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On p-Adic Log-Γ-Functions Associated to the Lubin-Tate Formal Groups

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Introduction.

For a prime number p, let $\mathbf{Q}_p$ be the p-adic number field, $\mathbf{Z}_p$ be the ring of p-adic integers in $\mathbf{Q}_p$, and $\mathbf{C}_p$ be the completion of algebraic closure of $\mathbf{Q}_p$.

Let $F(X, Y) \in \mathbf{Z}_p[[X, Y]]$ be the Lubin-Tate formal group and $h(X) \in \Theta((X))^{\times}$ be a meromorphic series where Θ is the ring of p-adic integers in $\mathbf{C}_p$. In [9], Shiratani and Imada constructed a p-adic meromorphic function $\zeta_p(s, F, h)$ which was a generalization of the ordinary p-adic zeta function $\zeta_p(s)$. In fact, in the case that $F(X, Y) = G_m(X, Y) = (X+1)(Y+1)-1$ and $h(X) = X$, we have $\zeta_p(s, G_m, X) = \zeta_p(s)$. In the case that $F(X, Y) = \xi(X, Y)$ which is the formal group associated with elliptic curves with complex multiplication defined over $\mathbf{Z}$ with ordinary reduction, $\zeta_p(s, \xi, X)$ coincides with the p-adic zeta function defined by Lichtenbaum in [5].

In the present paper, we construct a function $T_{p,c}(s, F, h)$ for $c \in \mathbf{Z}_p^{\times}$, which we can regard as a generalization of the Morita p-adic log-Γ-function (cf. [7]) twisted by c. By using $T_{p,c}(s, F, h)$, we describe the values of $\zeta_p(s, F, h)$ at positive integers (see §3).

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1. Notations.

According to [2], [6], [9] and [10], we prepare some notations with respect to the formal groups. Let $k/\mathbf{Q}_p$ be a finite unramified extension and Θ_k be the ring of p-adic integers in k. Let π be a prime element in Θ_k, and $f(x) \in \Theta_k[[X]]$ be the Frobenius power series determined by π, namely $f(X)$ is a power series which satisfies

$$f(X) \equiv \pi X \ (\text{mod degree } 2) \quad \text{and} \quad f(X) \equiv X^p \ (\text{mod}\, \pi)\,. \tag{1.1}$$

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There exists a unique formal group $F(X, Y)\in \Theta_k[[X, Y]]$ such that f is an endomorphism of F. So $F(X, Y)$ is called the relative Lubin-Tate formal group associated with f (see [6]). We denote by $\lambda_F(X)$ and $e_F(X)$ respectively the logarithmic series and the exponential series of $F(X, Y)$ (cf. [2] Chap. 4 §1). Namely $\lambda_F(X)$ satisfies $\lambda_F(F(X, Y))=\lambda_F(X)+\lambda_F(Y)$ and $\lambda'_F(0)=1$, and $e_F(X)$ is the inverse series of $\lambda_F(X)$. In the case that $F(X, Y)=G_m(X, Y)$,

$$\lambda_{G_m}(X)=\log(1+X) \quad \text{and} \quad e_{G_m}(X)=e^X-1\,. \tag{1.2}$$

We use the same notation as that in [9] and [10]. Let K be the maximal unramified extension of k, $\bar{K}$ be the completion of K, and φ be the Frobenius automorphism of $\bar{K}$ over k. There is an isomorphism $\phi_F\colon G_m\simeq F$ over $\Theta_{\bar{K}}^{\times}$ such that $\kappa^{\varphi-1}=p/\pi$ where $\kappa=\phi'_F(0)^{-1}\in\Theta_{\bar{K}}$. Note that p is a prime element in k, since $k/\mathbf{Q}_p$ is a finite unramified extension. Then we have the following (see [10] Introduction):

$$\phi_F(e^z-1)=e_F(\kappa^{-1}z)\,. \tag{1.3}$$

2. The Shiratani-Imada p-adic zeta-function $\zeta_p(s, F, h)$.

Now we reconstruct the Shiratani-Imada function $\zeta_p(s, F, h)$ by using the theory of p-adic Γ-transform.

Shiratani and Imada defined the numbers $\{B_n(F, h)\}$ by

$$G(z, F, h)=\frac{zh'(e_F(z))}{\lambda'_F(e_F(z))h(e_F(z))}=\sum_{n=0}^{\infty}B_n(F, h)\frac{z^n}{n!}$$

for $h(X)\in\Theta((X))^{\times}$. By (1.2), we have $B_n(G_m, X)=B_n$, where $\{B_n\}$ is the ordinary Bernoulli numbers. We let

$$g(T, F, h)=\frac{h'(\phi_F(T))}{\lambda'_F(\phi_F(T))h(\phi_F(T))}\,.$$

Since $\lambda'_F(X)\in\Theta[[X]]$ (see [9] §2) and $h(X)\in\Theta((X))^{\times}$, we have

$$g(T, F, h)\in\frac{1}{T}\,\Theta[[T]]\,.$$

By (1.3), we have

$$\kappa^{-1}zg(e^z-1, F, h)=G(\kappa^{-1}z, F, h)\,. \tag{2.1}$$

Select $c\in\mathbf{Z}_p^{\times}$ with $c\neq 1$, and let

$$g_c(T, F, h)=cg((1+T)^c-1, F, h)-g(T, F, h)\,. \tag{2.2}$$

We can prove that $g_c(T, F, h)\in\Theta[[T]]$. By (2.1) and (2.2), we have the following.

LEMMA 2.1. *For $c\in\mathbf{Z}_p^{\times}$ with $c\neq 1$,*

$$\kappa^{-1}zg_c(e^z-1, F, h)=G(c\kappa^{-1}z, F, h)-G(\kappa^{-1}z, F, h).$$

Now we recall Coleman's norm operator N_F associated with F. Namely, for any $h(X)\in\Theta((X))$, we can uniquely determine $N_F h(X)\in\Theta((X))$ which satisfies

$$N_F h([\pi]_F(X))=\prod_{a\in A_0} h(F(X, a)),$$

where $A_0=\{\phi_F(\xi-1); \xi^p=1\}$ and $[\pi]_F(X)=f(X)$ (see [1] Theorem 11).

LEMMA 2.2 (Shiratani-Imada).

$$\pi g((1+T)^\pi-1, F, N_F h)=\sum_{\xi^p=1} g(\xi(1+T)-1, F, h).$$

PROOF. See [9] Lemma 7.

LEMMA 2.3.

$$\kappa^{-1}zUg_c(e^z-1, F, h)=G(c\kappa^{-1}z, F, h)-G(\kappa^{-1}z, F, h)$$
$$-\frac{1}{p}\{G(c\kappa^{-1}\pi z, F, N_F h)-G(\kappa^{-1}\pi z, F, N_F h)\}.$$

PROOF. By (2.2) and Lemma 2.2, we have

$$Ug_c(T, F, h)=g_c(T, F, h)-\frac{\pi}{p} g_c((1+T)^\pi-1, F, N_F h).$$

By Lemma 2.1, we have the assertion.

Let $\mu_{c,F,h}$ be a Θ-valued measure which corresponds to $g_c(T, F, h)$. By [12], we have the following.

LEMMA 2.4. $Ug_c(T, F, h)=\int_{\mathbf{Z}_p^\times}(1+T)^x d\mu_{c,F,h}(x)$.

PROOF. See [12] Proposition 12.8.

PROPOSITION 2.5. *For $n\in\mathbf{Z}$ with $n\geq 1$,*

$$\int_{\mathbf{Z}_p^\times} x^{n-1}d\mu_{c,F,h}(x)=\frac{(c^n-1)\kappa^{1-n}}{n}\left\{B_n(F, h)-\frac{\pi^n}{p}B_n(F, N_F h)\right\}.$$

PROOF. By Lemma 2.3 and Lemma 2.4, we have

$$\kappa^{-1}z\int_{\mathbf{Z}_p^\times} e^{xz}d\mu_{c,F,h}(x)=\kappa^{-1}zUg_c(e^z-1, F, h)$$
$$=\sum_{m=0}^{\infty}(c^m-1)\kappa^{-m}\left\{B_m(F, h)-\frac{\pi^m}{p}B_m(F, N_F h)\right\}\frac{z^m}{m!}.$$

Hence we have the assertion.

REMARK. By (2.2), we have

$$g_c(T, G_m, X) = \frac{c(1+T)^c}{(1+T)^c - 1} - \frac{1+T}{(1+T)-1} = \sum_{\substack{\xi^c=1 \\ \xi \neq 1}} \frac{1+T}{(1+T)-\xi}. \tag{2.3}$$

Hence we can see that $\mu_{c,G_m,X}$ is essentially equal to the measure defined by Koblitz in [3]. So $\mu_{c,F,h}$ can be regarded as a generalization of the Koblitz measure.

For $x \in \mathbf{Z}_p^\times$, we use the notation $x = \omega(x)\langle x \rangle$ corresponding to the usual decomposition $\mathbf{Z}_p^\times = V \times (1 + p\mathbf{Z}_p)$, where V is the group of roots of unity in $\mathbf{Z}_p^\times$. Since $\kappa \in \Theta_{\bar{K}}^\times$, we can select $\kappa_0 \in \Theta_K^\times$ such that $\kappa \equiv \kappa_0 (\bmod p)$. Moreover we can select κ_0 on condition that $[\mathbf{Q}_p(\kappa_0) : \mathbf{Q}_p]$ is the lowest. Let $E = \mathbf{Q}_p(\kappa_0)$ and Θ_E be the ring of p-adic integers in E. For $x \in \Theta_E^\times$, we also use the same notation $x = \omega(x)\langle x \rangle$ corresponding to the usual decomposition $\Theta_E^\times = V_E \times (1 + p\Theta_E)$. Since $\kappa \equiv \kappa_0 (\bmod p)$, we define $\omega(\kappa) = \omega(\kappa_0)$ and $\langle \kappa \rangle = \kappa / \omega(\kappa)$. We denote by $r(\kappa)$ the number of elements of V_E. Note that if $n \equiv 0 (\bmod r(\kappa))$, then $\kappa^n = \langle \kappa \rangle^n$. We define the following function.

$$\zeta_p(s, F, h) = \frac{\langle \kappa \rangle^{1-s}}{\kappa(1 - \langle c \rangle^{1-s})} \int_{\mathbf{Z}_p^\times} \langle x \rangle^{-s} \omega^{-1}(x) d\mu_{c,F,h}(x). \tag{2.4}$$

By Proposition 2.5, we can immediately prove the following.

PROPOSITION 2.6. *For $n \in \mathbf{Z}$ with $n \geq 1$ and $n \equiv 0 (\bmod r(\kappa))$,*

$$\zeta_p(1-n, F, h) = -\frac{1}{n} \left\{ B_n(F, h) - \frac{\pi^n}{p} B_n(F, N_F h) \right\}.$$

REMARK 1. We can see that $\zeta_p(s, F, h)$ coincides with the Shiratani-Imada p-adic ζ-function defined in [9]. In fact, the result in Proposition 2.6 is the same as the one in Theorem 9 in [9].

REMARK 2. As a generalization of the p-adic L-function $L_p(s, \omega^j)$ for $j \in \mathbf{Z}$, we define

$$L_p(s, \omega^j, F, h) = \frac{\langle \kappa \rangle^{1-s}}{\kappa(1 - \langle c \rangle^{1-s} \omega^j(c))} \int_{\mathbf{Z}_p^\times} \langle x \rangle^{-s} \omega^{j-1}(x) d\mu_{c,F,h}(x), \tag{2.5}$$

which is almost the same as the one defined by Kozuka in [4]. By the Koblitz result (see [3] (1.12)), we can see that $L_p(s, \omega^j, G_m, X) = L_p(s, \omega^j)$.

3. p-adic log-Γ-functions $T_{p,c}(z, F, h)$.

Now we define the function $T_{p,c}(z, F, h)$ by

$$T_{p,c}(z, F, h) = -\int_{\mathbf{Z}_p^\times} \log(x+z) d\mu_{c,F,h}(x)$$

for $z \in \mathbf{C}_p - \mathbf{Z}_p^\times$. Later on, we will be able to see that $T_{p,c}(z, F, h)$ is a generalization of the Morita p-adic log-Γ-function twisted by c (see Proposition 3.3). Let $\mathbf{P}^1(\mathbf{C}_p)$ be the one dimensional projective space over $\mathbf{C}_p$. In [8], Morita investigated the properties of analytic functions on an open subset of $\mathbf{P}^1(\mathbf{C}_p)$. According to Morita's result, we prove the following proposition.

PROPOSITION 3.1. *$(d/dz)T_{p,c}(z, F, h)$ is an analytic function on $\mathbf{P}^1(\mathbf{C}_p) - \mathbf{Z}_p^\times$.*

PROOF. For $m \in \mathbf{Z}$ with $m \geq 1$, let

$$C_m = \{z \in \mathbf{C}_p \,;\, |z+a| > p^{-m},\, a = 1, 2, \cdots, p^{m+1}-1,\, (a, p) = 1\}.$$

For any $m \geq 1$,

$$\begin{aligned} \frac{d}{dz} T_{p,c}(z, F, h) &= -\int_{\mathbf{Z}_p^\times} \frac{1}{x+z} d\mu_{c,F,h}(x) \\ &= -\sum_{j=1}^{p^{m+1}}{}^{*} \int_{j+p^{m+1}\mathbf{Z}_p} \frac{1}{x+z} d\mu_{c,F,h}(x). \end{aligned}$$

If $x = j + p^{m+1}y$ with $y \in \mathbf{Z}_p$, then

$$\frac{1}{x+z} = \frac{1}{j+z} \sum_{n=0}^{\infty} (-1)^n \frac{p^{n(m+1)}}{(j+z)^n} y^n.$$

So we have

$$\int_{j+p^{m+1}\mathbf{Z}_p} \frac{1}{x+z} d\mu_{c,F,h}(x) = \frac{1}{j+z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{p^m}{j+z}\right)^n p^n \int_{\mathbf{Z}_p} y^n d\mu'_{c,F,h}(y), \tag{3.1}$$

where $\mu'_{c,F,h}(y) = \mu_{c,F,h}(j + p^{m+1}y)$. Since $|p^m/(j+z)| < 1$ for $z \in C_m$, and

$$\left| \int_{\mathbf{Z}_p} y^n d\mu'_{c,F,h}(y) \right| \leq 1,$$

we can see that the right-hand side of (3.1) is uniformly convergent on C_m for $m \geq 1$. Note that $\mathbf{P}^1(\mathbf{C}_p) - \mathbf{Z}_p^\times = \bigcup_{m \geq 1} C_m$. By Morita's result (see [8] §2, §3), we can verify that $(d/dz)T_{p,c}(z, F, h)$ is an analytic function on $\mathbf{P}^1(\mathbf{C}_p) - \mathbf{Z}_p^\times$.

PROPOSITION 3.2 (p-adic Stirling expansions). *For $z \in \mathbf{C}_p$ with $|z| > 1$,*

$$\frac{d}{dz} T_{p,c}(z, F, h) = \sum_{n=0}^{\infty} \frac{(c^{n+1}-1)\kappa^{-n}}{n+1} \left\{ B_{n+1}(F, h) - \frac{\pi^{n+1}}{p} B_{n+1}(F, N_F h) \right\} \frac{(-1)^{n+1}}{z^{n+1}}.$$

PROOF. If $|z| > 1$, then we have

$$\frac{d}{dz}T_{p,c}(z,F,h)=-\int_{\mathbf{Z}_p^\times}\frac{1}{x+z}d\mu_{c,F,h}(x)=-\frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}\int_{\mathbf{Z}_p^\times}x^n d\mu_{c,F,h}(x)\,.$$

By Proposition 2.5, we have the assertion.

Now we recall the Morita p-adic log-Γ-function $\log\Gamma_p(z+1)$ (cf. [7], [8]). By [7] Theorem 5, the following relation holds:

$$\left(\frac{d}{dz}\right)^2\log\Gamma_p(z+1)=\sum_{m=1}^{\infty}mL_p(1-m,\omega^m)\frac{(-1)^{m+1}}{z^{m+1}} \tag{3.2}$$

for $z\in\mathbf{C}_p$ with $|z|>1$. By Remark 2 of Proposition 2.6, we have the following:

PROPOSITION 3.3.

$$\left(\frac{d}{dz}\right)^2 T_{p,c}(z,G_m,X)=\left(\frac{d}{dz}\right)^2\left\{\log\Gamma_p(z+1)-c\log\Gamma_p\left(\frac{z}{c}+1\right)\right\}.$$

PROOF. By (2.5), we have

$$\begin{aligned}\left(\frac{d}{dz}\right)^2 T_{p,c}(z,F,h)&=-\sum_{m=1}^{\infty}m\int_{\mathbf{Z}_p^\times}x^{m-1}d\mu_{c,F,h}(x)\frac{(-1)^m}{z^{m+1}}\\&=\sum_{m=1}^{\infty}m\int_{\mathbf{Z}_p^\times}\langle x\rangle^{m-1}\omega^{m-1}(x)d\mu_{c,F,h}(x)\frac{(-1)^{m+1}}{z^{m+1}}\,,\\&=\sum_{m=1}^{\infty}\kappa\langle\kappa\rangle^{-m}(1-c^m)mL_p(1-m,\omega^m,F,h)\frac{(-1)^{m+1}}{z^{m+1}}\,,\end{aligned}$$

for $z\in\mathbf{C}_p$ with $|z|>1$. If $F=G_m$ and $h(X)=X$, then we have $\kappa=1$. By (3.2), we have the assertion.

By the relation in Proposition 3.3, we can regard $T_{p,c}(z,F,h)$ as a generalization of $\log\Gamma_p(z+1)-c\log\Gamma_p(z/c+1)$. Finally, we describe the values of $\zeta_p(s,F,h)$ at positive integers, by using $T_{p,c}(z,F,h)$.

PROPOSITION 3.4. *For $m\in\mathbf{Z}$ with $m\geq 2$ and $m\equiv 1\,(\mathrm{mod}\,r(\kappa))$,*

$$\zeta_p(m,F,h)=\frac{(-1)^m\kappa^{-m}}{(m-1)!(1-c^{1-m})}\left(\frac{d}{dz}\right)^m T_{p,c}(z,F,h)\bigg|_{z=0}\,.$$

PROOF. By induction, we can prove that

$$\left(\frac{d}{dz}\right)^m T_{p,c}(z,F,h)=(-1)^m(m-1)!\int_{\mathbf{Z}_p^\times}\frac{1}{(x+z)^m}d\mu_{c,F,h}(x)\,,$$

for $m\geq 2$. By (2.4), we have the assertion.

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