TOKYO J. MATH.
VOL. 19, NO. 1, 1996

The Complexity of Generalized Sturmian Sequences

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(Communicated by M. Sugiura)

1. Introduction.

Let $\{b_i\}_{n=1}^{\infty}$ be a sequence with finite state and let

$$P_n(\{b_i\}_{i=1}^{\infty}) = \#\{(b_j, b_{j+1}, \cdots, b_{j+n-1}) \mid j=1, 2, \cdots\} . \qquad (1)$$

We call $P_n(\{b_i\}_{i=1}^{\infty})$ the complexity of $\{b_i\}_{i=1}^{\infty}$. The complexity of the sequence $\{[ix+y]-[(i-1)x+y] \mid i=1, 2, \cdots\}$ for an irrational number x and a real number y called a sturmian sequence is known as $(n+1)$ ([1]). In several cases, the explicit forms of the complexity are calculated ([2] and [3]). The purpose of this paper is to introduce a sequence $\{Q_i^k(x_1, \cdots, x_k) \mid i=1, 2, \cdots\}$ which we call a generalized Sturmian sequence: for each $(x_1, \cdots, x_k) \in \mathbf{R}^k$

$$Q_i^k(x_1, \cdots, x_k) = [x_1[x_2 \cdots [x_k(i+1)] \cdots]] - [x_1[x_2 \cdots [x_k i] \cdots]] \quad (i=1, 2, \cdots) , \qquad (2)$$

and to give the explicit form of the complexity for the generalized sturmian sequence as follows.

THEOREM 1. *Let* $(x_1, \cdots, x_k)$ *be the k-dimensional positive real vector satisfying* $x_i > 2$ $(1 \le i \le k)$. *Assume that*

$$1, \quad \frac{1}{x_1}, \quad \cdots, \quad \frac{1}{x_1 \cdots x_i}, \quad \cdots, \quad \frac{1}{x_1 \cdots x_k} \qquad (3)$$

be linearly independent over $\mathbf{Q}$. *Then the complexity of generalized Sturmian sequence* $\{Q_i(x_1, \cdots, x_k)\}_{i=1}^{\infty}$ *is given by* $(n+1)^k$, *that is,*

$$P_n(\{Q_i^k(x_1, \cdots, x_k)\}_{i=1}^{\infty}) = (n+1)^k .$$

2. Dynamical system.

In this section, we will show that the generalized sturmian sequence is related to a dynamical system.

Received October 17, 1994

For $(i_1, i_2, \cdots, i_k) \in \{0, 1\}^k$, put

$$Q^k(i_1, i_2, \cdots, i_k) = [x_1[x_2 \cdots [x_k + i_k] \cdots + i_2] + i_1] .$$

Then we have

LEMMA 1. *For $(i_1, i_2, \cdots, i_k) \in \{0, 1\}^k$, we have*

$$Q^k(i_1, \cdots, i_k) < [x_1 \cdots [x_{k-1}[2x_k]] \cdots] .$$

PROOF. To obtain the assertion, it is sufficient to prove the following relation:

$$Q^k(1, \cdots, 1) < [x_1 \cdots [x_{k-1}[2x_k]] \cdots] . \tag{4}$$

We prove this by induction on k. Let us assume that $k=1$. Because $x_1 > 2$, it is easy to see that

$$Q^1(1) = [x_1 + 1] \leq [x_1 + x_1 - 1] < [2x_1] .$$

Therefore (4) holds on $k=1$. Let us assume that $k>1$. By the inductive assumption,

$$[x_2[\cdots [x_k + 1] \cdots] + 1] < [x_2 \cdots [x_{k-1}[2x_k]] \cdots] . \tag{5}$$

Because $x_1 > 2$, we have

$$\begin{aligned} Q^k(1, \cdots, 1) &= [x_1[x_2[\cdots [x_k + 1] \cdots] + 1] + 1] \\ &< [x_1([x_2 \cdots [x_{k-1}[2x_k]] \cdots] - 1) + x_1] \\ &< [x_1 \cdots [x_{k-1}[2x_k]] \cdots] . \end{aligned}$$

Hence, we completed the proof.

Let $\beta = (1/x_1, 1/(x_1x_2), \cdots, 1/(x_1 \cdots x_k))$. We introduce the dynamical system T on $[-1, 0]^k$ as follows:

$$Tx = x + \beta \qquad \text{mod } 1 .$$

For any natural number i, put

$$P_i^k(x_1, \cdots, x_k) = [x_1[x_2 \cdots [x_k i] \cdots]] .$$

Let M be the following $k \times k$ matrix

$$M = \begin{pmatrix} 1/x_1 & & & & \\ 1/x_1x_2 & 1/x_2 & & & \\ \cdots & \cdots & \ddots & & \\ \cdots & \cdots & \cdots & \ddots & \\ 1/x_1 \cdots x_k & \cdots & \cdots & 1/x_{k-1}x_k & 1/x_k \end{pmatrix} .$$

Define the domain B as the image of $[-1, 0]^k$ by the linear map M, that is, $B = M([-1, 0]^k)$. Then we have

LEMMA 2. *The successive return time of* $n\beta = T^n(0)$ *into* B *for a natural number* n *is characterized by* $P_n^k(x_1, \cdots, x_k)$. *That is,* $m\beta \bmod 1 \in B$ *for some integer* $m>0$ *if and only if there exists an integer* n *such that* $m = P_n^k(x_1, \cdots, x_k)$. *Moreover,* $m\beta \bmod 1$ *are not on the boundary of* B *for every* $m \geq 1$, *that is*

$$m\beta \notin \partial B \quad \text{for} \quad m = 1, 2, \cdots .$$

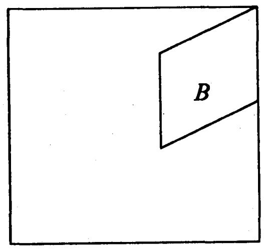

FIGURE 2.1. Figure of B $(k=2)$

PROOF. We show that $P_n^k(x_1, \cdots, x_k)\beta \in B \bmod 1$. By the definition we have

$$P_n^k(x_1, \cdots, x_k)\beta = \left(\frac{P_n^k(x_1, \cdots, x_k)}{x_1}, \cdots, \frac{P_n^k(x_1, \cdots, x_k)}{x_1 \cdots x_k} \right).$$

We will show that for $1 \leq j \leq k$,

$$\frac{P_n^k(x_1, \cdots, x_k)}{x_1 \cdots x_j} = P_n^{k-j}(x_{j+1}, \cdots, x_k) - \frac{\varepsilon_j}{x_j} - \frac{\varepsilon_{j-1}}{x_j x_{j-1}} - \cdots - \frac{\varepsilon_1}{x_j \cdots x_1}, \tag{6}$$

where $\varepsilon_j = x_j P_n^{k-j}(x_{j+1}, \cdots, x_k) - [x_j P_n^{k-j}(x_{j+1}, \cdots, x_k)]$ for $1 \leq j \leq k-1$ and $\varepsilon_k = nx_k - [nx_k]$. We prove (6) by induction on j. Let us assume that $j=1$, then we have

$$\begin{aligned} \frac{P_n^k(x_1, \cdots, x_k)}{x_1} &= \frac{[x_1 P_n^{k-1}(x_2, \cdots, x_k)]}{x_1} \\ &= \frac{x_1 P_n^{k-1}(x_2, \cdots, x_k) - \varepsilon_1}{x_1} = P_n^{k-1}(x_2, \cdots, x_k) - \frac{\varepsilon_1}{x_1} . \end{aligned} \tag{7}$$

Let us assume that $j > 1$. By inductive assumption, we have

$$\frac{P_n^k(x_1, \cdots, x_k)}{x_1 \cdots x_{j-1}} = P_n^{k-(j-1)}(x_j, \cdots, x_k) - \frac{\varepsilon_{j-1}}{x_{j-1}} - \frac{\varepsilon_{j-2}}{x_{j-1} x_{j-2}} - \cdots - \frac{\varepsilon_1}{x_{j-1} \cdots x_1} .$$

Therefore, we have

$$\frac{P_n^k(x_1, \cdots, x_k)}{x_1 \cdots x_j} = \frac{P_n^{k-(j-1)}(x_j, \cdots, x_k)}{x_j} - \frac{\varepsilon_{j-1}}{x_j x_{j-1}} - \cdots - \frac{\varepsilon_1}{x_j \cdots x_1}$$

$$=\frac{[x_jP_n^{k-j}(x_{j+1},\cdots,x_k)]}{x_j}-\frac{\varepsilon_{j-1}}{x_jx_{j-1}}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1}$$

$$=P_n^{k-j}(x_{j+1},\cdots,x_k)-\frac{\varepsilon_j}{x_j}-\frac{\varepsilon_{j-1}}{x_jx_{j-1}}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1}.$$

From the fact that $P_n^{k-j}(x_{j+1},\cdots,x_k)$ is an integer, we have

$$\frac{P_n^k(x_1,\cdots,x_k)}{x_1\cdots x_j}\equiv-\frac{\varepsilon_j}{x_j}-\frac{\varepsilon_{j-1}}{x_jx_{j-1}}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1}\quad \bmod 1\,.$$

That is,

$$P_n^k(x_1,\cdots,x_k)^t\beta\equiv\begin{pmatrix}1/x_1 & & & & \\ 1/x_1x_2 & 1/x_2 & & & \\ \cdots & \cdots & \ddots & & \\ \cdots & \cdots & \cdots & \ddots & \\ 1/x_1\cdots x_k & \cdots & \cdots & 1/x_{k-1}x_k & 1/x_k\end{pmatrix}\begin{pmatrix}-\varepsilon_1\\ \cdot\\ \cdot\\ \cdot\\ -\varepsilon_k\end{pmatrix}\quad \bmod 1\,.$$

Conversely let us assume that $m\beta\in B \bmod 1$. By the definition of B, there exists $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_k)$ such that for $1\le i\le k$, $0\le\varepsilon_i\le 1$ and $m^t\beta\equiv M(-{}^t\varepsilon) \bmod 1$. We show that there exist integers $P_1>0,\cdots,P_{k+1}>0$ satisfying the following conditions

(1) $P_1=m$,

(2) $P_j=[x_jP_{j+1}]$ for $j=1,\cdots,k$,

(3) $\varepsilon_j=x_jP_{j+1}-[x_jP_{j+1}]$ for $j=1,\cdots,k$.

If we can show above assertion, then we have $m=[x_1[x_2\cdots[x_kP_{k+1}]\cdots]]$, that is $m=P_{P_{k+1}}^k(x_1,\cdots,x_k)$ and $m\beta\notin\partial B$. We construct P_j $(j=1,2,\cdots,k+1)$ by induction on j. By the assumption, $m/x_1\equiv-\varepsilon_1/x_1 \bmod 1$. Therefore there exists an integer $P_2>0$ such that

$$m/x_1=P_2-\varepsilon_1/x_1. \tag{8}$$

Therefore we have

$$m=[x_1P_2]\,,\qquad \varepsilon_1=x_1P_2-[x_1P_2]\,. \tag{9}$$

From the above equation (9) we know that $0<\varepsilon_1<1$. So, we have the assertion (2) and (3) for the case $j=1$. Let us assume that $P_1,\cdots,P_j$ are constructed. Then we have

$$\frac{m}{x_1\cdots x_j}=\frac{P_2x_1-\varepsilon_1}{x_1\cdots x_j}=\frac{P_2}{x_2\cdots x_j}-\frac{\varepsilon_1}{x_1\cdots x_j}$$

$$=\frac{P_j}{x_j}-\frac{\varepsilon_{j-1}}{x_jx_{j-1}}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1}. \tag{10}$$

By the assumption, we have

$$\frac{m}{x_1\cdots x_j}\equiv -\frac{\varepsilon_j}{x_j}-\frac{\varepsilon_{j-1}}{x_jx_{j-1}}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1} \qquad \mathrm{mod}\, 1\,. \tag{11}$$

From (10) and (11), we see $P_j/x_j\equiv -\varepsilon_j/x_j$ mod 1. Therefore there exists an integer $P_{j+1}>0$ such that $P_j/x_j=P_{j+1}-\varepsilon_j/x_j$. Then we see easily that P_{j+1} satisfies the above conditions.

LEMMA 3. *Let us denote* $S=\{0, 1\}^k$ *and* $Q_a=Q^k(i_1, \cdots, i_k)$ *for* $a=(i_1, \cdots, i_k)\in S$. *Then we have the following assertion*:

(1) *there exists* $(\varepsilon_1, \cdots, \varepsilon_k)$ *uniquely such that* $0<\varepsilon_j<1$ *for* $j=1, 2, \cdots, k$ *and*

$$M(-{}^ta)+Q_a{}^t\beta\equiv M^t(-\varepsilon_1, \cdots, -\varepsilon_k) \qquad \mathrm{mod}\, 1\,. \tag{12}$$

(2) *Denoting* $(\varepsilon_1, \cdots, \varepsilon_k)$ *in* (1) *by* ε^a, *if* $a=(i_1, \cdots, i_k)$, $b=(l_1, \cdots, l_k)\in S$ *and* $i_j=l_j$ *for* $q\le j\le k$, *then* $\varepsilon_j^a=\varepsilon_j^b$ *for* $q-1\le j\le k$.

(3) *If* $a\ne b$, *then* $M(-{}^ta)+Q_b{}^t\beta\notin B$.

(4) *If* $a\ne b$, *then* $Q_a\ne Q_b$.

PROOF. Let ${}^t(u_1, \cdots, u_k)=M(-{}^ta)+Q_a{}^t\beta$. Then, we can show by the analogous method in the proof of Lemma 2 that for $1\le j\le k$

$$u_j=Q^{k-j}(i_{j+1}, \cdots, i_k)-\frac{\varepsilon_j}{x_j}-\frac{\varepsilon_{j-1}}{x_jx_{j-1}}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1}\,, \tag{13}$$

where ε_j are given as follows:

$$\begin{aligned}&\varepsilon_j=x_jQ^{k-j}(i_{j+1}, \cdots, i_k)-[x_jQ^{k-j}(i_{j+1}, \cdots, i_k)] \qquad (1\le j\le k-1)\,,\\ &\varepsilon_k=x_k-[x_k]\,.\end{aligned} \tag{14}$$

Therefore we have

$$u_j\equiv -\frac{\varepsilon_j}{x_j}-\frac{\varepsilon_{j-1}}{x_jx_{j-1}}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1} \qquad \mathrm{mod}\, 1\,,$$

that is $M(-{}^ta)+Q_a{}^t\beta\equiv M^t(-\varepsilon_1, \cdots, -\varepsilon_k)$ mod 1. For the uniqueness, we prove that if for $x, y\in[0, 1]^k$, $x\ne y$, then $Mx\ne My$ mod 1. For this purpose it is enough to know that $B\subset[-1, 0]^k$. Note that $2<x_i$ for $j=1, 2, \cdots, k$. We get for $(a_1, \cdots, a_k)\in[-1, 0]^k$

$$0\ge M^t(a_1, \cdots, a_k)_j=\frac{a_1}{x_1\cdots x_j}+\frac{a_2}{x_2\cdots x_j}+\cdots+\frac{a_j}{x_j}>-\sum_{i=1}^{j}\frac{1}{2^i}>-1\,.$$

Therefore we have $B\subset[-1, 0]^k$. From the definition of ε_j, the number ε_j is determined by $i_{j+1}, \cdots, i_k$ and $x_j, \cdots, x_k$. Therefore the second assertion (2) of the lemma is justified. For the assertion (3), let us assume that $a\ne b\in S$ and $i_j\ne l_j$ and $i_n=l_n$ for $1\le n\le j-1$, where $a=(i_1, \cdots, i_k)$, $b=(l_1, \cdots, l_k)$. We shall show that $M(-{}^ta)+Q_b^t\beta\notin B$. Let $(u_1, \cdots, u_k)=M(-{}^ta)+Q_b^t\beta$. Recursively, we get

$$u_j \equiv \frac{-i_j+l_j-\varepsilon_j}{x_j} - \frac{\varepsilon_{j-1}}{x_j x_{j-1}} - \cdots - \frac{\varepsilon_1}{x_j \cdots x_1} \quad \bmod 1\,, \tag{15}$$

where for $n=1, 2, \cdots, j$, $\varepsilon_n = x_n Q(l_{n+1}, \cdots, l_k) - [x_n Q(l_{n+1}, \cdots, l_k)]$ for $n=1, 2, \cdots, j$. Suppose that $(u_1, \cdots, u_k) \in B$ and put

$$(u_1, \cdots, u_k) \equiv M(-e_1, \cdots, -e_k) \quad \bmod 1\,,$$

where $0 \le e_n \le 1$, for $n=1, 2, \cdots, k$. Then we have

$$\frac{-i_j+l_j-\varepsilon_j+e_j}{x_j} \equiv 0 \quad \bmod 1\,.$$

Let $i_j=0$ and $l_j=1$. Then we see

$$\frac{1-\varepsilon_j+e_j}{x_j} \equiv 0 \quad \bmod 1\,.$$

This is contradictory to $x_j>2$. In the case of $i_j=1$ and $l_j=0$ we can have also a contradiction. Therefore we get $M(-{}^t a)+Q_b^t \beta \notin B$. The proof of (4) is easily derived from (3).

LEMMA 4. *For $a=(i_1, \cdots, i_k) \in S$, let m be a natural number such that $M(-{}^t a)+m{}^t\beta \in in(B)$ mod 1, where $in(X)$ is the set of any interior points of X. Then, there exists a natural number n such that $m=[x_1 \cdots [x_{k-1}[x_k n+i_k]+i_{k-1}] \cdots +i_1]$.*

PROOF. The proof is obtained as same as the proof of Lemma 2.

To observe when $M(-{}^t a)+m{}^t\beta$ belongs to the boundary ∂B of B we introduce the notation $G_{(i_1,\cdots,i_n)}^{(a_1,\cdots,a_n)}$ as follows. For an integer $n \ge 0$ and $(a_1, \cdots, a_n) \in \mathbf{R}^n$ and $(i_1, \cdots, i_n) \in S$, define $G_{(i_1,\cdots,i_n)}^{(a_1,\cdots,a_n)}$ by

$$G_{(i_1,\cdots,i_n)}^{(a_1,\cdots,a_n)} = [a_1[\cdots[a_n+i_n]\cdots]+i_1] \qquad \text{if} \quad n>0\,,$$

$$G_\phi^\phi = 1 \qquad \text{if} \quad n=0\,.$$

LEMMA 5. *For $a=(i_1, \cdots, i_k) \in S$,*

$$\{m \in N \mid M(-{}^t a)+m{}^t\beta \in \partial B \bmod 1\} = \{G_{(i_1,\cdots,i_{l-1})}^{(x_1,\cdots,x_{l-1})} \mid i_l=1\}\,.$$

Moreover let us assume that $M(-{}^t a)+G_{(i_1,\cdots,i_{l-1})}^{(x_1,\cdots,x_{l-1})}{}^t\beta \equiv M^t(-\varepsilon_1, \cdots, -e_k) \in \partial B$ (mod 1), where $i_l=1$. Then $0<\varepsilon_j<1$ for $1 \le j<l$ and $\varepsilon_l=0$ and $\varepsilon_j=i_j$ for $l<j$.

PROOF. Let m be a natural number such that $M(-{}^t a)+m{}^t\beta \in \partial B$ mod 1. Then, there exists $(-\varepsilon_1, \cdots, -\varepsilon_k) \in [-1, 0]^k$ and $M(-{}^t a)+m{}^t\beta \equiv M^t(-\varepsilon_1, \cdots, -\varepsilon_k)$ mod 1. Let l be a natural number such that $0<\varepsilon_j<1$ for $1 \le j<l$ and $\varepsilon_l \in \{0, 1\}$. Then analogously in the proof of Lemma 3 we get the integers $P_1, \cdots, P_l>0$ such that (1) $P_1=m$, (2) $P_j=[x_j P_{j+1}+i_j]$ for $j=1, \cdots, l-1$, (3) $\varepsilon_j=x_j P_{j+1}-[x_j P_{j+1}]$ for

$j=1, \cdots, l-1$. Therefore, inductively we get

$$\begin{aligned}\frac{m}{x_1\cdots x_l}-\frac{i_l}{x_l}-\cdots-\frac{i_1}{x_1\cdots x_l}&=\frac{[x_1P_2+i_1]}{x_1\cdots x_l}-\frac{i_l}{x_l}-\cdots-\frac{i_1}{x_1\cdots x_l}\\&=\frac{x_1P_2-\varepsilon_1+i_1}{x_1\cdots x_l}-\frac{i_l}{x_l}-\cdots-\frac{i_1}{x_1\cdots x_l}\\&=\frac{P_2}{x_2\cdots x_l}-\frac{\varepsilon_1}{x_1\cdots x_k}-\frac{i_l}{x_l}-\cdots-\frac{i_2}{x_2\cdots x_l}\\&=\frac{P_l}{x_l}-\frac{\varepsilon_{l-1}}{x_{l-1}\cdots x_1}-\cdots-\frac{\varepsilon_1}{x_1\cdots x_k}-\frac{i_l}{x_l}.\end{aligned}\tag{16}$$

Therefore we get

$$P_l/x_l-i_l/x_l\equiv -\varepsilon_l/x_l \qquad \bmod 1 .$$

Let us assume that $\varepsilon_l=1$, then we have

$$\frac{P_l-i_l+1}{x_l}\equiv 0 \qquad \bmod 1 .$$

This contradicts the irrationality of x_l. Let us assume that $\varepsilon_l=0$, then, we have

$$\frac{P_l-i_l}{x_l}\equiv 0 \qquad \bmod 1 .$$

Therefore we have $P_l=i_l=1$. By using the fact we have $m=[x_1P_2+i_1]=\cdots=[x_1[\cdots[x_{l-1}+i_{l-1}]\cdots]+i_1]=G^{(x_1,\cdots,x_{l-1})}_{(i_1,\cdots,i_{l-1})}$. Hence, we get

$$\{m\in\mathbf{N} \mid M(-{}^ta)+m{}^t\beta\in\partial B \bmod 1\}\subset\{G^{(x_1,\cdots,x_{l-1})}_{(i_1,\cdots,i_{l-1})} \mid i_l=1\} .$$

Conversely, for $1\le j\le k$, if we set

$$\varepsilon_j=\begin{cases} x_jG^{(x_{j+1},\cdots,x_{l-1})}_{(i_{j+1},\cdots,i_{l-1})}-[x_jG^{(x_{j+1},\cdots,x_{l-1})}_{(i_{j+1},\cdots,i_{l-1})}] & 1\le j<l-1\\ x_{l-1}-[x_{l-1}] & j=l-1\\ 0 & j=l\\ i_j & l<j\le k ,\end{cases}\tag{17}$$

then we get

$$-\frac{i_j}{x_j}-\cdots-\frac{i_1}{x_j\cdots x_1}+G^{(x_1,\cdots,x_{l-1})}_{(i_1,\cdots,i_{l-1})}\frac{1}{x_j\cdots x_1}\equiv-\frac{\varepsilon_j}{x_j}-\cdots-\frac{\varepsilon_1}{x_j\cdots x_1} \qquad \bmod 1 .$$

Therefore, we have

$$G^{(x_1,\cdots,x_{l-1})}_{(i_1,\cdots,i_{l-1})}\in\{m\in\mathbf{N} \mid M(-{}^ta)+m{}^t\beta\in\partial B \bmod 1\} .$$

We completed the first half of the lemma. And the last half of the lemma is derived easily from (17) and the uniqueness of $\varepsilon_1, \cdots, \varepsilon_k$.

We see easily the uniqueness of the expression of the numbers $G^{(x_1,\cdots,x_l)}_{(i_1,\cdots,i_l)}$ as follows.

LEMMA 6. *If* $(i_1, \cdots, i_l) \neq (j_1, \cdots, j_l)$, *then* $G^{(x_1,\cdots,x_l)}_{(i_1,\cdots,i_l)} \neq G^{(x_1,\cdots,x_l)}_{(j_1,\cdots,j_l)}$.

LEMMA 7. *If there exists* $e=(\varepsilon_1, \cdots, \varepsilon_k) \in \mathbf{R}^k$ *satisfying the both conditions*:

(A) $[0, 1]^k \cap [0, 1]^k + e$ *has an inner point.*

(B) *If the edge point of* $[0, 1]^k$ *is in* $[0, 1]^k + e$, *then it is in the boundary of* $[0, 1]^k + e$. *Then, there exists a natural number m satisfying the following conditions*:

(a) $\#\{a \in S \mid a \in \partial([0, 1]^k + e)\} = 2^m$.

(b) *If* $a=(i_1, \cdots, i_k) \in S$ *belongs to* $\partial([0, 1]^k + e)$, *then* $\#\{j \mid 1 \leq j \leq k, i_j - e_j \in \{0, 1\}\} = m$.

PROOF. From (A), we can easily derive that $|\varepsilon_j| \leq 1$ for $j=1, \cdots, k$. And from (B) we have $\varepsilon_i = 0$ for some i. To simplify, we assume that $\varepsilon_1 = \cdots = \varepsilon_m = 0$ and $\varepsilon_j \neq 0$ for $j > m$. Let $a=(i_1, \cdots, i_k) \in S$ satisfying $a - e \in [0, 1]^k$. Then we can easily derive that for $m < j$,

$$i_j = \begin{cases} 1 & \text{if } \varepsilon_j > 0 \\ 0 & \text{if } \varepsilon_j < 0 . \end{cases}$$

Therefore, we get

$$\{a \in S \mid a \in \partial([0, 1]^k + e)\} = \{a = (i_1, \cdots, i_k) \in S \mid i_j = sg(\varepsilon_j) \text{ for } m < j\} , \tag{18}$$

where

$$sg(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 . \end{cases}$$

From (18), we can derive (a) and (b).

In general for a recurrent transformation $F: X \to X$, the transformation F_A on A is said to be the induced transformation to the set A if the following relation holds

$$F_A x = F^{n(x)} x \qquad \text{for any} \quad x \in A , \tag{19}$$

where $n(x) = \min\{n \mid F^n x \in A, n \geq 1\}$. We will introduce the induced transformation T_B to the set B of the transformation $(\mathbf{R}^k/\mathbf{Z}^k, T)$, where $T: x \to x + \beta \bmod 1$. We denote the interval $[min(a, b), max(a, b)]$ by $\langle a, b \rangle$. We have the following lemma.

LEMMA 8. *For* $a=(i_1, \cdots, i_k) \in S$, *let* $B_a = M(\prod_{j=1}^k \langle -i_j, -1+\varepsilon_j^a \rangle)$. *Then we have*

(1) $B = \bigcup_{a \in S} B_a$ *and if* $a \neq b \in S$, $in(B_a) \cap in(B_b) = \emptyset$.

(2) $T_B x \equiv x + Q_a \beta \bmod 1$ *for* $x \in in(B_a)$ *and* $T_B(B_a) = M(\prod_{j=1}^k \langle -1+i_j, -\varepsilon_j^a \rangle)$.

(3) $m\beta \notin \partial B_a$, *for any natural number* $m > 0$ *and any* $a \in S$.

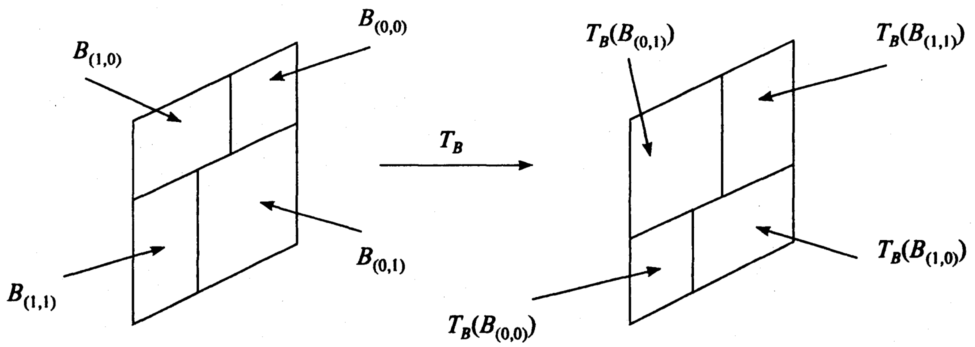

FIGURE 2.2. The induced transformation T_B $(k=2)$

PROOF. (1) We will show firstly that $\bigcup_{a\in S}\prod_{j=1}^{k}\langle -i_j, -1+\varepsilon_j^a\rangle=[-1,0]^k$. Let $(a_1, \cdots, a_k)$ is any element of $[-1,0]^k$. Note from Lemma 3 that ε_j^a is determined by $i_{j+1}, \cdots, i_k$ for $1\le j\le k-1$ and ε_k^a is the independent value for any a. Hence, we denote ε_j^a for $1\le j\le k-1$ by $\varepsilon_{(i_{j+1},\cdots,i_k)}$, where $a\in S$ and $a=(*, \cdots, *, i_{j+1}, \cdots, i_k)$, and we denote ε_k^a by ε. We will inductively construct $i_k, i_{k-1}, \cdots, i_1$ satisfying the following relation, for $j=k, k-1, \cdots, 1$,

$$a_j\in\langle -i_j, -1+\varepsilon_{(i_{j+1},\cdots,i_k)}\rangle . \tag{20}$$

Firstly, let i_k be

$$i_k=\begin{cases}0 & \text{if } a_k\in(-1+\varepsilon, 0]\\ 1 & \text{if } a_k\in[-1, -1+\varepsilon) .\end{cases}$$

And assume that $i_k, \cdots, i_{j+1}$ can be constructed satisfying (20). Then put i_j as follows:

$$i_j=\begin{cases}0 & \text{if } a_j\in(-1+\varepsilon_{(i_{j+1},\cdots,i_k)}, 0]\\ 1 & \text{if } a_j\in[-1, -1+\varepsilon_{(i_{j+1},\cdots,i_k)}) .\end{cases}$$

Then we have

$$a_j\in\langle -i_j, -1+\varepsilon_{(i_{j+1},\cdots,i_k)}\rangle .$$

Therefore we can construct by induction $i_1, \cdots, i_k$. Hence we get that

$$(a_1, \cdots, a_k)\in\prod_{j=1}^{k}\langle -i_j, -1+\varepsilon_j^a\rangle .$$

The proof of disjointness of B_a and B_b is easy.

(2) Let x be any element of B_a, where $a=(i_1, \cdots, i_k)$. By Lemma 3, we know $M(-{}^t a)+Q_a\beta\equiv M^t(-\varepsilon_1^a, \cdots, -\varepsilon_k^a) \bmod 1$. Hence, by the definition of B_a we get

$$B_a+Q_a\beta=M\left(\prod_{j=1}^{k}\langle -1+i_j, -\varepsilon_j^a\rangle\right) \quad \bmod 1 .$$

Hence, we have $x+Q_a\beta\in B$. We will show that $x+Q_a\beta$ is the first return point to B. Let $m>0$ be natural number such that $x+m\beta\in B \bmod 1$. Then, $B+m\beta\cap B\neq\emptyset$ in $\mathbf{R}^k/\mathbf{Z}^k$. Hence, there exists $b=(j_1,\cdots,j_k)\in S$ such that $M(-{}^tb)+m^t\beta\in B$. Firstly, we assume that $M(-{}^tb)+m^t\beta\in in(B)$. From Lemma 4, there exists a natural number $n>0$ such that $m=[x_1[\cdots[x_kn+j_k]\cdots]+j_1]$. If $n\geq 2$, by Lemma 1 we get,

$$m=[x_1[\cdots[x_kn+j_k]\cdots]+j_1]\geq[x_1[\cdots[x_k2]\cdots]]>Q_a\,.$$

Let us consider the case of $n=1$. We assume that $a\neq b$. Then, we know that $x\in B_b$. But from (2) of this lemma, it is impossible. Therefore $m=Q_a$. Let us consider the case of $M(-{}^tb)+m^t\beta\in\partial B$. Then we see from Lemma 5, $m=G^{(x_1,\cdots,x_{l-1})}_{(i_1,\cdots,i_{l-1})}$, where $l\leq k$ and $i_l=1$. And from Lemma 5 and Lemma 6, we know

$$\{b\in S \mid M(-{}^tb)+m^t\beta\in\partial B\}=\{(n_1,\cdots,n_k)\in S \mid n_j=i_j \text{ for } 1\leq j\leq l\}\,.$$

Therefore, we have

$$\#\{b\in S \mid M(-{}^tb)+m^t\beta\in\partial B\}=2^{k-l}\,,$$

and for $b=(n_1,\cdots,n_k)\in\{b\in S \mid M(-{}^tb)+m^t\beta\in\partial B\}$ we get

$$\#\{j \mid 1\leq j\leq k,\ \varepsilon_j\in\{0,1\}\}=k-l+1\,,$$

where $M^t(-\varepsilon_1,\cdots,\varepsilon_k)=M(-{}^tb)+m^t\beta$.

On the other hand from the fact that $B+m\beta\cap B\neq\emptyset$ in $\mathbf{R}^k/\mathbf{Z}^k$ and Lemma 7 we know that there exists $r\in\mathbf{N}$ such that

$$\#\{b\in S \mid M(-{}^tb)+m^t\beta\in\partial B\}=2^r\,,$$

and for $b=(n_1,\cdots,n_k)\in\{b\in S \mid M(-{}^tb)+m^t\beta\in\partial B\}$ we get

$$\#\{j \mid 1\leq j\leq k,\ \varepsilon_j\in\{0,1\}\}=r\,,$$

where $M^t(-\varepsilon_1,\cdots,-\varepsilon_k)=M(-{}^tb)+m^t\beta$. However, this result contradicts the previous result. Thus the proof is completed.

(3) Suppose that $m\beta\in\partial B_a \bmod 1$ for some $m>0\in\mathbf{N}$ and some $a=(i_1,\cdots,i_k)\in S$. And let $(-\varepsilon_1,\cdots,-\varepsilon_k)\equiv m\beta \bmod 1$, where $0\leq\varepsilon_j\leq 1$, for $j=1,\cdots,k$. Then, as same as Lemma 2, we know that there exists $n\in\mathbf{N}$ such that $m=P^k_n(x_1,\cdots,x_k)$ and satisfying the following relation: for $j=1,\cdots,k$,

$$\varepsilon_j=x_jP^{k-j}_n(x_{j+1},\cdots,x_k)-[x_jP^{k-j}_n(x_{j+1},\cdots,x_k)]\,.$$

Therefore we know that $0<\varepsilon_j<1$, for $j=1,\cdots,k$. From the definition of B_a, there exists a natural number j such that $-1+\varepsilon^a_j=-\varepsilon_j$. From the proof of Lemma 3, we get

$$\begin{aligned}&-x_jP^{k-j}_n(x_{j+1},\cdots,x_k)+[x_jP^{k-j}_n(x_{j+1},\cdots,x_k)]\\&\quad=-1+x_jQ^{k-j}(i_{j+1},\cdots,i_k)-[x_jQ^{k-j}(i_{j+1},\cdots,i_k)]\,.\end{aligned}$$

From the fact that x_j is the irrational number the above equation is impossible. Therefore we have $m\beta \notin \partial B \mod 1$.

LEMMA 9. *The following relation holds.*

$$P_n(\{Q_i^k(x_1, \cdots, x_k)\}_{i=1}^{\infty}) = \#\left\{ \bigcap_{j=0}^{n-1} T_B^{-j}(B_{b_j}) \;\middle|\; b_j \in S \text{ and } in\left(\bigcap_{j=0}^{n-1} T_B^{-j}(B_{b_j}) \right) \neq \varnothing \right\}. \quad (21)$$

PROOF. We show that there exists a one to one and onto map between

$$\{(Q_i^k(x_1, \cdots, x_k), \cdots, Q_{i+n-1}^k(x_1, \cdots, x_k)) \mid i=1, \cdots\}$$

and

$$\left\{ \bigcap_{j=0}^{n-1} T_B^{-j}(B_{b_j}) \;\middle|\; b_j \in S \text{ and } in\left(\bigcap_{j=0}^{n-1} T_B^{-j}(B_{b_j}) \right) \neq \varnothing \right\}.$$

From Lemma 8 (2), for any natural number $i>0$ there exists $a_i \in S$ such that $Q_{a_i} = Q_i^k(x_1, \cdots, x_k)$. We make $(Q_i^k(x_1, \cdots, x_k), \cdots, Q_{i+n-1}^k(x_1, \cdots, x_k))$ corresponded to $\bigcap_{j=0}^{n-1} T_B^{-j}(B_{a_{i+j}})$. This mapping is denoted by ϕ. Firstly, we show that ϕ is well defined. From Lemma 2 and Lemma 8, we get

$$P_i^k(x_1, \cdots, x_k)\beta \in B_{a_i} \qquad \mod 1 . \quad (22)$$

From Lemma 2, we know

$$P_{i+j}^k(x_1, \cdots, x_k)\beta \equiv T_B^j(P_i^k(x_1, \cdots, x_k)\beta) \qquad \mod 1 .$$

From (22) we have $T_B^j(P_i^k(x_1, \cdots, x_k)\beta) \in B_{a_{i+j}} \mod 1$. Therefore, we get

$$P_i^k(x_1, \cdots, x_k)\beta \in \bigcap_{j=0}^{n-1} T_B^{-j}(B_{a_{i+j}}) \qquad \mod 1 .$$

This gives the well definedness of ϕ. Next we will show that ϕ is the onto mapping. Let

$$in\left(\bigcap_{j=0}^{n-1} T_B^{-j}(B_{b_j}) \right) \neq \varnothing ,$$

where $b_j \in S$. By the Kronecker Approximation Theorem (for example [4]) and formula (1) we know that the set $\{m\beta \mid m \in \mathbf{N}\}$ is dense in $\mathbf{R}^k/\mathbf{Z}^k$. Therefore from Lemma 3, we see that $\{P_i^k(x_1, \cdots, x_k)\beta \mid i=1, 2, \cdots\}$ is dense in B. We know that there exists natural number i such that

$$P_i^k(x_1, \cdots, x_k)\beta \in in\left(\bigcap_{j=0}^{n-1} T_B^{-j}(B_{b_j}) \right).$$

Therefore, we have $P_{i+j}^k(x_1, \cdots, x_k)\beta \in B_{b_j}$ for $j=0, \cdots, n-1$. From Lemma 8, we get $Q_{i+j}^k(x_1, \cdots, x_k) = Q_{b_j}$ for $j=0, \cdots, n-1$. Therefore we know that ϕ is the onto mapping. And it is easily shown that ϕ is the one to one mapping.

Define the transformation $T_{(x_1,\cdots,x_k)}$ on $[-1, 0]^k$ by the following equation:

$$T_{(x_1,\cdots,x_k)}=M^{-1}T_B M .$$

Then, from Lemma 8, $T_{(x_1,\cdots,x_k)}$ is also defined as the following formula:

$$T_{(x_1,\cdots,x_k)}(x)=x+a-\varepsilon^a \qquad \text{if } x\in B_a^{(x_1,\cdots,x_k)} \text{ for } a\in S , \tag{23}$$

where $B_a^{(x_1,\cdots,x_k)}=\prod_{j=1}^k \langle -i_j, -1+\varepsilon_j^a \rangle$. Let π be the projection $\mathbf{R}^k \to \mathbf{R}^{k-1}$ satisfying $\pi(y_1, \cdots, y_k)=(y_2, \cdots, y_k)$. From now on we denote $S=\{(i_1, \cdots, i_k) \mid i_j\in\{0, 1\}$ for $j=1, \cdots, k\}$ by S_k.

LEMMA 10. *The following commutative relation holds.*

$$\begin{array}{ccc} [-1, 0]^k & \xrightarrow{T_{(x_1,\cdots,x_k)}} & [-1, 0]^k \\ \pi\downarrow & & \pi\downarrow \\ [-1, 0]^{k-1} & \xrightarrow{T_{(x_2,\cdots,x_k)}} & [-1, 0]^{k-1} \end{array} \quad . \tag{24}$$

And, for $a\in S$,

$$\pi(B_a^{(x_1,\cdots,x_k)})=B_{\pi(a)}^{(x_2,\cdots,x_k)} . \tag{25}$$

PROOF. From the equation (14), we can derive that for $a\in S$, $\varepsilon_j^a=\varepsilon_{j-1}^{\pi(a)}$ for $2\le j\le k$. Therefore, from the definition of $B_a^{(x_1,\cdots,x_k)}$, we can easily get the lemma.

For a natural number $n>0$ let us introduce the partition $\Delta_n^{(x_1,\cdots,x_k)}$ as follows:

$$\left\{ \bigcap_{j=0}^{n-1} T^j_{(x_1,\cdots,x_k)}(B_{b_j}^{(x_1,\cdots,x_k)}) \;\middle|\; b_j\in S_k \text{ and } in\left(\bigcap_{j=0}^{n-1} T^j_{(x_1,\cdots,x_k)}(B_{b_j}^{(x_1,\cdots,x_k)}) \right) \neq \varnothing \right\} .$$

Then we have the following lemma.

LEMMA 11. *The mapping* $\Pi : \Delta_n^{(x_1,\cdots,x_k)} \to \Delta_n^{(x_2,\cdots,x_k)}$ *is defined by*

$$\Pi(x)=\pi(x) \qquad \textit{for} \quad x\in \Delta_n^{(x_1,\cdots,x_k)} .$$

Then, for any $y\in \Delta_n^{(x_2,\cdots,x_k)}$ *we have*

$$\sharp \Pi^{-1}(y)=n+1 \quad \textit{and} \quad \left(\bigcup_{x\in\Pi^{-1}(y)} x \right)=[-1, 0]\times y .$$

PROOF. From Lemma 10, Π is well defined. And we note that for any j and $a\in S$, the i-coordinate of $\partial(T^j_{(x_1,\cdots,x_k)}B_a^{(x_1,\cdots,x_k)})$ are in the set $\{0, 1\} \cup \{\{-mx_i\} \mid m\in\mathbf{N}\}$. It is concluded by the formula (23). We will prove this lemma by the induction on n. Let $n=1$. Then, we have

$$\Delta_1^{(x_1,\cdots,x_k)}=\{B_a^{(x_1,\cdots,x_k)} \mid a\in S_k\} , \qquad \Delta_1^{(x_2,\cdots,x_k)}=\{B_a^{(x_2,\cdots,x_k)} \mid a\in S_{k-1}\} .$$

Therefore, from Lemma 10, we get

$$\Pi^{-1}(B^{(x_2,\cdots,x_k)}_{(i_2,\cdots,i_k)})=\{B^{(x_1,x_2,\cdots,x_k)}_{(0,i_2,\cdots,i_k)},\ B^{(x_1,x_2,\cdots,x_k)}_{(1,i_2,\cdots,i_k)}\}\,.$$

And from the formula (17), we get $\varepsilon_1^{(0,i_2,\cdots,i_k)}=\varepsilon_1^{(1,i_2,\cdots,i_k)}$. Therefore, we have

$$\langle 0,\ -1+\varepsilon_1^{(0,i_2,\cdots,i_k)}\rangle\cup\langle -1,\ -1+\varepsilon_1^{(1,i_2,\cdots,i_k)}\rangle=[-1,0]\,.$$

Hence, we get

$$B^{(x_1,x_2,\cdots,x_k)}_{(0,i_2,\cdots,i_k)}\cup B^{(x_1,x_2,\cdots,x_k)}_{(1,i_2,\cdots,i_k)}=[-1,0]\times B^{(x_2,\cdots,x_k)}_{(i_2,\cdots,i_k)}\,.$$

This means that the case that $n=1$ is verified. We assume that $n>1$. By the inductive assumption, for any $y\in\Delta_{n-1}^{(x_2,\cdots,x_k)}$

$$\sharp\Pi^{-1}(y)=n \quad\text{and}\quad \bigcup_{x\in\Pi^{-1}(y)} x=[-1,0]\times y\,.$$

From the definition of $\Delta_n^{(x_1,\cdots,x_k)}$, we get

$$\begin{aligned}\Delta_n^{(x_1,\cdots,x_k)}&=\{B_a^{(x_1,\cdots,x_k)}\cap T_{(x_1,\cdots,x_k)}(x)\mid a\in S_k \text{ and } x\in\Delta_{n-1}^{(x_1,\cdots,x_k)}\}\,,\\ \Delta_n^{(x_2,\cdots,x_k)}&=\{B_a^{(x_2,\cdots,x_k)}\cap T_{(x_2,\cdots,x_k)}(y)\mid a\in S_{k-1} \text{ and } y\in\Delta_{n-1}^{(x_2,\cdots,x_k)}\}\,.\end{aligned}\tag{26}$$

Then, we get the following fact for $y\in\Delta_{n-1}^{(x_2,\cdots,x_k)}$ and $a=(i_2,\cdots,i_k)\in S_{k-1}$

$$\begin{aligned}&\Pi^{-1}(T_{(x_2,\cdots,x_k)}(y)\cap B_a^{(x_2,\cdots,x_k)})\\&=\{T_{(x_1,\cdots,x_k)}(x)\cap B_{a^0}^{(x_1,\cdots,x_k)}\mid x\in\Pi^{-1}(y)\}\bigcup\{T_{(x_1,\cdots,x_k)}(x)\cap B_{a^1}^{(x_1,\cdots,x_k)}\mid x\in\Pi^{-1}(y)\}\,,\end{aligned}$$

where $a^i=(i,i_2,\cdots,i_k)$. Therefore, we get

$$\begin{aligned}\bigcup_{x\in\Pi^{-1}(T_{(x_2,\dots,x_k)}(y)\cap B_a^{(x_2,\cdots,x_k)})} z&=(B_{a^0}^{(x_1,\cdots,x_k)}\cup B_{a^1}^{(x_1,\cdots,x_k)}\cap\bigcup_{x\in\Pi^{-1}(y)}T_{(x_1,\cdots,x_k)}(x)\\&=[-1,0]\times B_a^{(x_2,\cdots,x_k)}\cap[-1,0]\times y=[-1,0]\times(B_a^{(x_2,\cdots,x_k)}\cap y)\,.\end{aligned}\tag{27}$$

There exist real numbers p, q such that

$$T_{(x_1,\cdots,x_k)}(x)=[p,q]\times y\,.$$

We may assume that there exists numbers $p_0<p_1,\cdots<p_n$ such that

$$\{T_{(x_1,\cdots,x_k)}(x)\mid x\in\Pi^{-1}(y)\}=\{[p_i,p_{i+1}]\times y\mid i=0,1,\cdots,n-1\}\,.$$

From the note before, we derive that $p_i\in\{0,1\}\cup\{\{-mx_i\}\mid m\in\mathbf{N}\}$. Therefore, we get

$$-1+\varepsilon_1^{a^0}\notin\{p_i\mid i=0,1,\cdots,n\}\,.$$

Therefore, there exists natural number l such that $0\le l\le n$ and $p_l<-1+\varepsilon_1<p_{l+1}$. Hence we get

$$\begin{aligned}&\{T_{(x_1,\cdots,x_k)}(x)\cap B_{a^i}^{(x_1,\cdots,x_k)}(\neq\varnothing)\mid i\in\{0,1\},\ x\in\Pi^{-1}(y)\}\\&=\{[p_m,p_{m+1}]\times y\cap B_a^{(x_2,\cdots,x_k)}\mid 0\le i<l \text{ or } l<m\le n-1\}\end{aligned}$$

$$\cup\{[p_l, -1+\varepsilon_1]\times y\cap B_a^{(x_2,\cdots,x_k)}\}\cup\{[-1+\varepsilon_1, p_{l+1}]\times y\cap B_a^{(x_2,\cdots,x_k)}\}\,.$$

Therefore, we get

$$\sharp\Pi^{-1}(T_{(x_2,\cdots,x_k)}(y)\cap B_a^{(x_2,\cdots,x_k)})=n+1\,.$$

Hence the proof is completed.

LEMMA 12. *For any irrational number* $x>1$, *we know* $\sharp\Delta_n^{(x)}=n+1$.

The lemma is proved as same as Lemma 11.

PROOF OF THE THEOREM. From Lemma 10, we get

$$\begin{aligned}P_n(\{Q_i^k(x_1,\cdots,x_k)\}_{i=1}^{\infty})&=\sharp\left\{\bigcap_{j=0}^{n-1}T_B^{-j}(B_{b_j})\;\middle|\;b_j\in S \text{ and } in\left(\bigcap_{j=0}^{n-1}T_B^{-j}(B_{b_j})\right)\neq\varnothing\right\}\\&=\sharp\left\{\bigcap_{j=0}^{n-1}T_B^{j}(B_{b_j})\;\middle|\;b_j\in S \text{ and } in\left(\bigcap_{j=0}^{n-1}T_B^{j}(B_{b_j})\right)\neq\varnothing\right\}\\&=\sharp\Delta_n^{(x_1,\cdots,x_k)}\,.\end{aligned}\qquad(28)$$

From Lemma 11 and Lemma 12, we get $\sharp\Delta_n^{(x_1,\cdots,x_k)}=(n+1)^k$. Therefore, we get

$$P_n(\{Q_i^k(x_1,\cdots,x_k)\}_{i=1}^{\infty})=(n+1)^k\,.$$

ACKNOWLEDGEMENT. We wish to thank Prof. Shunji Ito for providing nice working condition and his helpful comments. We wish to thank Prof. Shigeru Tanaka and Prof. Makoto Mori for their helpful comments.

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