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Linkage Property of Witt Ring and Torsion Quadratic Forms with Trivial Witt Invariant

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1. Introduction.

Let F be a field of characteristic different from 2. Let $W(F)$ be the Witt ring of quadratic forms over F. We denote by $W_t(F)$ and IF the maximal torsion subgroup of $W(F)$ and a fundamental ideal of $W(F)$ generated by all even dimensional quadratic forms respectively. For a positive integer $n \geqq 2$, we denote n-th power of IF by I^nF. For elements $a_1, a_2, \cdots, a_n$ of the multiplicative group $\dot{F}$ of F, $\langle a_1, a_2, \cdots, a_n\rangle$ denotes a diagonal quadratic form $a_1X_1^2 + \cdots + a_nX_n^2$. A quadratic form of the form $\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ is called an n-fold Pfister form.

Let φ_1 and φ_2 be two n-fold Pfister forms. If there exist an $(n-1)$-fold Pfister form σ and two 1-fold Pfister forms τ_1 and τ_2 such that $\varphi_i \cong \sigma \otimes \tau_i$ $(i=1, 2)$, we say φ_1 and φ_2 are linked (cf. Definition 2.2 of [4]). Further we say I^nF (resp. $I^nF \cap W_t(F)$) is linked if any pair of n-fold Pfister forms (resp. torsion n-fold Pfister forms) is linked. If F is non-formally real, then we know $W(F) = W_t(F)$ (cf. [4]). Thus, in this case, the notion I^nF is linked coincides with the notion $I^nF \cap W_t(F)$ is linked. Let us denote by $u(F)$ the u-invariant of F, which is defined to be the maximal dimension of anisotropic torsion quadratic forms over F. Elman and Lam showed following results (cf. Theorems 3.4, 4.3, of [4]).

(1.1) Let I^2F be linked. Then possible value of $u(F)$ is $0, 1, 2, 4$ or 8. Further $u(F) \leqq 4$ if and only if $I^3F \cap W_t(F) = \{0\}$,

(1.2) Let F be formally real. Assume $I^nF \cap W_t(F)$ is linked for every positive integer n. Then possible value of $u(F)$ is $0, 2, 4, 8, 16$ or 18.

If the condition of (1.1) is replaced by that I^3F is linked, then the linkage property of I^3F seems to limit $cu(F)$, which is the maximal dimension of anisotropic torsion forms with trivial Witt invariant, rather than $u(F)$. We show in this paper:

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THEOREM 1.1. *Let I^3F be linked. Then we have*

(1) $cu(F)=0, 1, 2, 8$ *or* 16,

(2) $cu(F)=0, 1, 2$ *or* $8 \Leftrightarrow I^4F \cap W_t(F)=\{0\}$.

Further we determine possible value of $cu(F)$ under a condition weaker than that of (1.2). We know the condition of (1.2) is equivalent to the assertion (T) (cf. Proposition 4.1 of [4]):

(T) For any positive integer n, any form of $I^nF \cap W_t(F)$ is congruent to an n-fold Pfister form modulo $I^{n+1}F$.

Let us consider the following assertion (T_l) which is weaker than (T), where l is a positive integer $\geqq 3$.

(T_l) For any positive integer $n \geqq l$, any form of $I^nF \cap W_t(F)$ is congruent to an n-fold Pfister form modulo $I^{n+1}F$.

Then we show:

THEOREM 1.2. *Let F be formally real. Assume* (T_3). *Then we have*

(1) $cu(F)=0, 2, 8, 16$ *or* 18,

(2) $cu(F)=0, 2$ *or* $8 \Leftrightarrow I^4F \cap W_t(F)=\{0\}$.

We prove these theorems in §3 and §4 respectively. In §2, we give auxiliary results needed to prove the theorems. We use the following notation. For a quadratic form φ, the dimension (resp. the signed determinant, the Witt invariant) of φ is denoted by $\dim\varphi$ (resp. $d_{\pm}\varphi$, $c(\varphi)$). For $a_1, \cdots, a_n \in \dot{F}=F-\{0\}$, an n-fold Pfister form $\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ is denoted by $\langle\langle a_1, \cdots, a_n\rangle\rangle$. We refer to Lam [5] for other basic definitions and notation concerning quadratic forms.

2. Preliminaries.

First of all, we recall some basic results needed below in the following theorems 2.1~2.4.

THEOREM 2.1 (Corollary 2.8, 2.11 of [4]). *Suppose I^nF is linked. Then*

(1) $I^{n+2}F \cap W_t(F)=\{0\}$,

(2) *if φ is an anisotropic form of $I^nF \cap W_t(F)$, then* $\dim\varphi=0, 2^n$ *or* 2^{n+1} *and* $\varphi=\langle x\rangle \otimes \mu_1 - \mu_2$ *in $W(F)$, where $x \in \dot{F}$, μ_1 and μ_2 are n-fold Pfister forms.*

THEOREM 2.2 ([6]). *Let φ be a quadratic form of $W(F)$. Then we have $\varphi \in I^3F \Leftrightarrow$* $\dim\varphi$ *is even*, $d_{\pm}\varphi=1$, $c(\varphi)=1$.

THEOREM 2.3 ([1]). *Let n be a positive integer. Then the dimension of any anisotropic form in I^nF is at least 2^n.*

THEOREM 2.4 (Theorem 4.8 of [2]). *Let $\varphi_1, \varphi_2, \varphi_3$ be n-fold Pfister forms such that $\varphi_1+\varphi_2+\varphi_3 \in I^{n+1}F$. Then any pair of φ_1, φ_2 and φ_3 is linked.*

Let us denote by $cu(F)$ the maximal dimension of anisotropic torsion forms of which Witt invariant is 1. Let $I_3(F)$ be the maximal dimension of anisotropic forms in $I^3F\cap W_t(F)$. Since the field F is fixed throughout this paper, henceforth we write simply u, cu and I_3 instead of $u(F)$, $cu(F)$ and $I_3(F)$. By the definition of cu and by Theorem 2.2, we have

(2.1) $I_3\leqq cu\leqq u$.

If the field F is formally real, then a torsion form is of even dimension (cf. Satz 22 of [7]). Therefore we have

(2.2) if F is formally real, then cu is even.

LEMMA 2.5. *For a quadratic form φ over F with $c(\varphi)=1$, define a form φ^{ass} by*

(2.3) $$\varphi^{ass}=\begin{cases}\langle -1, d_{\pm}\varphi\rangle\perp\varphi, & \text{if } \dim\varphi \text{ is even,}\\ \langle -d_{\pm}\varphi\rangle\perp\varphi, & \text{if } \dim\varphi \text{ is odd.}\end{cases}$$

Then we have $\varphi^{ass}\in I^3F$. Further if φ is a torsion form, then $\varphi^{ass}\in I^3F\cap W_t(F)$.

PROOF. By the properties of signed determinant and Witt invariant (cf. p. 38, p. 120 of [5]), we easily obtain $d_{\pm}\varphi^{ass}=1$, $c(\varphi^{ass})=1$. Thus by Theorem 2.2, we have $\varphi^{ass}\in I^3F$. If F is non-formally real, then we see $W(F)=W_t(F)$. Thus $\varphi^{ass}\in I^3F\cap W_t(F)$. Let F be formally real. Assume φ is a torsion form. Then φ is hyperbolic over any real closure of F. Therefore we have $d_{\pm}\varphi$ is totally positive. This shows a form $\langle -1, d_{\pm}\varphi\rangle$ is torsion (cf. [7]). Hence we have $\varphi^{ass}\in I^3F\cap W_t(F)$. □

PROPOSITION 2.6. *If cu is even, then $cu=I_3$ or I_3+2. If cu is odd, then $cu=I_3+1$.*

PROOF. Let φ be a cu-dimensional anisotropic torsion form with $c(\varphi)=1$. Let φ^{ass} be the form defined by (2.3). Then $\varphi^{ass}\in I^3F\cap W_t(F)$ by Lemma 2.5. Since $\dim\varphi^{ass}>cu$ and $c(\varphi^{ass})=1$, we know φ^{ass} is isotropic. Let m be the Witt index of φ^{ass}. Then $m\geqq 1$ and we have an isometry:

$$\varphi^{ass}\cong m\langle 1, -1\rangle\perp\varphi_0\,,$$

where φ_0 is an anisotropic form of $I^3F\cap W_t(F)$. Let cu be even. If $m>2$, then we have

$$\begin{aligned}\varphi^{ass}&=\langle -1, d_{\pm}\varphi\rangle\perp\varphi\\ &\cong\langle -1, 1\rangle\perp\langle -d_{\pm}\varphi, d_{\pm}\varphi\rangle\perp(m-2)\langle -1, 1\rangle\perp\varphi_0\,.\end{aligned}$$

By Witt's cancellation theorem, we have

$$\varphi\cong\langle 1, -d_{\pm}\varphi\rangle\perp(m-2)\langle -1, 1\rangle\perp\varphi_0\,.$$

Since φ is anisotropic, we have a contradiction. Hence $m=1$ or 2. Since the dimension of φ_0 is $cu-2(m-1)$, we have

$$cu-2(m-1)\leqq I_3\leqq cu\,.$$

Thus we see $cu=I_3$ or I_3+2. Let cu be odd. Then a similar argument shows $m=1$.

Thus we have

$$cu-1\leqq I_3\leqq cu\,.$$

Since cu is odd and I_3 is even, we have $cu=I_3+1$. □

PROPOSITION 2.7. *If I^4F is torsion free and $cu>2$, then there exists an anisotropic cu-dimensional form φ in $I^3F\cap W_t(F)$. In particular we have $cu=I_3$.*

PROOF. Let φ be an anisotropic torsion form of dimension cu with $c(\varphi)=1$ and φ^{ass} the form defined by (2.3). First of all, we prove cu is even. Suppose cu is odd. Since in the proof of Proposition 2.6 we see the Witt index of $\varphi^{ass}=1$, we have

$$\varphi^{ass}=\langle -d_\pm\varphi\rangle\perp\varphi\cong\langle -d_\pm\varphi, d_\pm\varphi\rangle\perp\varphi_0\,,$$

where φ_0 is an anisotropic form in $I^3F\cap W_t(F)$. By Witt's cancellation theorem, we have $\varphi\cong\langle d_\pm\varphi\rangle\perp\varphi_0$. Since I^4F is torsion free, we know φ_0 is universal. This shows φ is isotropic. Hence cu is even and Witt index m of φ^{ass} is 1 or 2. Suppose $m=2$. Then we have

$$\begin{aligned}\varphi^{ass}&=\langle -1, d_\pm\varphi\rangle\perp\varphi\\ &\cong\langle 1,-1\rangle\perp\langle d_\pm\varphi, -d_\pm\varphi\rangle\perp\varphi_0\,,\end{aligned}$$

where φ_0 is anisotropic and $\varphi_0\in I^3F\cap W_t(F)$. By Witt's cancellation theorem, we have

$$\varphi\cong\langle 1, -d_\pm\varphi\rangle\perp\varphi_0\,.$$

By assumption, we know φ_0 is universal. Thus φ is isotropic. This is a contradiction. Hence $m=1$ and the anisotropic part of φ^{ass} is of dimension cu and is in $I^3F\cap W_t(F)$. This shows our assertion. □

COROLLARY 2.8. *Assume I^4F is torsion free. Then we have $cu=0,1,2$ or $cu\geqq 8$.*

PROOF. Suppose $cu>2$. Then by Proposition 2.7 we know there exists an anisotropic form φ of dimension cu in $I^3F\cap W_t(F)$. By Theorem 2.3, we have $cu\geqq 8$. □

PROPOSITION 2.9. *The number cu equals $0,1,2$ or 8 if and only if I^4F is torsion free and every anisotropic form φ of $I^3F\cap W_t(F)$ is a 3-fold Pfister form.*

PROOF. Let $cu=0$, 1 or 2. Then our assertion is trivial because of $I^3F\cap W_t(F)=\{0\}$. Assume now $cu=8$. Then Theorem 2.3 shows $I^4F\cap W_t(F)=\{0\}$ and the dimension of any anisotropic form of $I^3F\cap W_t(F)$ is 8. Let φ be an anisotropic form $\in I^3F\cap W_t(F)$. Then there exists $a\in\dot F$ and a 3-fold Pfister form σ such that

$$\varphi\cong\langle a\rangle\otimes\sigma$$

(cf. Chapter 10 of [5]). Since $\langle 1, -a\rangle\otimes\varphi\in I^4F\cap W_t(F)=\{0\}$, we have

$$\varphi\cong\langle a\rangle\otimes\varphi\,.$$

Therefore we have

$$\varphi \cong \langle a\rangle \otimes \varphi \cong \langle a\rangle \otimes \langle a\rangle \otimes \sigma \cong \sigma .$$

Hence φ is a 3-fold Pfister form. The converse part of our assertion is deduced from Proposition 2.7. □

3. The proof of Theorem 1.1.

In this section we give a proof of Theorem 1.1 in §1.

First of all, we shall prove the assertion (2) of Theorem 1.1. The only if part of (2) is obvious by Theorem 2.3. To prove the if part, by Proposition 2.9, it is sufficient to show the following:

If I^3F is linked and I^4F is torsion free, then any anisotropic form $\varphi \in I^3F \cap W_t(F)$ is a 3-fold Pfister form.

Let φ be an anisotropic form of $I^3F \cap W_t(F)$. Since I^3F is linked, we may write

$$\varphi = \langle\langle a_1, a_2, a_3\rangle\rangle \perp \psi ,$$

where $a_1, a_2, a_3 \in \dot{F}$ and $\psi \in I^4F$ by Proposition 2.1 of [4]. Since $2\varphi \in I^4F \cap W_t(F)$ and I^4F is torsion free, we have

$$2\langle\langle a_1, a_2, a_3\rangle\rangle = -2\psi \in I^5F .$$

Since $2\langle\langle a_1, a_2, a_3\rangle\rangle \in I^5F$, $2\langle\langle a_1, a_2, a_3\rangle\rangle$ is isotropic by Theorem 2.3. Now it is known that an isotropic Pfister form is 0 in $W(F)$ (cf. Corollary 2.3 of [2]), we deduce

$$2\langle\langle a_1, a_2, a_3\rangle\rangle = 0 = -2\psi .$$

This implies ψ is a torsion form. Therefore $\psi \in I^4F \cap W_t(F) = \{0\}$. Hence we have φ is a 3-fold Pfister form $\langle\langle a_1, a_2, a_3\rangle\rangle$.

Next we shall prove the assertion (1). Let I^4F be torsion free. Then we have $cu = 0, 1, 2$ or 8 by (2). Let I^4F be not torsion free. Then we know $cu \geqq 16$. Thus we have only to show $cu \leqq 16$. To obtain $cu \leqq 16$, we prove first the following:

(3.1) if ψ is an anisotropic form of $IF \cap W_t(F)$ with $c(\psi) = 1$, then $\dim\psi \leqq 16$.

Let ψ be an anisotropic form of $IF \cap W_t(F)$ with $c(\psi) = 1$. Let ψ^{ass} be the form defined by (2.3). By Lemma 2.5, we have $\psi^{ass} \in I^3F \cap W_t(F)$. By (2) of Theorem 2.1, we may write $\psi^{ass} = \langle x\rangle \otimes \mu_1 - \mu_2$, where $x \in \dot{F}$, μ_1 and μ_2 are 3-fold Pfister forms. Since we have

$$\psi = \langle 1, -d_{\pm}\psi\rangle \perp \psi^{ass} = \langle 1, -d_{\pm}\psi\rangle - \mu_2 + \langle x\rangle \otimes \mu_1$$

in $W(F)$, and the form at RHS contains at least one hyperbolic plane, we know $\dim\psi \leqq 16$. This shows (3.1).

Next we prove that cu is even under the assumption. Suppose cu is odd. Let φ be an anisotropic form of dimension cu with $c(\varphi) = 1$. If necessary, after replacing φ by

$\langle d_{\pm}\varphi\rangle\otimes\varphi$, we may assume that $d_{\pm}\varphi=1$. Let φ^{ass} be the form defined by (2.3). Then $\varphi^{ass}\in I^3F\cap W_t(F)$ and the Witt index of $\varphi^{ass}=1$. Thus we have:

$$\varphi^{ass}=\varphi\perp\langle -1\rangle\cong\langle 1,-1\rangle\perp\varphi_0\,,$$

where φ_0 is an anisotropic form $\in I^3F\cap W_t(F)$ (cf. the proof of Proposition 2.6). Therefore we know

$$\varphi\cong\langle 1\rangle\perp\varphi_0\,.$$

Since $\varphi_0\in I^3F\cap W_t(F)$, we have $\varphi_0=\langle x\rangle\otimes\mu_1-\mu_2$ for $x\in\dot{F}$, μ_1 and μ_2 are 3-fold Pfister forms. Therefore we see

$$\varphi\cong\langle 1\rangle-\mu_2+\langle x\rangle\otimes\mu_1\,.$$

Since μ_2 contains a subform $\langle 1\rangle$, we have $\dim\varphi\leqq 15$. This contradicts $cu\geqq 16$. Thus cu is even. Hence by (3.1) we have $cu\leqq 16$. □

REMARK. In the case I^2F is linked, by (1.1), Theorem 2.1, Corollary 2.8, we know $cu=0,1,2$ or 8. Further in this case, we see $cu=8\Leftrightarrow u=8\Leftrightarrow I^3F$ is not torsion free. If $u\leqq 4$, then we have $cu=0,1$ or 2.

4. The proof of Theorem 1.2.

In this section, we give a proof of Theorem 1.2 in §1.

We begin with the proof of some Lemmas.

LEMMA 4.1. *Let n be a positive integer $\geqq 2$. Assume that any form of $I^nF\cap W_t(F)$ is congruent to an n-fold Pfister form modulo $I^{n+1}F$. Then any pair of torsion n-fold Pfister forms is linked.*

PROOF. Let φ_1 and φ_2 be two torsion n-fold Pfister forms. Then by assumption, there exists an n-fold Pfister form φ_3 such that

$$\varphi_1+\varphi_2+\varphi_3\in I^{n+1}F\,.$$

By Theorem 2.4, we have φ_1 and φ_2 are linked. □

LEMMA 4.2. *Let l be a positive integer $\geqq 3$. Assume the condition* (T_l) *given in* §1. *Then $I^{l+2}F\cap W_t(F)=\{0\}$.*

PROOF. Assume $I^{l+2}F\cap W_t(F)\neq\{0\}$. Then there exists a form $\varphi\in I^{l+2}F$ of order 2 (cf. Satz 10 of [7]). Let $m\geqq l+2$ be the largest integer such that $\varphi\in I^mF$. By assumption, we know $\varphi=\varphi_1+\varphi_2$, where φ_1 is an m-fold Pfister form and $\varphi_2\in I^{m+1}F$. Since $2\varphi=0$, we have $2\varphi_1=-2\varphi_2\in I^{m+2}F$. Since $\dim(2\varphi_1)=2^{m+1}$, Theorem 2.3 implies $2\varphi_1$ is isotropic. Thus we have $2\varphi_1=0$. By Corollary 2.3 of [3], we have an isometry:

$$\varphi_1 \cong \langle\langle -\omega, a_2, \cdots, a_m \rangle\rangle ,$$

where $a_2, \cdots, a_m \in \dot{F}$ and ω is a sum of two squares. Consider two torsion $(m-2)$-fold Pfister forms:

$$\psi_1 \cong \langle\langle -\omega, a_2, \cdots, a_{l-2}, a_{l-1}, a_l, a_{l+3}, \cdots, a_m \rangle\rangle ,$$

$$\psi_2 \cong \langle\langle -\omega, a_2, \cdots, a_{l-2}, a_{l+1}, a_{l+2}, a_{l+3}, \cdots, a_m \rangle\rangle .$$

Since any pair of torsion l-fold Pfister forms is linked by Lemma 4.1, we have:

$$\psi_1 \cong \langle\langle -\omega, a_2, \cdots, a_{l-2}, b_{l-1}, c_l, a_{l+3}, \cdots, a_m \rangle\rangle ,$$

$$\psi_2 \cong \langle\langle -\omega, a_2, \cdots, a_{l-2}, b_{l-1}, d_l, a_{l+3}, \cdots, a_m \rangle\rangle$$

(cf. proof of Proposition 4.2 of [4]). Therefore we have following isometries:

$$\begin{aligned} \varphi_1 &\cong \langle\langle a_{l-1}, a_l \rangle\rangle \otimes \psi_2 \\ &\cong \langle\langle b_{l-1}, d_l \rangle\rangle \otimes \psi_1 \\ &\cong \langle\langle -\omega, a_2, \cdots, a_{l-2}, b_{l-1}, b_{l-1}, c_l, d_l, a_{l+3}, \cdots, a_m \rangle\rangle \\ &\cong 2\langle\langle -\omega, a_2, \cdots, a_{l-2}, b_{l-1}, c_l, d_l, a_{l+3}, \cdots, a_m \rangle\rangle . \end{aligned}$$

Since ω is a sum of two squares, the above isometries imply $\varphi_1 = 0$. Hence $\varphi = \varphi_2 \in I^{m+1}F$. This contradicts the choice of m. □

LEMMA 4.3. *Assume the condition* (T_3). *Let φ be any form of $I^3F \cap W_t(F)$. Then there exists a 3-fold torsion Pfister form ψ_1 and a 4-fold torsion Pfister form ψ_2 such that $\varphi = \psi_1 + \psi_2$ in $W_t(F)$.*

PROOF. Let φ be any form of $I^3F \cap W_t(F)$. Then we write $\varphi = \psi_1 + \phi_2$ where ψ_1 is a 3-fold Pfister form and $\phi_2 \in I^4F$. By Lemma 4.2 we have I^5F is torsion free. Since $4\varphi \in I^5F \cap W_t(F)$, we have $4\varphi = 0$. Therefore $4\psi_1 = -4\phi_2 \in I^6F$. We can verify $4\psi_1 = 0$ by Theorem 2.3 as in the first step of the proof of Theorem 1.1. Therefore ϕ_2 is also a torsion form. By the assumption, we can write $\phi_2 = \psi_2 + \phi_3$ where ψ_2 is a 4-fold Pfister form and $\phi_3 \in I^5F$. By the same argument as above, ψ_2 and ϕ_3 are torsion forms. Thus $\phi_3 \in I^5F \cap W_t(F) = \{0\}$. This implies $\phi_2 = \psi_2$. Therefore we obtain $\varphi = \psi_1 + \psi_2$. Hence we have our assertion. □

Now we prove Theorem 1.2. Assume $I^4F \cap W_t(F) = \{0\}$. Then by Lemma 4.3, we know any form of $I^3F \cap W_t(F)$ is a 3-fold Pfister form. Thus from Proposition 2.9 we deduce $cu = 0, 2$ or 8. Assume $I^4F \cap W_t(F) \neq \{0\}$. Then by Theorem 2.3, we have $I_3 \geqq 16$. If we prove $I_3 \leqq 16$, then Proposition 2.6 gives $cu = 16$ or 18. Let φ be an anisotropic form of $I^3F \cap W_t(F)$. Then by Lemma 4.3, we can write $\varphi = \psi_1 + \psi_2$, where ψ_1 is a torsion 3-fold Pfister form and ψ_2 is a torsion 4-fold Pfister form. Since $I^5F \cap W_t(F) = \{0\}$ by Lemma 4.2, we have $2\psi_2 = 0$. Thus we have an isometry from Corollary 2.3 of [3]:

$$\psi_2 \cong \langle\langle -\omega, a_2, a_3, a_4 \rangle\rangle ,$$

where $a_2, a_3, a_4 \in \dot{F}$ and ω is a sum of two squares. Since $I^3F \cap W_t(F)$ is linked by Lemma 4.1, ψ_1 and $\langle\langle -\omega, a_2, a_3 \rangle\rangle$ are linked. Therefore there exists a 2-fold Pfister form σ, 1-fold Pfister forms τ_1 and τ_2 such that

$$\psi_1 \cong \sigma \otimes \tau_1 , \qquad \langle\langle -\omega, a_2, a_3 \rangle\rangle \cong \sigma \otimes \tau_2 .$$

Thus we have, in $W(F)$,

$$\varphi = \psi_1 + \psi_2 = \psi_1 - \psi_2 = \sigma \otimes \tau_1 - \sigma \otimes \tau_2 \otimes \langle\langle a_4 \rangle\rangle = \sigma \otimes (\tau_1 - \tau_2 \otimes \langle\langle a_4 \rangle\rangle) .$$

Since τ_1 and $\tau_2 \otimes \langle\langle a_4 \rangle\rangle$ both contain a subform $\langle 1 \rangle$, the dimension of the anisotropic part of $\sigma \otimes (\tau_1 - \tau_2 \otimes \langle\langle a_4 \rangle\rangle) \leqq 16$. Therefore we have $\dim \varphi \leqq 16$. Hence we have $I_3 \leqq 16$. This completes the proof of Theorem 1.2. □

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