* $\tilde{\gamma}$ *is a horizontal curve (with arc-length parameter* s*) on* $S^5(1)$. *Moreover* $\pi(\tilde{\gamma})$ *is a helix in* $CP^2(4)$ *with the first curvature* k *and with the first complex torsion* 0 *satisfying* (4.3), *where* $\pi: S^5(1)\to CP^2(4)$ *is the Hopf fibration.*

SKETCH OF THE PROOF. A direct computation yields

$$\langle\tilde{\gamma},\tilde{\gamma}\rangle=A^2+B^2+C^2=1\,,$$
$$\langle\dot{\tilde{\gamma}},\dot{\tilde{\gamma}}\rangle=\alpha^2A^2+\beta^2B^2+\varepsilon^2C^2=1\,,$$
$$\langle\dot{\tilde{\gamma}},J\tilde{\gamma}\rangle=\alpha A^2+\beta B^2+\varepsilon C^2=0\,.$$

Hence $\tilde{\gamma}$ is a horizontal curve with arc-length parameter s in $S^5(1)$. In addition, a long calculation yields that $\pi(\tilde{\gamma})$ is a helix with the first curvature k and with the complex torsion 0 in $CP^2(4)$ satisfying (4.3). Here t and v are expressed as

$$t=i(\alpha Ae^{i\alpha s}, \beta Be^{i\beta s}, \varepsilon Ce^{i\varepsilon s})\,,$$
$$v=(1/k)((1-\alpha^2)Ae^{i\alpha s}, (1-\beta^2)Be^{i\beta s}, (1-\varepsilon^2)Ce^{i\varepsilon s})\,.$$

Finally, needless to say, we note that $F(\gamma)(=F(\tilde{\gamma}))$ is of 2-type in C^9 with eigen-functions $e^{i(\alpha-\beta)s}$ and $e^{i(\varepsilon-\alpha)s}$. □

As an immediate consequence of Theorem 4, we obtain the following.

THEOREM 5. *Let* γ *be a curve in* $CP^n(4)$. *Then* γ *is a circle with complex torsion* 0 *in* $CP^n(4)$ *if and only if* γ *lies on totally geodesic* $RP^n(1)$ *in* $CP^n(4)$ *and* $F(\gamma)$ *is of* 2*-type in* $R^{n(n+2)}$.

PROOF. Since $\gamma \subset RP^n(1)$, each complex torsion of γ is zero. In particular the first complex torsion τ_1 of γ is zero. So our curve satisfies the assumption of Theorem 4. Note that (4.3) implies that the helix γ of case (2) does not lie on $RP^n(1)$. Therefore the result follows. □

5. Circles with complex torsion $\tau \neq 0, \pm 1$.

The purpose of this section is to characterize circles with complex torsion $\tau \neq 0, \pm 1$ in CP^n. First we give the following.

PROPOSITION 4. *Let γ be a curve in $CP^n(4)$ satisfying that $F(\gamma)$ is of 3-type in $R^{n(n+2)}$. If the first curvature k_1 of γ is constant, then the second curvature k_2 of a horizontal lift $\tilde{\gamma}$ of γ on $S^{2n+1}(1)$ is constant. Moreover, if the first complex torsion τ_1 of γ is constant, then the second curvature of γ is also constant.*

PROOF. We use the same terminologies as in the proof of Theorem 4. Note that $\nabla_t(A_H t) = 2k_1k_1' t = 0$. By using this equality repeatedly, we obtain from (3.1) and (3.2) that

$$
\begin{aligned}
(5.1)\qquad & \{(k_1^2+1)^2 + k_1^2k_2^2\}(H\otimes\bar z + z\otimes\bar H) + 2k_1^2k_2k_2'(t\otimes\bar z + z\otimes\bar t) \\
& -(A_{D_t^3H}t\otimes\bar z + z\otimes\overline{A_{D_t^3H}t}) + \{32(k_1^2+1)^2 + 12k_1^2k_2^2\}(t\otimes\bar t) \\
& -26(k_1^2+1)(D_tH\otimes\bar t + t\otimes\overline{D_tH}) - (k_1^2+1)(D_t^2H\otimes\bar z + z\otimes\overline{D_t^2H}) \\
& -30(k_1^2+1)(H\otimes\bar H) + 20(D_tH\otimes\overline{D_tH}) + 15(D_t^2H\otimes\bar H + H\otimes\overline{D_t^2H}) \\
& +6(D_t^3H\otimes\bar t + t\otimes\overline{D_t^3H}) + (D_t^4H\otimes\bar z + z\otimes\overline{D_t^4H}) \\
& +a_1\{-8(k_1^2+1)(t\otimes\bar t) - (k_1^2+1)(H\otimes\bar z + z\otimes\bar H) + 6H\otimes\bar H \\
& +4(D_tH\otimes\bar t + t\otimes\overline{D_tH}) + (D_t^2H\otimes\bar z + z\otimes\overline{D_t^2H})\} \\
& +a_2\{(H\otimes\bar z + z\otimes\bar H) + 2(t\otimes\bar t)\} + a_3(z\otimes\bar z) + \eta = 0\,.
\end{aligned}
$$

Now we shall differentiate (5.1) in the direction of t with respect to $\tilde\nabla$ and we pay particular attention to $z\otimes\bar z$-term. Then we have

$$
(5.2)\qquad 2k_1^2(k_2^2)'(H\otimes\bar z + z\otimes\bar H)_{z\otimes\bar z} - (\sigma(A_{D_t^3}t, t)\otimes\bar z + z\otimes\overline{\sigma(A_{D_t^3}t, t)})_{z\otimes\bar z} = 0\,,
$$

by virtue of $D_t^l H \perp z$, where $(*)_{z\otimes\bar z}$ is the $(z\otimes\bar z)$-component of $(*)$. From (3.5) we get $D_tH = k_1k_2w$. So, the Frenet formulas imply

$$
D_t^2H = k_1k_2'w + k_1k_2D_tw = k_1k_2'w + k_1k_2\tilde\nabla_t w = k_1k_2'w + k_1k_2(-k_2v + k_3w_2)\,,
$$

so that

$$
\begin{aligned}
D_t^3H &= k_1k_2''w + k_1k_2'(-k_2v + k_3w_2) - k_1(k_2^2)'v \\
&\quad - k_1k_2^2(-k_1t + k_2w) + (k_1k_2k_3)'w_2 + k_1k_2k_3(-k_3w + k_4w_3)\,.
\end{aligned}
$$

Hence we get

$$
(5.3)\qquad \sigma(t, A_{D_t^3H}t) = -3k_1^2k_2k_2'H = -(3/2)k_1^2(k_2^2)'(k_1v - z)\,.
$$

It follows from (5.2) and (5.3) that

$$-4k_1^2(k_2^2)'-3k_1^2(k_2^2)'=0\,.$$

Therefore, the second curvature k_2 of $\tilde{\gamma}$ is constant, since k_1 is nonzero constant. Combining this with (1.4) and using the hypothesis that the first complex torsion τ_1 of γ is constant, we conclude that the second curvature of γ is constant. □

We are now in a position to prove the following.

THEOREM 6. *Let γ be a curve in $CP^n(4)$. Then γ is a circle with the complex torsion $\tau\neq 0, \pm 1$ in $CP^n(4)$ if and only if γ satisfies the following five conditions.*

(i) *$F(\gamma)$ is of 3-type in $R^{n(n+2)}$,*
(ii) *γ lies on some totally geodesic $CP^2(4)$ in $CP^n(4)$,*
(iii) *the first curvature of γ is constant,*
(iv) *the first complex torsion τ_1 of γ is constant but $-1<\tau_1(\neq 0)<1$, and*
(v) *the second complex torsion τ_2 of γ is zero.*

PROOF. Let $\tilde{\gamma}$ be a horizontal lift of γ on $S^{2n+1}(1)$. We choose a totally real orthonormal frame $\{z, t(=\dot{\tilde{\gamma}}), e\}$ along $\tilde{\gamma}$ in C^3. On the other hand from the Frenet formula for $\tilde{\gamma}$ in $S^5(1)$ we may put

$$\tilde{\nabla}_t t=k_1 v\,,\quad \tilde{\nabla}_t v=-k_1 t+k_2 w\,,\quad \tilde{\nabla}_t w=-k_2 v+k_3 w_2\,,$$
$$\tilde{\nabla}_t w_2=-k_3 w+k_4 w_3\,,\qquad \tilde{\nabla}_t w_3=-k_4 w_2\,.$$

Note that k_1 and k_2 are constant (see, Proposition 4). Put $\cos\beta=\langle Jt, v\rangle(=-\tau_1)$. Since v is horizontal, we get

$$v=(\cos\beta)Jt+(\sin\beta)e\,. \tag{5.4}$$

Since w is perpendicular to z, t and v, we have

$$w=aJz+(\mu\sin\beta)Jt-(\mu\cos\beta)e+\nu Je\,.$$

By assumption (v) and (1.4), we get $k_2\langle w, Jv\rangle=0$. It follows from Proposition 4 that k_2 is nonzero constant. If $k_2\equiv 0$, then $\tilde{\gamma}$ is of 1-type in C^{n+1}. Hence Theorem 1 implies that $F(\gamma)$ is of 2-type in $C^{(n+1)^2}$, which is a contradiction. Therefore, $\langle w, Jv\rangle=\nu\sin\beta=0$ on $\tilde{\gamma}$. Hence the assumption (iv) yields $\nu=0$ on $\tilde{\gamma}$. Hence we have

$$w=aJz+(\mu\sin\beta)Jt-(\mu\cos\beta)e\,, \tag{5.5}$$

where a and μ are real-valued functions on $\tilde{\gamma}$ satisfying $a^2+\mu^2=1$. Our next aim is to prove that $\mu\equiv 0$.

From (5.5) we get

$$\begin{aligned}\bar{\nabla}_t v=\tilde{\nabla}_t v&=-k_1 t+k_2 w\\ &=-k_1 t+ak_2 Jz+(\mu k_2\sin\beta)Jt-(\mu k_2\cos\beta)e\,.\end{aligned}$$

On the other hand, (5.4) yields

$$\begin{aligned}\bar{\nabla}_t v &= \cos\beta JH + \sin\beta \bar{\nabla}_t e\\ &= \cos\beta k_1 Jv - \cos\beta Jz + \sin\beta \tilde{\nabla}_t e\\ &= -\cos^2\beta k_1 t + (k_1\cos\beta\sin\beta) Je - \cos\beta Jz + \sin\beta\tilde{\nabla}_t e\,.\end{aligned}$$

Since the assumption (iv) shows that $\sin\beta \neq 0$, these equalities yield

$$\tilde{\nabla}_t e = (ak_2 \operatorname{cosec}\beta + \cot\beta) Jz - \sin\beta k_1 t + \mu k_2 Jt - (\mu k_2 \cot\beta) e - (k_1\cos\beta) Je\,. \tag{5.6}$$

Similarly we find

$$\begin{aligned}\bar{\nabla}_t w = \tilde{\nabla}_t w &= -k_2 v + k_3 w_2\\ &= -(k_2\cos\beta) Jt - (k_2\sin\beta) e + k_3 w_2\end{aligned}$$

as well as

$$\bar{\nabla}_t w = aJt + \mu\sin\beta(k_1 Jv - Jz) - (\mu\cos\beta)\bar{\nabla}_t e + a' Jz + (\mu'\sin\beta) Jt - (\mu'\cos\beta) e\,.$$

It follows from these relations and (5.6) that

$$\begin{aligned}k_3 w_2 = &\{a' - \mu\sin\beta - \mu(ak_2\cot\beta + \cos\beta\cot\beta)\}(Jz)\\ &+ (k_2\cos\beta + \mu'\sin\beta + a - \mu^2 k_2\cos\beta)(Jt)\\ &+ (k_2\sin\beta - \mu'\cos\beta + \mu^2 k_2\cos\beta\cot\beta) e + k_1\mu(Je)\,.\end{aligned}$$

Since $\langle v, k_3 w_2\rangle = 0$, this asserts

$$k_2 + a\cos\beta = 0\,. \tag{5.7}$$

Hence a and b are constant. Thus from these relations we obtain

$$k_3 w_2 = (1 - k_2^2)\{-(\mu\operatorname{cosec}\beta)(Jz) + a(Jt) + (k_2\operatorname{cosec}\beta) e\} + k_1\mu Je\,. \tag{5.8}$$

This shows that k_3 is constant. It follows from (5.6) and (5.7) that

$$\tilde{\nabla}_t e = (\mu^2\cot\beta) Jz - (k_1\sin\beta) t + \mu k_2 Jt + (\mu k_2\cot\beta) e - (k_1\cos\beta) Je\,. \tag{5.9}$$

Now from (5.5) we have

$$\begin{aligned}k_3\tilde{\nabla}_t w_2 &= -k_3^2 w + k_3 k_4 w_3\\ &= -k_3^2 aJz - (\mu k_3^2\sin\beta) Jt + (\mu k_3^2\cos\beta) e + k_3 k_4 w_3\,.\end{aligned}$$

On the other hand, since k_3 is constant, (5.8) and (5.9) imply

$$\begin{aligned} k_3\tilde{\nabla}_t w_2 = & -(1-k_2^2)\mu \operatorname{cosec}\beta Jt - ak_1(1-k_2^2)\cos\beta t \\ & + ak_1(1-k_2^2)\sin\beta Je - a(1-k_2^2)Jz \\ & + (1-k_2^2)k_2 \operatorname{cosec}\beta\{(\mu^2\cot\beta)(Jz) \\ & - (k_1\sin\beta)t + \mu k_2 Jt - (\mu k_2\cot\beta)e - (k_1\cos\beta)Je\} \\ & + k_1\mu\{-(\mu^2\cot\beta)z - (k_1\sin\beta)Jt - (\mu k_2)t \\ & - (\mu k_2\cot\beta)Je + (k_1\cos\beta)e\}\,. \end{aligned}$$

Thus we obtain

$$\begin{aligned} k_3k_4w_3 = & -(k_1\mu^3\cot\beta)z \\ & + \{ak_3^2 - a(1-k_2^2) + \mu^2\operatorname{cosec}\beta k_2(1-k_2^2)\cot\beta\}Jz \\ & - \{ak_1(1-k_2^2)\cos\beta - \mu^2 k_1k_2 - k_1k_2(1-k_2^2)\}t \\ & + \{\mu k_3^2\sin\beta - (1-k_2^2)\mu\operatorname{cosec}\beta + \mu k_2^2(1-k_2^2)\operatorname{cosec}\beta - k_1^2\mu\sin\beta\}Jt \\ & + \{-\mu k_2^2(1-k_2^2)\operatorname{cosec}\beta\cot\beta - \mu k_3^2\operatorname{cosec}\beta + k_1^2\mu\cos\beta\}e \\ & + \{ak_1(1-k_2^2)\sin\beta - k_1k_2(1-k_2^2)\operatorname{cosec}\beta\cos\beta - k_1\mu^2k_2\cot\beta\}Je\,. \end{aligned}$$

Since $k_3k_4w_3 \perp z$, it follows that $k_1\mu^3\cot\beta=0$, so that, $\mu=0$. Therefore $w=\pm Jz$. □

REMARKS. (1) Theorem 6 does not hold if we remove the condition (v). In fact, by a direct calculation we can establish the following:

PROPOSITION 5. *Let*

$$\tilde{\gamma}(s)=\left(\frac{\sqrt{3}}{3}e^{is}, \frac{\sqrt{14}}{14}e^{2is}, \frac{5\sqrt{42}}{42}e^{-4is/5}\right)$$

be a curve in C^3. *Then* $\pi(\tilde{\gamma})$ *is a helix with the second complex torsion* $\tau_2=-\sqrt{2}/2$ *in* $CP^n(4)$ *satisfying the conditions* (i), (ii), (iii) *and* (iv) *in Theorem* 6. *The Frenet formula for* $\pi(\tilde{\gamma})$ *in* $CP^n(4)$ *is given by*

$$\begin{cases} \nabla_{\dot\gamma}u_1 = \dfrac{3\sqrt{2}}{5}u_2\,, \\ \nabla_{\dot\gamma}u_2 = -\dfrac{3\sqrt{2}}{5}u_1 + \dfrac{11\sqrt{2}}{10}u_3\,, \\ \nabla_{\dot\gamma}u_3 = -\dfrac{11\sqrt{2}}{10}u_2 + \dfrac{\sqrt{2}}{2}u_4\,, \\ \nabla_{\dot\gamma}u_4 = -\dfrac{\sqrt{2}}{2}u_3\,, \end{cases}$$

where

$$\begin{cases} u_1 = i\left(\dfrac{\sqrt{3}}{3}e^{is}, \dfrac{\sqrt{14}}{7}e^{2is}, -\dfrac{2\sqrt{42}}{21}e^{-4is/5}\right), \\ u_2 = \dfrac{5\sqrt{2}}{6}\left(0, -\dfrac{3\sqrt{14}}{14}e^{2is}, \dfrac{3\sqrt{42}}{70}e^{-4is/5}\right), \\ u_3 = i\left(\dfrac{\sqrt{3}}{3}e^{is}, -\dfrac{3\sqrt{14}}{14}e^{2is}, -\dfrac{\sqrt{42}}{42}e^{-4is/5}\right), \\ u_4 = \left(-\dfrac{\sqrt{6}}{3}e^{is}, \dfrac{\sqrt{7}}{14}e^{2is}, \dfrac{5\sqrt{21}}{42}e^{-4is/5}\right). \end{cases}$$

(2) It is known that every helix in a Euclidean space R^m is a curve of finite type. But the converse is not true. The class of curves of finite type in R^m is too large to classify. We remark that for a circle γ with the complex torsion τ in $CP^n(4)$ the curve $F(\gamma)$ is a helix of order 2, 4 or 6 in $R^{n(n+2)}$ according as $\tau = \pm 1$, $\tau = 0$ or $\tau \neq 0, \pm 1$. Furthermore, the curve $F(\gamma)$ is not necessarily closed when $\tau \neq 0, \pm 1$ (see, [1]).

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