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Complete Intersections Which Are Abelian Extensions of a Factorial Domain

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Introduction.

Let A be a Noetherian local factorial domain having a field of fractions K and L be an Abelian extension of K with $G=Gal(L/K)$. An Abelian extension R of A is an integral closure of A in L.

Assume that $ch(A)$ does not divide $n=|G|$ and A has a primitive n-th root of unity. Roberts [12] showed that R is Cohen-Macaulay, if A is a Cohen-Macaulay factorial domain. Also Itoh [8] studied a condition for a cyclic extension of a formal power series ring to be Gorenstein. Furthermore, Griffith [7] showed that if A is regular and if R is factorial, then R is a complete intersection.

Our purpose of this article is to give a condition for R to be a complete intersection. We shall show in section 4 that an Abelian extension of a local factorial domain A which is a complete intersection, is completely determined by a *datum* of A. A datum is a pair (Γ, w) of a finite subset Γ of $\mathrm{Div}(A)\otimes_{\mathbf{Z}}\mathbf{Q}$ $(=\mathrm{P}(A)\otimes_{\mathbf{Z}}\mathbf{Q}$ since A is factorial) and a map $w:\Gamma\to\mathbf{N}_+$ satisfying the following condition.

(1) For $D, E\in\Gamma$ $(D\neq E)$, one of the following cases occurs;
(a) $\mathrm{Supp}(D)\subsetneq\mathrm{Supp}(E)$, (b) $\mathrm{Supp}(D)\supsetneq\mathrm{Supp}(E)$, (c) $\mathrm{Supp}(D)\cap\mathrm{Supp}(E)=\varnothing$.

(2) For $E\in\Gamma$, there is a relation $w(E)E=\sum_{i=1}^{k}E_i+\sum_{j=1}^{l}\mathfrak{p}_j$ where $\{E_1,\cdots,E_k\}=\{D\in\Gamma \mid D\prec E\}$ and $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_l\}=\mathrm{Supp}(E)\setminus\bigcup_{i=1}^{k}\mathrm{Supp}(E_i)$.

Here we denote by $\mathrm{Supp}(D)$ $(D\in\mathrm{Div}(A)_{\mathbf{Q}})$ the set of prime divisors of A which appear in D with non-zero coefficients. We write $D\prec E$, if $\mathrm{Supp}(D)\subsetneq\mathrm{Supp}(E)$ and if there is no element $D'\in\Gamma$ such that $\mathrm{Supp}(D)\subsetneq\mathrm{Supp}(D')\subsetneq\mathrm{Supp}(E)$.

Then we state our main result as follows.

Theorem. *Let A be as above and R be a local ring such that $R\supset A$. Then the following conditions are equivalent.*

(1) *R is an Abelian extension of A which is a complete intersection.*

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(2) *There exists a datum* (Γ, w) *such that*

$$R \cong A[Y_E \mid E \in \Gamma] \Big/ \Big(Y_E^{w(E)} - a_E \prod_{D \prec E} Y_D \mid E \in \Gamma \Big)$$

where $A[Y_E \mid E \in \Gamma]$ *is a polynomial ring and* a_E *is an element of* A *satisfying* $\operatorname{div}(a_E) = w(E)E - \sum_{D \prec E} D$.

A datum was first defined by Watanabe [15] for normal simplicial semigroup rings (cf. Definition 1.1 of [15]). He used it to determine normal simplicial semigroup rings which are complete intersections. In his case, an affine semigroup ring can be regarded as a $\mathbf{Z}^n$-graded ring in natural way. In our case, an Abelian extension R is graded by its Galois group. (See (4.1).) By this reason, the concept of *rings graded by an Abelian group* is essential for us. In particular, we state a *group graded* version of the divisor group theory (in section 1).

1. Preliminaries.

Let us recall some concepts of graded rings and graded modules from [6], [11] and [9].

Let G be an Abelian group. We say that a ring R is a *G-graded ring*, if there exists a family $\{R_g\}_{g \in G}$ of additive subgroups of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subset R_{g+h}$ for every $g, h \in G$. Similarly, a *G-graded R-module* is an R-module M for which there is given a family $\{M_g\}_{g \in G}$ of additive subgroups of M such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subset M_{g+h}$ for every $g, h \in G$.

A *homomorphism* $f: M \to N$ of G-graded R-modules in an R-linear map such that $f(M_g) \subset N_g$ for all $g \in G$.

Let R be a G-graded ring and M a G-graded R-module. For $g \in G$, we define a G-graded R-module $M(g)$ by $M = M(g)$ as the underlying R-module and graded by $[M(g)]_h = M_{g+h}$ for all $h \in G$. We say that M is *free*, if it is isomorphic to a direct sum of G-graded R-modules of the form $R(g)$ $(g \in G)$.

We denote by $\underline{\operatorname{Hom}}_R(M, N)_g$ the Abelian group of all the G-graded homomorphisms from M to $N(g)$. We put $\underline{\operatorname{Hom}}_R(M, N) = \bigoplus_{g \in G} \underline{\operatorname{Hom}}_R(M, N)_g$ and consider it as a G-graded R-module.

The elements of $\bigcup_{g \in G} M_g$ are called *homogeneous elements* of M. Every non-zero element $x \in M_g$ is said to be *homogeneous of degree g*, and we denote $\deg(x) = g$. For a subset $N \subset M$, we set $h(N) = \bigcup_{g \in G} (N \cap M_g)$. Any element $x \in M$ has a unique expression as a sum of homogeneous elements, $x = \sum_{g \in G} x_g$ where $x_g \in M_g$ and $x_g = 0$ for almost all $g \in G$. With this notation, we call nonzero x_g the *homogeneous component* (of degree g) of x.

Let H be a subgroup of G and $g \in G$. We define $R^{(H)} = \bigoplus_{h \in H} R_h$ and $M^{(g,H)} = \bigoplus_{h \in H} M_{g+h}$. Then $R^{(H)}$ is a subring of R and $M^{(g,H)}$ is an $R^{(H)}$-submodule of M. We define a G-grading on $M^{(g,H)}$ as

$$[M^{(g,H)}]_{g'} = \begin{cases} M_{g'}, & \text{if } g-g' \in H \\ (0), & \text{if } g-g' \notin H \end{cases}$$

for all $g' \in G$. If $g-g' \in H$, then we have $M^{(g,H)} = M^{(g',H)}$ as a G-graded $R^{(H)}$-module. Hence M has the following decomposition as a G-graded $R^{(H)}$-module

$$M = \bigoplus_{i \in \Gamma} M^{(g_i,H)}$$

where $\{g_i\}_{i \in \Gamma}$ is a system of representatives of $G \bmod H$. Also, we have $R^{(g_i,H)} M^{(g_j,H)} \subset M^{(g_i+g_j,H)}$ for all $i, j \in \Gamma$. Hence a G-graded ring R (resp. G-graded R-module M) can be regarded as a G/H-graded ring (resp. G/H-graded R-module).

We say that R is a *G-domain* (resp. *G-simple*), if every nonzero G-homogeneous element of R is a nonzero divisor of R (resp. a unit of R).

A G-graded ideal $\mathfrak{P}$ of R is said to be a *G-prime* ideal (resp. a *G-maximal* ideal), if the G-graded ring $R/\mathfrak{P}$ is a G-domain (resp. G-simple). Note that a G-prime ideal is not necessarily a prime ideal, if G has a torsion. We denote by $V_G(R)$ the set of all G-prime ideals of R. For $\mathfrak{P} \in V_G(R)$, we denote by $M_{(\mathfrak{P})}$ the module of fractions of M with respect to the multiplicatively closed subset $h(R \setminus \mathfrak{P})$ and call it the *homogeneous localization* of M at $\mathfrak{P}$. We set $V_G(M) = \{\mathfrak{P} \in V_G(R) \mid M_{(\mathfrak{P})} \neq (0)\}$. For an ideal P of R, we denote by P^* the maximal graded ideal of R contained in P (or the graded ideal generated by $h(P)$). If P is a prime ideal of R, then P^* is a G-prime ideal of R. Furthermore, for a G-graded R-module M and $P \in \mathrm{Spec}(R)$, $P \in \mathrm{Supp}_R(M)$ if and only if $P^* \in V_G(M)$. We denote by $\underline{\dim}(M)$ the largest length of a chain of G-prime ideals in $V_G(M)$ (cf. section 2 of [9]).

REMARK 1.1. Let H be a subgroup of G such that G/H is torsion. If $\mathfrak{P} \in V_G(R)$, then $\mathfrak{P}^{(H)} \in V_H(R^{(H)})$. Furthermore, if $\mathfrak{p} \in V_H(R^{(H)})$, then a G-graded ideal $(\sqrt{\mathfrak{p}R})^*$ is G-prime. This gives a bijective correspondence between $V_H(R^{(H)})$ and $V_G(R)$. $M_{(\mathfrak{P})} \cong M \otimes_{R^{(H)}} (R^{(H)})_{(\mathfrak{P}^{(H)})}$ holds for a G-graded R-module M and $\mathfrak{P} \in V_G(R)$.

DEFINITION 1.2. R is said to be a *G-Noetherian* graded ring, if every strictly ascending chain of G-graded ideals of R has finite length.

DEFINITION 1.3. We say that R is a *G-local* graded ring, if it has a unique G-maximal ideal. Often we use the notation $(R, \mathfrak{M})$ to say that R is a G-local ring with the unique G-maximal ideal $\mathfrak{M}$.

Next, we state a G-graded version of the theory of *divisors*. All propositions shall be proved in the same way as in the non graded case or $\mathbf{Z}^n$-graded case (cf. Anderson [1]). Therefore, we omit proofs.

Let R be a G-domain and K be the homogeneous localization of R at (0).

DEFINITION 1.4. A G-domain R is called *G-normal*, if every element of $h(K)$, which is integral over R, is in R.

DEFINITION 1.5. R is said to be *completely G-normal*, if it satisfies the following condition; if $x \in h(K)$ and $R[x]$ is contained in a finitely generated G-graded R-submodule of K, then $x \in R$.

A G-graded R-submodule $0 \neq I$ of K is said to be a *G-fractional* ideal of R, if there exists $a \in R$ such that $aI \subset R$. We denote by $\underline{\mathrm{I}}(R)$ the set of all G-fractional ideals of R and by $\underline{\mathrm{P}}(R)$ the set of all homogeneous principal idelas. We set $\underline{\mathrm{div}}_R(I) = \bigcap_{0 \neq x \in h(K), I \subset R_x} R_x$ for $I \in \underline{\mathrm{I}}(R)$.

DEFINITION 1.6. A G-fractional ideal I is said to be a *G-divisorial* ideal of R, if $I = \underline{\mathrm{div}}_R(I)$. We denote by $\underline{\mathrm{Div}}(R)$ the set of all G-divisorial ideals of R.

REMARK 1.7. Let $I, J \in \underline{\mathrm{I}}(R)$. Then the following hold.

(0) $\underline{\mathrm{Hom}}_R(I, J) \cong [J : I]$.

(1) $[I : J]_K = \bigcap_{x \in h(J)} x^{-1} I$.

(2) If $I \in \underline{\mathrm{Div}}(R)$, then so is $[I : J]_K$.

(3) $\underline{\mathrm{div}}_R(I) = [R : [R : I]_K]_K$.

(4) $\underline{\mathrm{div}}_R(I) \subset \underline{\mathrm{div}}_R(J)$ if and only if $[R : I]_K \supset [R : J]_K$.

DEFINITION 1.8. We define a commutative monoid structure on $\underline{\mathrm{Div}}(R)$ by $\underline{\mathrm{div}}_R(I) + \underline{\mathrm{div}}_R(J) := \underline{\mathrm{div}}_R(IJ)$ for $I, J \in \underline{\mathrm{I}}(R)$. Also, we denote $\underline{\mathrm{div}}_R(I) \sim \underline{\mathrm{div}}_R(J)$, if $\underline{\mathrm{div}}_R(I) = \underline{\mathrm{div}}_R(J) + \underline{\mathrm{div}}_R(a)$ for some $0 \neq a \in h(K)$. Here we denote $\underline{\mathrm{div}}_R(a) = \underline{\mathrm{div}}_R(aR)$ for $0 \neq a \in h(K)$. We put $\underline{Cl}(R) = \underline{\mathrm{Div}}(R)/\sim$.

The following proposition is proved in the same way as in the non-graded case (cf. Chap. VII, §1.2, Theorem 1 of [3]).

PROPOSITION 1.9. *R is completely G-normal if and only if $\underline{\mathrm{Div}}(R)$ is an Abelian group.*

DEFINITION 1.10. Let Γ be an ordered Abelian group and $v : h(K \setminus \{0\}) \to \Gamma$ be a map. We call v a *G-valuation* on K, if it satisfies the following two conditions; for $x, y \in h(K \setminus \{0\})$,

(1) $v(xy) = v(x) + v(y)$

(2) $v(x+y) \geq \min(v(x), v(y))$, if $\deg(x) = \deg(y)$ and $x + y \neq 0$.

Let v be a G-valuation on K. Then $v|_{K_0'}$ is a valuation on $K_0' := K_0 \setminus \{0\}$. We denote by R_{v0} the valuation ring of $v|_{K_0'}$. We set $R_v = R_{v0}[x \in h(K \setminus \{0\}) \mid v(x) \geq 0]$ and call it the *G-valuation ring* of v.

A G-valuation $v : h(K \setminus \{0\}) \to \Gamma$ is said to be *equivalent* to a G-valuation v': $h(K \setminus \{0\}) \to \Gamma'$, if there exists an isomorphism $\phi : Im(v) \to Im(v')$ of ordered Abelian groups such that $v' = \phi \circ v$.

DEFINITION 1.11. A G-valuation v on K is said to be *discrete*, if $Im(v) \cong \mathbf{Z}$.

We say that R is a *G-discrete valuation ring* (*G-DVR*), if there exists a discrete G-valuation v on K such that $R = R_v$.

As in the non-graded case, the following hold.

PROPOSITION 1.12. *Let* $(R, \mathfrak{M})$ *be a G-Noetherian G-local G-domain of* $\underline{\dim}(R) = 1$. *Then the following are equivalent.*

(1) *R is G-DVR.*

(2) *R is G-normal.*

(3) *$\mathfrak{M}$ is a principal ideal.*

(4) *Every proper G-homogeneous ideal is a power of* $\mathfrak{M}$.

EXAMPLE 1.13. Let $(R, \mathfrak{M})$ be a G-DVR and $x \in h(R)$ such that $\mathfrak{M} = xR$. We put $H = \{g \in G \mid R_g \neq \mathfrak{M}_g\}$ $(= \{g \in G \mid R_g$ contains a unit of $R\})$. (Note that H is a subgroup of G.) Then $\mathfrak{M}^{(H)} = \mathfrak{M}_0 R^{(H)}$ (cf. (1.4) of [9]).

• If $\mathfrak{M}_0 = (0)$ (i.e. $\mathfrak{M}^{(H)} = (0)$), then $R^{(H)}$ is H-simple and $x^n \notin R^{(H)}$ for every $n > 0$. Hence $n \deg(x) \notin H$ for every $n > 0$ (or $H \oplus \mathbf{Z} \deg(x) \subset G$) and $R = R^{(H)}[x]$ the polynomial ring over $R^{(H)}$.

• If $\mathfrak{M}_0 \neq 0$, then $(R_0, \mathfrak{M}_0)$ is a DVR. We take $x_0 \in \mathfrak{M}_0$ such that $\mathfrak{M}_0 = x_0 R_0$. Then there exists a positive integer $e > 0$ such that $x_0 = ux^e$ for some homogeneous unit $u \in R^{(H)}$. Hence $R = \bigoplus_{i=0}^{e-1} R^{(H)} x^i$.

EXAMPLE 1.14. Let $K = \bigoplus_{n \in \mathbf{Z}} K_n$ be a simple $\mathbf{Z}$-graded ring. We assume that $K \neq K_0$. Then there exists $t \in h(K \setminus \{0\})$ such that t is algebraically independent over K_0 and $K = K_0[t, t^{-1}]$. We define $\mathbf{Z}$-valuations as follows;

(i) $v_t : h(K \setminus \{0\}) \to \mathbf{Z}$ by $v_t(at^n) = n$ $(a \in K_0)$,

(ii) $v_{t^{-1}} : h(K \setminus \{0\}) \to \mathbf{Z}$ by $v_t(at^n) = -n$ $(a \in K_0)$,

(iii) $v_0^{d,e} : h(K \setminus \{0\}) \to \mathbf{Z}(1/e)$ $(\subset \mathbf{Q})$ by $v_0^{d,e}(at^n) = v_0(a) + n(d/e)$ $(a \in K_0)$ where v_0 is a discrete valuation of K_0 and $d, e \in \mathbf{Z}$ such that $e > 0$ and either $d = 0$ or $(e, d) = 1$.

Let v be a discrete $\mathbf{Z}$-valuation of K and (R, xR) be a $\mathbf{Z}$-valuation ring of v. We put $m \geq 0$ such that $\mathbf{Z}m = \{n \in \mathbf{Z} \mid R_n$ contains a unit of $R\}$. Then, by (1.13), either (1) $R = R^{(m)}[x]$ or (2) $R = \bigoplus_{i=0}^{e-1} R^{(m)} x^i$.

Case (1) Since $\mathbf{Z}m \oplus \mathbf{Z} \deg(x) \subset \mathbf{Z}$, $m = 0$ and $R = R_0[x]$. On the other hand, K is a homogeneous localization of R at (0) (i.e. $K = R_0[x, x^{-1}]$). Hence $R_0 = K_0$ and either $x = t$ or $x = t^{-1}$. Namely, either $v = v_t$ or $v = v_{t^{-1}}$.

Case (2) We put $v(t) = d$ and $v_0 = (1/e) v|_{K_0}$. Then v_0 is a normalized discrete valuation of R_0. Furthermore, for $a \in K_0 \setminus \{0\}$ and $n \in \mathbf{Z}$, $v(at^n) = e(v_0(a) + n(d/e) = ev_0^{d,e}(at^n)$. Hence v is equivalent to $v_0^{d,e}$ and $R = R_{v_0^{d,e}} = \bigoplus_{n \in \mathbf{Z}} R_0 x_0^{[-nd/e]} t^n$ where x_0 is the primitive element of R_0 and $[a]$ is the largest integer not larger than a for $a \in \mathbf{Q}$.

We set $V_G^1(R) = \{\mathfrak{P} \in V_G(R) \mid \underline{\dim}(R_{(\mathfrak{P})} = 1\}$.

DEFINITION 1.15. A G-domain R is said to be *G-Krull*, if it satisfies the following conditions:

(1) $R_{(\mathfrak{P})}$ is G-DVR for every $\mathfrak{P} \in V_G^1(R)$,

(2) $R = \bigcap_{\mathfrak{P} \in V_G^1(R)} R_{(\mathfrak{P})}$,

(3) every nonzero element of $h(R)$ is contained in only a finite members of $V_G^1(R)$.

PROPOSITION 1.16 (cf. Chap. VII, §1 of [3]). (1) *A G-domain R is G-Krull, if R is G-Noetherian and G-normal. Conversely, a G-Krull R is completely G-normal.*

(2) *A G-graded Krull domain is G-Krull.*

(3) *Let R be a G-Krull G-domain and S be a multiplicatively closed subset of* $h(R)$. *Then a G-graded ring* $S^{-1}R$ *is G-Krull and* $S^{-1}R = \bigcap_{\mathfrak{P} \in \Lambda} R_{(\mathfrak{P})}$ *where* $\Lambda = \{\mathfrak{P} \in V_G^1(R) \mid \mathfrak{P} \cap S = \varnothing\}$.

(4) *Let R be a G-Krull G-domain and K be the homogeneous localization of R at* (0). *If K' is a simple graded subring of K, then* $R \cap K'$ *is a G-Krull G-domain. In particular,* $R^{(H)}$ *is an H-Krull H-domain for a subgroup H of G.*

Throughout this section, we assume that R is G-Krull.

PROPOSITION 1.17 (cf. Chap. VII, §1 of [3]). (1) *For* $I \in \underline{\mathrm{I}}(R)$, $\underline{\mathrm{div}}_R(I) = \bigcap_{\mathfrak{P} \in V_G^1(R)} IR_{(\mathfrak{P})}$.

(2) *For* $I, J \in \underline{\mathrm{Div}}(R)$, $I = J$ *if and only if* $I_{(\mathfrak{P})} = J_{(\mathfrak{P})}$ *for every* $\mathfrak{P} \in V_G^1(R)$.

COROLLARY 1.18. *Let H be a subgroup of G such that G/H is torsion and* $\{g_i\}_{i \in \Gamma}$ *be a system of representatives of G mod H. Then a G-fractional ideal I is G-divisorial if and only if* $I^{(g_i, H)}$ *is H-divisorial for any* $i \in \Gamma$.

Let $\mathfrak{P} \in V_G^1(R)$. Then, by (1.12), $\mathfrak{P}R_{(\mathfrak{P})}$ defines a discrete G-valuation on K. We denote it by $v_{\mathfrak{P}}$.

We define a map $\mu : \underline{\mathrm{Div}}(R) \to \bigoplus_{\mathfrak{P} \in V_G^1(R)} \mathbf{Z}\mathfrak{P}$ by $\mu(\underline{\mathrm{div}}_R(I)) = \sum_{\mathfrak{P} \in V_G^1(R)} v_{\mathfrak{P}}(I)\mathfrak{P}$. Then μ is an isomorphism of Abelian groups and, for $I, J \in \underline{\mathrm{Div}}(R)$, $I \subset J$ if and only if $\mu(I) \geq \mu(J)$ (i.e. $v_{\mathfrak{P}}(I) \geq v_{\mathfrak{P}}(J)$ for all $\mathfrak{P} \in V_G^1(R)$).

Let H be a subgroup of G such that G/H is torsion.

DEFINITION 1.19. Let $\mathfrak{p} \in V_H^1(R^{(H)})$ and $\mathfrak{P} \in V_G^1(R)$ such that $\mathfrak{P}^{(H)} = \mathfrak{p}$. We put $e_R(\mathfrak{p}) = v_{\mathfrak{P}}(\mathfrak{p}R_{(\mathfrak{P})})$.

REMARK 1.20. The correspondence $I \mapsto \underline{\mathrm{div}}_R(IR)$ ($I \in \underline{\mathrm{Div}}(R^{(H)})$) defines a homomorphism $\underline{\mathrm{Div}}(R^{(H)}) \to \underline{\mathrm{Div}}(R)$ and there is the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \underline{P}(R^{(H)}) & \longrightarrow & \underline{\mathrm{Div}}(R^{(H)}) & \longrightarrow & \underline{Cl}(R^{(H)}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{P}(R) & \longrightarrow & \underline{\mathrm{Div}}(R) & \longrightarrow & \underline{Cl}(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G'/H' & \xrightarrow{\alpha} & \bigoplus_{\mathfrak{p}} \mathbf{Z}/(e_R(\mathfrak{p})) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $G'=\{g\in G \mid K_g\neq 0\}$, H' is a subgroup of G generated by $(G'\cap H)\cup\{g\in G \mid R_g$ contains a unit of $R\}$ and the map $\underline{P}(R)\to G/H'$ is defined by $x\in h(K)\mapsto$ the image of $\deg(x)$ in G'/H'.

EXAMPLE 1.21 (Demazure's construction of normal graded rings [4]). Let $R=\bigoplus_{n\geq 0} R_n$ be a Krull domain such that $R\neq R_0$ and $K=K_0[t, t^{-1}]$ be the homogeneous localization of R at (0) with $\deg(t)>0$.

We put $\{\mathfrak{P}_1, \cdots \mathfrak{P}_r\}=\{\mathfrak{Q}\in V_{\mathbf{Z}}^1(R) \mid \mathfrak{Q}\not\supset R_+\}$. Then, by (1.14), $v_{\mathfrak{P}_i}$ is equivalent to $v_0^{d_i,e_i}$ for some $d_i, e_i\in\mathbf{Z}$.

We set $X=Proj(R)$ and $D=\sum_{i=1}^r (d_i/e_i)V_i\in Div(X)\otimes\mathbf{Q}$ where V_i is the prime divisor of X corresponding to $\mathfrak{P}_i$.

Then, for $a\in K_0$ and $n\geq 0$,

$$\begin{aligned}
at^n\in R &\Longleftrightarrow \underline{\mathrm{div}}_R(a)+n\,\underline{\mathrm{div}}_R(t)\geq 0\\
&\Longleftrightarrow \mathrm{div}_X(a)+nD\geq 0\in Div(X)\otimes\mathbf{Q}\\
&\Longleftrightarrow a\in H^0(X, \mathcal{O}_X(nD)) .
\end{aligned}$$

(Note that if $R_+:=\bigoplus_{n>0} R_n\in V_{\mathbf{Z}}^1(R)$, then $v_{R_+}=v_t$.) Hence we have

$$R\cong R(X, D):=\bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))T^n \quad (\subset K_0[T]) .$$

We set $R(E)=\bigoplus_{n\in\mathbf{Z}} H^0(X, \mathcal{O}_X(E+nD))T^n$ for $E\in Div(X)$. Then, the correspondence $E\mapsto \underline{\mathrm{div}}_R(R(-E))$ $(E\in Div(X))$ defines a homomorphism $Div(X)\to\underline{\mathrm{Div}}(R)$ and we have the following commutative diagram;

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P(X) & \longrightarrow & Div(X) & \longrightarrow & Cl(X) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \underline{P}(R) & \longrightarrow & \underline{\mathrm{Div}}(R) & \longrightarrow & Cl(R) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & I & \longrightarrow & \bigoplus_{i=1}^{r} \mathbf{Z}/(e_i) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where $I \cong \mathbf{Z}$ and is generated by the image of $\underline{\mathrm{div}}_R(T)$. (See, for instance, Watanabe [16].)

2. A description of R from $R^{(H)}$.

Let R be a G-Noetherian G-normal G-domain and K be the homogeneous localization of R at (0). We assume that $K_g \neq 0$ for every $g \in G$.

Let H be a subgroup of G such that $G/H \cong \bigoplus_{i=1}^{r} \mathbf{Z}/(d_i)$ and put $A = R^{(H)}$. We put $\underline{\mathrm{Div}}(A)_{\mathbf{Q}} = \underline{\mathrm{Div}}(A) \otimes_{\mathbf{Z}} \mathbf{Q}$. For $D = \sum_{\mathfrak{p} \in V_G^1(A)} a_{\mathfrak{p}} \mathfrak{p} \in \underline{\mathrm{Div}}(A)_{\mathbf{Q}}$, we set

$$\mathrm{Supp}(D) = \{\mathfrak{p} \mid a_{\mathfrak{p}} \neq 0\}, \qquad D(\mathfrak{p}) = a_{\mathfrak{p}},$$
$$[D] = \sum_{\mathfrak{p} \in V_G^1(R)} [a_{\mathfrak{p}}]\mathfrak{p}, \qquad \{D\} = D - [D],$$

where we denote by $A(D)$ the H-divisorial ideal of A generated by $\{a \in h(K) \mid \underline{\mathrm{div}}_A(a) + D \geq 0\}$. Note that $A(D) = A([D])$ and $\underline{\mathrm{div}}_A(A(D)) = -[D]$.

Tomari-Watanabe [14] constructed normal $\mathbf{Z}_r$-graded rings using an element of $\underline{P}(A)_{\mathbf{Q}}$. In this section, we repeat Tomari-Watanabe's construction.

We put K' the homogeneous localization of A at (0). Then $K' = K^{(H)}$ and $K = R \otimes_A K'$. Furthermore, there exists $x_i \in h(K \setminus \{0\})$ $(1 \leq i \leq r)$ such that $x_i^{d_i} = f_i \in A$ and

$$K = \bigoplus_{\substack{1 \leq i \leq r \\ 0 \leq m_i < d_i}} K' x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \cong K'[X_1, \cdots, X_r]/(X^{d_1} - f_1, \cdots, X^{d_r} - f_r)$$

where $K'[X_1, \cdots, X_r]$ is a polynomial ring over K' (cf. (1.6) of [9]).

We put $D_i = (1/d_i)\underline{\mathrm{div}}_A(f_i) \in \underline{\mathrm{Div}}(A)_{\mathbf{Q}}$ for $1 \leq i \leq r$.

PROPOSITION 2.1. *We have*

$$R = \bigoplus_{\substack{1 \leq i < r \\ 0 \leq m_i < d_i}} A\left(\sum_{i=1}^{r} m_i D_i\right) \prod_{i=1}^{r} x_i^{m_i}.$$

We can give a proof in the same way as in Proposition 1.4 of Tomari-Watanabe [14].

REMARK 2.2. (1) The converse of (2.1) is also true. Let A be an H-Noetherian H-normal H-domain and $D_i \in \underline{\mathrm{Div}}(A)_{\mathbf{Q}}$ such that $d_i D_i = \underline{\mathrm{div}}_A(f_i)$ $(1 \leq i \leq r)$. Put $G =$

$H\oplus \mathbf{Z}^r/\sum_{i=1}^r \mathbf{Z}(\deg(f_i), -d_ie_i)$ where $e_i=(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)\in \mathbf{Z}^r$. Then a G-graded subring

$$R(A\ ;\ D_1, \cdots, D_r, f_1, \cdots, f_r):= \bigoplus_{\substack{1\le i\le r\\ 0\le m_i<d_i}} A\left(\sum_{i=1}^r m_iD_i\right)\prod_{i=1}^r x_i^{m_i}$$

of $K'[X_1, \cdots, X_r]/(X_1^{d_1}-f_1, \cdots, X_r^{d_r}-f_r)$ is a G-normal G-domain where K' is the homogeneous localization of A at (0).

(2) It is easy to verify that $e_R(\mathfrak{p})>1$ if and only if $D_i(\mathfrak{p})\notin \mathbf{Z}$ for some $1\le i\le r$. Furthermore, if we denote $D_i(\mathfrak{p})=a_i/b_i\in \mathbf{Q}$ with $(a_i, b_i)=1$, then $e_R(\mathfrak{p})$ is equal to $LCM(\{b_i \mid a_i\neq 0, 1\le i\le r\})$.

EXAMPLE 2.3. Let k be a field and $A=k[s^6, t^{12}]\subset k[s, t]$ and $D_1=\frac{1}{2}\underline{\mathrm{div}}_A(s^6t^{12})$, $D_2=\frac{1}{3}\underline{\mathrm{div}}_A(s^6t^{24})$. Then we have

$$R(A\ ;\ D_1, D_2, s^{12}t^{12}, s^6t^{24})\cong k[s^6, st^{10}, s^2t^8, s^3t^6, s^4t^4, s^5t^2, t^{12}]\ .$$

For $E\in \underline{\mathrm{Div}}(A)_{\mathbf{Q}}$, we define a G-graded R-module $R(E\ ;\ D_1, \cdots, D_r, f_1, \cdots, f_r)$ as

$$R(E\ ;\ D_1, \cdots, D_r, f_1, \cdots, f_r)= \bigoplus_{\substack{1\le i\le r\\ 0\le m_i<d_i}} A\left(E+\sum_{i=1}^r m_iD_i\right)\prod_{i=1}^r x_i^{m_i}\ .$$

Then $R(E; D_1, \cdots, D_r, f_1, \cdots, f_r)$ is a G-divisorial ideal of R. We denote the G-divisorial ideal as above simply by $R(E)$, if no confusion is possible.

PROPOSITION 2.4. *We define a map*

$$\begin{aligned}\phi :\ \bigoplus_{\mathfrak{p}\in V_H^1(A)} \mathbf{Z}\frac{1}{e_R(\mathfrak{p})}\mathfrak{p} &\longrightarrow \underline{\mathrm{Div}}(R)\\ E\qquad &\longmapsto R(-E\ ;\ D_1, \cdots, D_r, f_1, \cdots, f_r)\ .\end{aligned}$$

Then ϕ is an isomorphism of Abelian groups.

PROOF. Since $\underline{\mathrm{div}}_R(R(D)R(E))=R(D+E)$ (cf. (1.17)), ϕ is a group homomorphism. Let $\mathfrak{P}\in V_G^1(R)$ and $\mathfrak{p}=\mathfrak{P}^{(H)}$. Then $\underline{\mathrm{div}}_R(\mathfrak{p}R)=R(-\mathfrak{p})$ (cf. (1.17) and (1.18)). Since $e_R(\mathfrak{p})\underline{\mathrm{div}}_R(\mathfrak{P})=\underline{\mathrm{div}}_R(\mathfrak{p}R)=\phi(\mathfrak{p})=e_R(\mathfrak{p})\phi((1/e_R(\mathfrak{p}))\mathfrak{p})$ (in $\underline{\mathrm{Div}}(R)$), we have $\phi((1/e_R(\mathfrak{p}))\mathfrak{p})=\underline{\mathrm{div}}_R(\mathfrak{P})$ and ϕ is an isomorphism. □

REMARK 2.5. (1) We note that $\phi(D_i)=\underline{\mathrm{div}}_R(Rx_i)$ for $1\le i\le r$ and, therefore, $\underline{\mathrm{P}}(R)$ is isomorphic to $\{E+\sum_{i=1}^r m_iD_i \mid E\in \underline{\mathrm{P}}(A), 0\le m_i<d_i, 1\le i\le r\}$.

(2) For $0\le m_i<d_i$ $(1\le i\le r)$, $A(\sum_{i=1}^r m_iD_i)\prod_{i=1}^r x_i^{m_i}$ contains a G-homogeneous unit of R if and only if $\sum_{i=1}^r m_iD_i\in \underline{\mathrm{P}}(A)$.

We put $D_R=\sum_{\mathfrak{p}\in V_{H}(A)}((e_R(\mathfrak{p})-1)/e_R(\mathfrak{p}))\mathfrak{p}\in \underline{\mathrm{Div}}(A)_{\mathbf{Q}}$.

LEMMA 2.6. *For $E\in \underline{\mathrm{Div}}(A)$, there is the following isomorphism*

$$\underline{\mathrm{Hom}}_A(R, A(E)) \cong R(E+D_R\,;\,D_1, \cdots, D_r, f_1, \cdots, f_r)\,.$$

PROOF. Let $0 \le m_i < d_i$ $(1 \le i \le r)$ and $g = \sum_{i=1}^{r} m_i \deg(x_i)$. Then, for $h \in H$, $[\underline{\mathrm{Hom}}_A(R, A(E))]_{g+h} = [\underline{\mathrm{Hom}}_A(R^{(-g,H)}, A(E))]_{g+h}$ and

$$R^{(-g,H)} = A\left(\sum_{m_i>0} (d_i - m_i)D_i\right) \prod_{m_i>0} x_i^{d_i - m_i} = A\left(\sum_{m_i>0} ((d_i - m_i)D_i - \underline{\mathrm{div}}_A(f_i))\right) \prod_{i=1}^{r} x_i^{-m_i}\,.$$

Hence we have

$$\begin{aligned} \underline{\mathrm{Hom}}_A(R^{(-g,H)}, A(E)) &= \underline{\mathrm{Hom}}_A\left(A\left(-\sum_{i=1}^{r} m_i D_i\right) \prod_{i=1}^{r} x_i^{-m_i}, A(E)\right) \\ &\cong \left[A(E) : A\left(-\sum_{i=1}^{r} m_i D_i\right)\right] \prod_{i=1}^{r} x_i^{m_i} = A\left(E - \left[-\sum_{i=1}^{r} m_i D_i\right]\right) \prod_{i=1}^{r} x_i^{m_i} \\ &= A\left(E + \left[D_R + \sum_{i=1}^{r} m_i D_i\right]\right) \prod_{i=1}^{r} x_i^{m_i} = A\left(E + D_R + \sum_{i=0}^{r} m_i D_i\right) \prod_{i=1}^{r} x_i^{m_i} \end{aligned}$$

and

$$\begin{aligned} \underline{\mathrm{Hom}}_A(R, A(E)) &= \bigoplus_{\substack{1 \le i \le r \\ 0 \le m_i < d_i}} \underline{\mathrm{Hom}}_A\left(A\left(-\sum_{i=1}^{r} m_i D_i\right) \prod_{i=1}^{r} x_i^{-m_i}, A(E)\right) \\ &\cong R(E + D_R\,;\,D_1, \cdots, D_r, f_1, \cdots, f_r)\,. \end{aligned}$$

□

If $(A, \mathfrak{m})$ is H-local, then an *H-canonical* module $\underline{K}_A$ is defined as an H-graded A-module satisfying $\underline{K}_A \otimes_{A_0} \hat{A}_0 \cong [H^d_{\hat{\mathfrak{m}}}(\hat{A})]^\vee$ where $\hat{A} = A \otimes_{A_0} \hat{A}_0$ and $d = \underline{\dim}(A)$ (cf. section 3 of [9]). Furthermore, if A is H-normal and $\underline{K}_A$ exists, then $\underline{K}_A$ is H-divisorial. We denote by $\mathfrak{K}_A \in \underline{\mathrm{Div}}(A)$ the H-divisor satisfying $\underline{K}_A = A(\mathfrak{K}_A)$, if $\underline{K}_A$ exists.

As a direct consequence of (2.6), we have the following:

PROPOSITION 2.7 (Theorem 3.2 of Tomari-Watanabe [14]). *Assume that A is H-local and has an H-canonical module $\underline{K}_A = \underline{\mathrm{div}}_A(\mathfrak{K}_A)$.*

(1) $\underline{K}_R = R(\mathfrak{K}_A + D_R\,;\,D_1, \cdots, D_r, f_1, \cdots, f_r)$.

(2) *$\underline{K}_R$ is free if and only if $\mathfrak{K}_A + D_R - \sum_{i=1}^{r} m_i D_i \in \underline{\mathrm{P}}(A)$ for some $0 \le m_i < d_i$ $(1 \le i \le r)$.*

3. Complete intersections.

We keep notations as in section 2. Suppose that R has a unique G-maximal ideal $\mathfrak{M}$. We put $\mathfrak{m} = \mathfrak{M}^{(H)} \subset A$ and assume that $A/\mathfrak{m} = A_0/\mathfrak{m}_0$.

In this section, we consider a condition for R to be (locally) a complete intersection under the following assumptions.

ASSUMPTION 3.1. (1) *A is a factorial domain and locally a complete intersection.*
(2) $R/\mathfrak{M}=A/\mathfrak{m}$ $(=A_0/\mathfrak{m}_0)$ (*or* $\sum_{i=1}^r m_iD_i\notin\underline{\mathrm{P}}(A)$ *for* $(m_1,\cdots,m_r)\neq(0,\cdots,0)$).

Throughout this section, we assume the above conditions (1) and (2).

We set $\mathrm{Deg}(R)=\{\{\sum_{i=1}^r m_iD_i\}\mid 0\leq m_i<d_i\ (1\leq i\leq r)\}$ and denote by $\mathrm{Fund}(R)$ the set of minimal elements of $\mathrm{Deg}(R)$ with respect to the order of $\underline{\mathrm{Div}}(A)_{\mathbf{Q}}$.

Since $[\sum_{i=1}^r m_iD_i]\in\underline{\mathrm{P}}(A)$, $\{\sum_{i=1}^r m_iD_i\}$ is a principal ideal of R and $\mathrm{Deg}(R)\subset\underline{\mathrm{P}}(R)$. For $E\in\mathrm{Fund}(R)$, we put $y_E\in h(R)$ a homogeneous element satisfying $\underline{\mathrm{div}}_R(y_E)=E$. Also, by (3.1), $\alpha: G/H\to\bigoplus_{\mathfrak{p}\in V_H^1(A)}\mathbf{Z}/(e_R(\mathfrak{p}))$ is injective where α is as in (1.20).

Put $\Lambda=\{\mathfrak{p}\in V_H^1(A)\mid e_R(\mathfrak{p})>1\}$. For $\Lambda'\subset\Lambda$, we denote by $G_{\Lambda'}$ a subgroup of G such that $G_{\Lambda'}\supset H$ and $G_{\Lambda'}/H=\alpha^{-1}(\bigoplus_{\mathfrak{p}\in\Lambda'}\mathbf{Z}/(e_R(\mathfrak{p})))$.

LEMMA 3.2. (1) *For $E\in\underline{\mathrm{P}}(R)$ with $E>0$ and $E\notin\underline{\mathrm{P}}(A)$, there exists $D\in\mathrm{Fund}(R)$ such that $D\leq E$ (i.e. $\mathrm{Fund}(R)$ is the set of minimal elements of $\{E\in\underline{\mathrm{P}}(R)\mid E>0$ and $E\notin\underline{\mathrm{P}}(A)\}$).*

(2) $R=A[y_E\mid E\in\mathrm{Fund}(R)]$.

(2′) *Every element of* $\mathrm{Deg}(R)$ *is a sum of elements of* $\mathrm{Fund}(R)$.

(3) $\mathrm{Fund}(R^{G_{\Lambda'}})=\{E\in\mathrm{Fund}(R)\mid\mathrm{Supp}(E)\subset\Lambda'\}$ *for* $\Lambda'\subset\Lambda$.

PROOF. (1) For every $E\in\underline{\mathrm{P}}(R)\setminus\underline{\mathrm{P}}(A)$ with $E>0$, there exist $0\leq m_i<d_i$ $(1\leq i\leq r)$ and $E'\in\underline{\mathrm{P}}(A)$ such that $E=E'+\sum_{i=1}^r m_iD_i$ and $E'+[\sum_{i=1}^r m_iD_i]\geq 0$ (cf. (1) of (2.5)). Hence $E\geq\{\sum_{i=1}^r m_iD_i\}$. This implies the assertion (1).

(2) Let $D\in\underline{\mathrm{P}}(R)$ such that $D>0$ and $D\notin\underline{\mathrm{P}}(A)$. Then there exists $E_1\in\mathrm{Fund}(R)$ such that $D\geq E_1$ by (1). Since $0\leq D-E_1\in\underline{\mathrm{P}}(R)$, if $D-E_1\notin\underline{\mathrm{P}}(A)$, then there exists $E_2\in\mathrm{Fund}(R)$ such that $D-E_1\geq E_2$. Continuing this process, we can write $D=\underline{\mathrm{div}}_A(a)+\sum_{i=1}^s p_iE_i$ where $a\in h(A)$, $E_i\in\mathrm{Fund}(R)$ and p_i's are non-negative integers. This implies that R is generated by $\{y_E\mid E\in\mathrm{Fund}(R)\}$ as an A-algebra.

(3) For $y\in h(R)$, $\deg(y)\in G_{\Lambda'}$ if and only if $\mathrm{Supp}(\{\underline{\mathrm{div}}_R(y)\})\subset\Lambda'$ by (1.20). Hence, by (1), $\mathrm{Fund}(R^{(G_{\Lambda'})})$ is the set of minimal elements of $\{E\in\underline{\mathrm{P}}(R)\mid[E]\neq E,\ E>0$ and $\mathrm{Supp}(\{E\})\subset\Lambda'\}$ and coincide with $\{E\in\mathrm{Fund}(R)\mid\mathrm{Supp}(E)\subset\Lambda'\}$. □

For $\Lambda'\subset\Lambda$, we define a polynomial ring $S_{\Lambda'}$ by $S_{\Lambda'}=A[Y_E\mid E\in\mathrm{Fund}(R^{(G_{\Lambda'})})]$ and $M_{\Lambda'}=\mathfrak{m}S_{\Lambda'}+(Y_E\mid E\in\mathrm{Fund}(R^{(G_{\Lambda'})}))$. We define a surjective A-algebra homomorphism

$$\psi_{\Lambda'}: S_{\Lambda'}\to R^{(G_{\Lambda'})}\qquad\text{by}\quad\psi_{\Lambda'}(Y_E)=y_E$$

for $E\in\mathrm{Fund}(R^{(G_{\Lambda'})})$ and put $J_{\Lambda'}=ker(\psi_{\Lambda'})$. It is easy to verify that $J_{\Lambda'}$ is generated by elements of the form $Y^\alpha-aY^\beta$ $(\neq 0)$ $(a\in h(A)$, Y^α, Y^β are monomials of $S_{\Lambda'})$ and $J_{\Lambda'}=J_\Lambda\cap S_{\Lambda'}$.

For $E\in\mathrm{Fund}(R)$, there exists an element of J_Λ of the form $Y_E^d-aY^\beta$, where $a\in h(A)$ and Y^β is a monomial of S_Λ, since G/H is torsion. Put $d_E=inf\{d\mid 0\neq Y_E^d-aY^\beta\in J_\Lambda\}$. We fix an element

$$F(E)=Y_E^{d_E}-a_EY^{\beta_E}\in J_\Lambda\,.$$

For $\Lambda' \subset \Lambda$, if $\mathrm{Supp}(E) \subset \Lambda'$, then $F(E) \in J_{\Lambda'}$. Moreover, $\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$ is a part of minimal basis of $(J_{\Lambda'})_{M_{\Lambda'}}$.

PROPOSITION 3.3. *For $\Lambda' \subset \Lambda$, $R^{(G_{\Lambda'})}$ is locally a complete intersection if and only if $J_{\Lambda'}$ is generated by $\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$.*

To prove Proposition 3.3, we need some preliminaries.

For $\Lambda' \subset \Lambda$, we define a directed graph $\mathcal{G}_{\Lambda'}$ of a vertex set $V(\mathcal{G}_{\Lambda'}) = \{Y_E \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$ and directed edges $DE(\mathcal{G}_{\Lambda'})$ as follows: an ordered pair (Y_E, Y_D) is in $DE(\mathcal{G}_{\Lambda'})$, if Y_D divides Y^{β_E} where Y^{β_E} is as above.

LEMMA 3.4. ((3.5) of Nakajima [10] and (2.1) of Eto [5]). *Suppose that $(J_{\Lambda'})_{M_{\Lambda'}}$ is generated by $\{F(E) \mid \mathrm{Fund}(R^{(G_{\Lambda'})})\}$.*

(1) *There exists a linear ordering $\leq_{\Lambda'}$ on $V(\mathcal{G}_{\Lambda'})$ such that $Y_E \geq_{\Lambda'} Y_D$, if $(Y_E, Y_D) \in DE(\mathcal{G}_{\Lambda})$.*

(2) *$\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$ forms a regular sequence (in any order).*

PROOF. (1) We have only to show that $\mathcal{G}_{\Lambda'}$ has no cycle. Therefore, we assume the contrary. Then there exist $(Y_{E_1}, Y_{E_2}), (Y_{E_2}, Y_{E_3}), \cdots, (Y_{E_{k-1}}, Y_{E_k}), (Y_{E_k}, Y_{E_1}) \in DE(\mathcal{G}_{\Lambda'})$. We may assume that $E_i \neq E_j$ for $i \neq j$. Then, by definition of $DE(\mathcal{G}_{\Lambda'})$, $\{F(E_1), \cdots, F(E_k)\}$ is contained in $(Y_{E_1}, \cdots, Y_{E_k})$. Since $(J_{\Lambda'})_{M_{\Lambda'}}$ is generated by $\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$, the number of minimal generators of $(J_{\Lambda'})_{M_{\Lambda'}} + (Y_{E_1}, \cdots, Y_{E_k})_{M_{\Lambda'}}$ is at most $|\mathrm{Fund}(R^{(G_{\Lambda'})})| - k + k = |\mathrm{Fund}(R^{(G_{\Lambda'})})|$. On the other hand, since $R^{(G_{\Lambda'})}$ is $G_{\Lambda'}$-domain, Y_{E_1} is not a zero divisor of $S_{\Lambda'}/J_{\Lambda'}$ and

$$|\mathrm{Fund}(R^{(G_{\Lambda'})})| = ht((J_{\Lambda'})_{M_{\Lambda'}}) < ht((J_{\Lambda'}, Y_{E_1}, \cdots, Y_{E_k})_{M_{\Lambda'}})) .$$

This is a contradiction. Hence $\mathcal{G}_{\Lambda'}$ has no cycle.

(2) We put $J' = (\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\})$ and extend $\leq_{\Lambda'}$ to a monomial ordering of $S_{\Lambda'}$, lexicographically. Then, for $E \in \mathrm{Fund}(R^{(G_{\Lambda'})})$, $Y_E^{d_E} >_{\Lambda'} Y^{\beta_E}$ and $\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$ forms a Gröbner basis of J' (cf. Proposition 9.2(b) of [13]). In particular, the initial term $in(J')$ of J' is generated by $\{Y_E^{d_E} \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$. By standard arguments of the theory of Gröbner basis, this implies $\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$ is a regular sequence. (See for instance [13].) □

PROOF OF (3.3). The "if" part follows from (3.4). We shall show the "only if" part.

We put $S = S_{\Lambda'}$, $M = M_{\Lambda'}$, and $J = J_{\Lambda'}$. Since $R^{(G_{\Lambda'})}$ is A-free, we have $J/\mathfrak{m}J \cong ker(\psi_{\Lambda'} \otimes_A A/\mathfrak{m})$. Thus the number $\mu(J_M)$ of minimal generators of J_M is equal to the number $\mu(J_M + \mathfrak{m}S_M/\mathfrak{m}S_M)$. Then, since $(R^{(G_{\Lambda'})})_M$ is complete intersection, $\mu(J_M) = |\mathrm{Fund}(R^{G_{\Lambda'}})|$ (cf. Theorem 2 of [2]). Namely, if we put $J' = (\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\})$, then $J_M = J'_M$ since $\{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$ is part of minimal generators of J_M. Thus $J = J' + MJ$ and $J = J' + M^n J$ for every $n > 0$. For $E \in \mathrm{Fund}(R^{(G_{\Lambda'})})$, we put $d = \prod_{Y_D \leq_{\Lambda'} Y_E} d_D$ where $<_{\Lambda'}$ is as in (1) of (3.4). By (2) of (3.1), there exists $D \in \mathrm{Fund}(R^{(G_{\Lambda'})})$

such that $Y_D <_{A'} Y_E$ and $F(D) = Y_D^{d_D} - a_D$ for some $a_D \in \mathfrak{m}$. This implies that $Y_E^d \in J' + \mathfrak{m}S$. Hence, for $n \gg 0$, $M_{A'}^n J \subset J' + \mathfrak{m}S$ and $J + \mathfrak{m}S/\mathfrak{m}S = J' + \mathfrak{m}S/\mathfrak{m}S$ (or $S/J' + \mathfrak{m}S \cong (S/J') \otimes_A A/\mathfrak{m} \cong R^{(G_{A'})} \otimes_A A/\mathfrak{m}$). On the other hand, S/J' is A-free, by the proof of (2) of (3.4). Then $rank_A(S/J') = \dim_{A/\mathfrak{m}}((S/J') \otimes_A A/\mathfrak{m}) = \dim_{A/\mathfrak{m}}(R^{(G_{A'})} \otimes_A A/\mathfrak{m}) = rank_A(R^{(G_{A'})})$ and the canonical surjection $S/J' \to R^{(G_{A'})}$ is isomorphism. Hence we have $J = J'$. □

COROLLARY 3.5. *If R is locally a complete intersection, then so is $R^{(G_{A'})}$ for any $\Lambda' \subset \Lambda$.*

PROOF. Suppose that R is locally a complete intersection. Then, by (3.3), J_Λ is generated by $\{F(E) \mid E \in \mathrm{Fund}(R)\}$. Also, by the proof of (3.4), $J_{\Lambda'} = J_\Lambda \cap S_{\Lambda'}$ is generated by $\{F(E) \mid E \in \mathrm{Fund}(R), F(E) \in S_{\Lambda'}\} = \{F(E) \mid E \in \mathrm{Fund}(R^{(G_{\Lambda'})})\}$ for $\Lambda' \subset \Lambda$. Thus $R^{(G_{\Lambda'})}$ is locally a complete intersection for $\Lambda' \subset \Lambda$. □

DEFINITION 3.6. Let Γ be a finite subset of $\underline{\mathrm{Div}}(A)_{\mathbf{Q}} \setminus \{0\}$ and $w : \Gamma \to \mathbf{N}_+$ be a map. Here we denote by $\mathbf{N}_+$ the set of all positive integers. We call that (Γ, w) is a *datum*, if it satisfies the following conditions;

(1) for $D, E \in \Gamma$ $(D \neq E)$, one of the following cases occurs;

(a) $\mathrm{Supp}(D) \subsetneq \mathrm{Supp}(E)$, (b) $\mathrm{Supp}(D) \supsetneq \mathrm{Supp}(E)$, (c) $\mathrm{Supp}(D) \cap \mathrm{Supp}(E) = \emptyset$,

(2) for $E \in \Gamma$, there is a relation $w(E)E = \sum_{i=1}^k E_i + \sum_{j=1}^l \mathfrak{p}_j$ where $\{E_1, \cdots, E_k\} = \{D \in \Gamma \mid D \prec E\}$ and $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_l\} = \mathrm{Supp}(E) \setminus \bigcup_{i=1}^k \mathrm{Supp}(E_i)$. (We write $D \prec E$ if $\mathrm{Supp}(D) \subsetneq \mathrm{Supp}(E)$ and there is no element $D' \in \Gamma$ such that $\mathrm{Supp}(D) \subsetneq \mathrm{Supp}(D') \subsetneq \mathrm{Supp}(E)$.)

Let (Γ, w) be a datum. For $E \in \Gamma$ and $\mathfrak{p} \in \mathrm{Supp}(E)$, we set

$$\Gamma_{E,\mathfrak{p}} = \{D \in \Gamma \mid \mathfrak{p} \in \mathrm{Supp}(D) \subset \mathrm{Supp}(E)\},$$

$$e_E(\mathfrak{p}) = \prod_{D \in \Gamma_{E,\mathfrak{p}}} w(D).$$

Then we have $E = \sum_{\mathfrak{p} \in \mathrm{Supp}(E)} (1/e_E(\mathfrak{p}))\mathfrak{p}$.

DEFINITION 3.7. Let (Γ, w) be a datum.

(1) Let $A[Y_E \mid E \in \Gamma]$ be a polynomial ring over A. We put $a_E \in h(A)$ a homogeneous element satisfying $\underline{\mathrm{div}}_A(a_E) = w(E)E - \sum_{D \prec E} D$ for $E \in \Gamma$ and set

$$R(\Gamma) = A[Y_E \mid E \in \Gamma] \Big/ \Big(Y_E^{w(E)} - a_E \prod_{D \prec E} Y_D \mid E \in \Gamma \Big).$$

(2) We put $G(\emptyset) = H$ and define an Abelian group $G(\Gamma)$ such that $R(\Gamma)$ is a $G(\Gamma)$-graded ring by

$$G(\Gamma) = \mathbf{Z}^{(\Gamma)} \Big/ \Big\langle w(E)\mathbf{e}_E - \sum_{D \prec E} \mathbf{e}_D \mid E \in \Gamma \Big\rangle,$$

where $\mathbf{Z}^{(\Gamma)}=\bigoplus_{E\in\Gamma}\mathbf{Z}\mathbf{e}_E$ is a free abelian group with free basis $\{\mathbf{e}_E \mid E\in \mathrm{Fund}(R)\}$.

(3) Let $\{E_1, \cdots, E_l\}=\{E\in\Gamma \mid \mathrm{Supp}(E)$ is maximal in $\{\mathrm{Supp}(D) \mid D\in\Gamma\}\}$ and $\mathfrak{p}\in V_H^1(A)$. We put

$$e(\mathfrak{p})=\begin{cases} e_{E_i}(\mathfrak{p}) & \text{if } \mathfrak{p}\in\mathrm{Supp}(E_i) \text{ for some } i \\ 1 & \text{otherwise}. \end{cases}$$

EXAMPLE 3.8. Let k be a field. We put $A=k[[a, b, c, d]]$ the formal power series ring and

$$E_1=\tfrac{1}{12}\mathrm{div}_A(ac)+\tfrac{1}{8}\mathrm{div}_A(bd),$$

$$E_2=\tfrac{1}{6}\mathrm{div}_A(a)+\tfrac{1}{4}\mathrm{div}_A(b), \qquad E_3=\tfrac{1}{6}\mathrm{div}_A(c)+\tfrac{1}{4}\mathrm{div}_A(d),$$

$$E_4=\tfrac{1}{3}\mathrm{div}_A(a), \qquad E_5=\tfrac{1}{2}\mathrm{div}_A(b), \qquad E_6=\tfrac{1}{3}\mathrm{div}_A(c), \qquad E_7=\tfrac{1}{2}\mathrm{div}_A(d),$$

and $\Gamma=\{E_1, \cdots, E_7\}$. We define the map $w:\Gamma\to\mathbf{N}_+$ by $w(E_1)=w(E_2)=w(E_3)=w(E_5)=w(E_7)=2$ and $w(E_4)=w(E_6)=3$. Then (Γ, w) is a datum and

$$R(\Gamma)=A[Y_{E_1}, \cdots, Y_{E_7}]/J$$

where $J=(Y_{E_1}^2-Y_{E_2}Y_{E_3}, Y_{E_2}^2-Y_{E_4}Y_{E_5}, Y_{E_3}^2-Y_{E_6}Y_{E_7}, Y_{E_4}^3-a, Y_{E_5}^2-b, Y_{E_6}^3-c, Y_{E_7}^2-d)$.

PROPOSITION 3.9. *Let (Γ, w) be a datum. Then the following hold.*

(1) *$R(\Gamma)$ is $G(\Gamma)$-normal and $\underline{\mathrm{Div}}(R(\Gamma))\cong\bigoplus_{\mathfrak{p}\in V_H^1(A)}\mathbf{Z}(1/e(\mathfrak{p}))\mathfrak{p}$.*

(2) *$\mathrm{div}_R(Y_E R(\Gamma))=E$ for $E\in\Gamma$ by the identification as in (2.4).*

(3) *$\mathrm{Fund}(R(\Gamma))=\Gamma$.*

The above proposition follows from the next remark.

REMARK 3.10. Let B be a G'-normal and $a\in h(B)$. We put $G''=G'\oplus\mathbf{Z}/\langle(\deg(a), -n)\rangle$. Then, by (2.1) and (2.2), $B':=B[X]/(X^n-a)$ is G''-normal if and only if $\underline{\mathrm{div}}_B(a)(\mathfrak{q})\le 1$ for every $\mathfrak{q}\in V_{G'}^1(B)$. In this case, $\underline{\mathrm{Div}}(B')=\bigoplus_{\mathfrak{q}\in V_{G'}^1(B)}\mathbf{Z}(1/e(\mathfrak{q}))\mathfrak{q}$ and $\underline{\mathrm{div}}_{B'}(XB')=(1/n)\underline{\mathrm{div}}_B(a)$ where $e(\mathfrak{q})=n$ (resp. $e(\mathfrak{q})=1$), if $\mathfrak{q}\in\mathrm{Supp}(\underline{\mathrm{div}}_B(a))$ (resp. $\mathfrak{q}\notin\mathrm{Supp}(\underline{\mathrm{div}}_B(a))$) (cf. (2.4)).

We state our main result as follows.

THEOREM 3.11. *The following conditions are equivalent.*

(1) *R is locally a complete intersection.*

(2) *There exists a datum (Γ, w) such that $G\cong G(\Gamma)$ and $R\cong R(\Gamma)$.*

The proof of (2) ⇒ (1) of (3.11) is easy. Therefore, it is enough to show the following proposition.

PROPOSITION 3.12. *Suppose that R is locally a complete intersection. We put $\Gamma=\mathrm{Fund}(R)$. Then the following hold.*

(1) *There exists a map $w:\Gamma\to\mathbf{N}_+$ such that (Γ, w) is a datum.*

(2) $R(\Gamma)=S_A/J_A\ (\cong R)$.
(3) $G\cong G(\Gamma)$.

PROOF. (1) We prove the assertion by induction on $|A|$. If $A=\varnothing$, then $R=A$ and $\mathrm{Fund}(R)=\varnothing$. We assume that $A\neq\varnothing$. Then, by (3.5), $R^{(G_{A'})}$ is locally a complete intersection and, by induction hypothesis, $\mathrm{Fund}(R^{(G_{A'})})$ is a datum for all $A'\subsetneq A$. Since A is factorial, $\sum_{\mathfrak{p}\in A}\mathfrak{p}\in\underline{\mathrm{P}}(A)$ and, by (2) of (2.7), $D:=\sum_{\mathfrak{p}\in A}(1/e_R(\mathfrak{p}))\mathfrak{p}\in\mathrm{Deg}(R)$. The proof is divided into two cases: (1) $D\notin\mathrm{Fund}(R)$, (2) $D\in\mathrm{Fund}(R)$.

Case (1). We can write $D=E_1+\cdots+E_p$ for $E_1,\cdots,E_p\in\mathrm{Fund}(R)$ $(p\geq 2)$ by (3.2), (2'). Then $\mathrm{Supp}(E_i)\cap\mathrm{Supp}(E_j)=\varnothing$ for $i\neq j$. We put $A_i=\mathrm{Supp}(E_i)$ for $1\leq i\leq p$.

Let $E'\in\mathrm{Fund}(R)$. If $A_i\subset\mathrm{Supp}(E')$, then $E_i\leq E'$ and, since $E'\in\mathrm{Fund}(R)$, we have $E_i=E'$. We assume that $A_i\not\subset\mathrm{Supp}(E')$ for $i=1,\cdots,p$. Since $\mathrm{Supp}(E')\subset A$, $\mathrm{Supp}(E_j)\cap\mathrm{Supp}(E')\neq\varnothing$ for some j. Then $A':=A_j\cup\mathrm{Supp}(E')\subsetneq A$ since $A_i\not\subset\mathrm{Supp}(E')$ for all i and $A_i\cap A_k=\varnothing$ for $i\neq k$. By induction hypothesis and the fact that E_j, $E'\in\mathrm{Fund}(R^{(G_{A'})})$ (cf. (3.2)), we have $\mathrm{Supp}(E')\subset\mathrm{Supp}(E_j)$. Hence $\mathrm{Fund}(R)$ is disjoint union of $\mathrm{Fund}(R^{(G_{A_i})})$ $(1\leq i\leq p)$. On the other hand, $\mathrm{Fund}(R^{(G_{A_i})})$ is a datum $(1\leq i\leq p)$, by induction hypothesis. Thus $\mathrm{Fund}(R)$ is also a datum.

Case (2) $D\in\mathrm{Fund}(R)$. We put $\{E_1,\cdots,E_p\}=\{E\in\mathrm{Fund}(R)\mid E\prec D\}$.

Claim 1. $\mathrm{Supp}(E_i)\cap\mathrm{Supp}(E_j)=\varnothing$ for $i\neq j$.

Assume that $\mathrm{Supp}(E_i)\cap\mathrm{Supp}(E_j)$ is not empty for some $i\neq j$. Then we have that $A=\mathrm{Supp}(E_i)\cup\mathrm{Supp}(E_j)$, since $\mathrm{Fund}(R^{(G_{A'})})$ is a datum for any $A'\subsetneq A$. Since $E_i+E_j\geq D$, we have a relation $E_i+E_j=nD+E$ where $0\leq E\in\underline{\mathrm{P}}(R)$ and $\mathrm{Supp}(E)\subsetneq A$. In other words, $Y_{E_i}Y_{E_j}-aY_D^nY^\beta\in J_A$ where $a\in h(A)$ such that $\underline{\mathrm{div}}_R(a\psi_A(Y^\beta))=E$. Then, by (3.3), there exists $E'\in\mathrm{Fund}(R)$ such that $Y^{\beta_{E'}}$ divides $Y_{E_i}Y_{E_j}$ and a_{E^i} is unit where $a_{E'}Y^{\beta_{E'}}$ is the term of the polynomial $F(E')\in S_A$. Since E_i, $E_j\in\mathrm{Fund}(R)$, we have $Y^{\beta_{E'}}=Y_{E_i}Y_{E_j}$ and $F(E')=Y_{E_i}^{d_{E_i}}-a_{E'}Y_{E_i}Y_{E_j}$. Also, since $\mathrm{Supp}(E')\supsetneq\mathrm{Supp}(E_i)$, we have $E'=D$. Namely, $d_DD=E_i+E_j$. On the other hand, if there exists $1\leq k\leq p$, $k\neq i,j$, then $\mathrm{Supp}(E_i)\cup\mathrm{Supp}(E_k)=A$ by induction hypothesis. Then $E_i+E_k=d_DD$ and $E_k=E_j$. This is a contradiction. Hence we have $p=2$.

We put $\mathrm{Supp}(E_1)\backslash\mathrm{Supp}(E_2)=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_s\}$, $\mathrm{Supp}(E_1)\cap\mathrm{Supp}(E_2)=\{\mathfrak{p}_{s+1},\cdots,\mathfrak{p}_t\}$ and $\mathrm{Supp}(E_2)\backslash\mathrm{Supp}(E_1)=\{\mathfrak{p}_{t+1},\cdots,\mathfrak{p}_m\}$. Since $d_DD=E_1+E_2$, we can write

$$E_1=\sum_{i=1}^{s}\frac{d_D}{e_R(\mathfrak{p}_i)}\mathfrak{p}_i+\sum_{i=s+1}^{t}\frac{n_i}{e_R(\mathfrak{p}_i)}\mathfrak{p}_i\quad\text{and}\quad E_2=\sum_{i=s+1}^{t}\frac{n_i'}{e_R(\mathfrak{p}_i)}\mathfrak{p}_i+\sum_{i=t+1}^{m}\frac{d_D}{e_R(\mathfrak{p}_i)}\mathfrak{p}_i$$

where $n_i+n_i'=d_D$ for $s+1\leq i\leq t$. Without loss of generality, we may assume that n_{s+1} is minimal in $\{n_{s+1},\cdots,n_t,n_{s+1}',\cdots,n_t'\}$. We put $E=E_1+\sum_{j=t+1}^{m}\mathfrak{p}_j-n_{s+1}D\in\underline{\mathrm{P}}(R)$. Then $E\not\geq E_i$ for $i=1,2$. Since $E=\{E\}$, there exists $E'\in\mathrm{Fund}(R)\backslash\{D,E_1,E_2\}$ such that $E'\leq E$ and $\mathrm{Supp}(E')\cap\{\mathfrak{p}_1,\cdots,\mathfrak{p}_s\}\neq\varnothing$. We note that, for every $E''\in\mathrm{Fund}(R)\backslash\{D,E_1,E_2\}$, either $\mathrm{Supp}(E'')\subset\mathrm{Supp}(E_1)$ or $\mathrm{Supp}(E'')\subset\mathrm{Supp}(E_2)$. Hence $\mathrm{Supp}(E')\subset\mathrm{Supp}(E_1)$ and $E'\leq\sum_{\mathfrak{p}\in\mathrm{Supp}(E_1)}E(\mathfrak{p})<E_1$. This contradicts $E_1\in\mathrm{Fund}(R)$.

The proof of Claim 1 is completed.

We put $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_q\} = \mathrm{Supp}(D) \setminus \bigcup_{i=1}^{p} \mathrm{Supp}(E_i)$.

Claim 2. $w(D)D = \sum_{i=1}^{p} E_i + \sum_{j=1}^{q} \mathfrak{p}_j$ for some $w(D) > 0$.

There exists an integer $w>0$ such that $E := \sum_{i=1}^{p} E_i + \sum_{j=1}^{q} \mathfrak{p} - wD \geq 0$ and $\mathrm{Supp}(E) \subsetneq \Lambda$. If $E \neq 0$, then there exists $E' \in \mathrm{Fund}(R) \setminus \{D\}$ such that $E' \leq E$. On the other hand, we have $\mathrm{Supp}(E') \subset \mathrm{Supp}(E_i)$ for some i. Since $\mathrm{Supp}(E_i) \cap \mathrm{Supp}(E_j) = \varnothing$ $(i \neq j)$, $E' \leq \sum_{\mathfrak{p} \in \mathrm{Supp}(E')} E(\mathfrak{p})\mathfrak{p} < E_i$. This is a contradiction. Hence $E=0$ and $wD = \sum_{i=1}^{p} E_i + \sum_{j=1}^{q} \mathfrak{p}$.

Combining Claims 1, 2 and the induction hypothesis, we have that $\mathrm{Fund}(R) = \{D\} \cup \bigcup_{i=1}^{p} \mathrm{Fund}(R^{(G_{\mathrm{Supp}(E_i)})})$ is a datum.

(2) We prove that $J_\Lambda = (Y_E^{w(E)} - a_E \prod_{E' \prec E} Y_{E'} \mid E \in \mathrm{Fund}(R))$ by induction on $|\mathrm{Fund}(R)|$. If $\mathrm{Fund}(R) = \varnothing$, then there is nothing to prove. We assume that $\mathrm{Fund}(R) \neq \varnothing$. Let $E \in \mathrm{Fund}(R)$ such that $\mathrm{Supp}(E)$ is maximal in $\{\mathrm{Supp}(D) \mid D \in \mathrm{Fund}(R)\}$. Then it suffices to show that locally a complete intersection $R^{(G_{\mathrm{Supp}(E)})}$ satisfies the above condition. Therefore we may assume that $E = \sum_{\mathfrak{p} \in \Lambda} (1/e_R(\mathfrak{p}))\mathfrak{p} \in \mathrm{Fund}(R)$. We put $\{E_1, \cdots, E_p\} = \{D \in \mathrm{Fund}(R) \mid D \prec E\}$. Then, by induction hypothesis, $R^{(G_{\mathrm{Supp}(E_i)})}$ satisfies the assertion (2) $(1 \leq i \leq p)$. Then we have only to show that $J_\Lambda = (Y_E^{w(E)} - a_E \prod_{i=1}^{p} Y_{E_i}) + \sum_{i=1}^{p} J_{\mathrm{Supp}(E_i)}$.

Let $E_1', \cdots, E_t' \in \mathrm{Fund}(R) \setminus \{E\}$. Assume that there exists a relation

$$\underline{\mathrm{div}}_\Lambda(a) + dE + \sum_{i=1}^{s} d_i E_i' = \underline{\mathrm{div}}_\Lambda(b) + \sum_{i=s+1}^{t} d_i E_i' .$$

Then, by definition of a datum, $E_1', \cdots, E_t' \in \bigoplus_{\mathfrak{p} \in \Lambda} \mathbf{Z}(w(E)/e_R(\mathfrak{p}))\mathfrak{p}$. Hence $dE \in \bigoplus_{\mathfrak{p} \in \Lambda} \mathbf{Z}(w(E)/e_R(\mathfrak{p}))\mathfrak{p}$ and $w(E)$ divides d. This implies $J_\Lambda = (Y_E^{w(E)} - a_E \prod_{i=1}^{p} Y_{E_i}) + \sum_{i=1}^{p} J_{\mathrm{Supp}(E_i)}$.

The assertion (3) follows from definition of $G(\mathrm{Fund}(R))$. □

4. Abelian extensions which are complete intersections.

Let A be a Noetherian normal domain with $K = Q(A)$ and L be a finite Abelian extension of K with $G = Gal(L/K)$. An integral closure R of A in L is called an Abelian extension of A with a Galois group G.

REMARK 4.1. We put $\hat{G} = \mathrm{Hom}(G, U(A))$, where $U(A)$ is the multiplicative group of units of A. Assume that $ch(A)$ does not divide $|G|$, if $ch(A)$ is positive and A contains a primitive $|G|$-th root of unity. Then R can be regarded as $\hat{G}$-graded ring in the following sense.

For $g \in \hat{G}$, we set $R_g = \{a \in R \mid \sigma(a) = g(\sigma)a$ for every $\sigma \in G\}$. Then

(1) $R_0 = R^G = A$,

(2) $R_g R_h \subset R_{g+h}$ for every $g, h \in \hat{G}$,

(3) $R = \sum_{g \in \hat{G}} R_g = \bigoplus_{g \in \hat{G}} R_g$.

(See 2 of Itoh [8].)

As a consequence of (3.11), we have the following.

THEOREM 4.2. *Let A be a complete intersection factorial local domain and G be a finite Abelian group of $n=|G|$. Assume that*

(i) *either $ch(A)=0$ or $ch(A)=p>0$ and $(p, n)=1$,*

(ii) *A contains a primitive n-th root of unity.*

Let $(R, \mathfrak{n})$ be a local ring such that $R\supset A$. Then the following are equivalent.

(1) *R is an Abelian extension of A with Galois group G such that $R/\mathfrak{n}\cong A/\mathfrak{m}$ and is a complete intersection.*

(2) *There exists a datum (Γ, w) (in $\mathrm{Div}(A)_{\mathbf{Q}}$) such that $G\cong G(\Gamma)$ and $R\cong R(\Gamma)$.*

PROOF. (1) $\Rightarrow$ (2): This follows from (4.1) and (3.11). (2) $\Rightarrow$ (1): Let $E\in\Gamma$ such that $\mathrm{Supp}(E)$ is maximal in $\{\mathrm{Supp}(D)\mid D\in\Gamma\}$ and $\{E_1, \cdots, E_s\}=\{D\in\Gamma\mid D\prec E\}$. By Proposition 1.12 of Tomari-Watanabe [14], if $R\{\Gamma\setminus\{E\}\}$ is normal domain, then so is

$$R(\Gamma)\cong R(\Gamma\setminus\{E\})[Y_E]\Big/\left(Y_E^{d_E}-a_E\prod_{i=1}^{s}y_{E_i}\right).$$

Hence R is normal domain. Also, by assumptions (i) and (ii), we have $Q(R)$ is Galois extension of K and $Gal(Q(R)/K)\cong G(\Gamma)\cong G$. □

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