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Nonlinear Stability of Travelling Waves for One-Dimensional Viscoelastic Materials with Non-Convex Nonlinearity

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Abstract. The aim of this paper is to study the stability of travelling wave solutions with shock profiles for one-dimensional viscoelastic materials with the non-degenerate and the degenerate shock conditions by means of an elementary weighted energy method. The stress function is not necessarily assumed to be convex or concave, and the third derivative of this stress function is also not necessarily assumed to be non-negative or non-positive. The travelling waves are proved to be stable for suitably small initial disturbance and shock strength, which improves recent stability results. The key points of our proofs are to choose the suitable weight function and weighted Sobolev spaces of the solutions.

1. Introduction.

In this paper we study the asymptotic stability of travelling wave solutions with shock profiles for the system of one-dimensional viscoelastic materials with non-convex nonlinearity in the form

$$v_t - u_x = 0\,, \tag{1.1}$$

$$u_t - \sigma(v)_x = \mu u_{xx}\,, \tag{1.2}$$

with the initial data

$$(v, u)|_{t=0} = (v_0, u_0)(x) \to (v_\pm, u_\pm) \qquad \text{as } x \to \pm\infty\,. \tag{1.3}$$

Here, $x \in R^1$ and $t \geq 0$, v is the strain, u the velocity, $\mu > 0$ the viscous constant, $\sigma(v)$ is the smooth stress function satisfying

$$\sigma'(v) > 0 \qquad \text{for all } v \text{ under consideration}\,, \tag{1.4}$$

$$\sigma''(v) \lesseqgtr 0 \qquad \text{for } v \lesseqgtr 0 \text{ under consideration}\,, \tag{1.5}$$

so that $\sigma(v)$ is neither convex nor concave, and has an inflection point at $v=0$. We find that the system (1.1), (1.2) with $\mu=0$ is strictly hyperbolic, with the characteristic roots

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$$\lambda=\pm\lambda(v)\,,\qquad \text{where}\quad \lambda(v)=\sqrt{\sigma'(v)}$$

and with the corresponding right eigenvectors

$$r_{\pm}(v)=\begin{pmatrix}1\\ \mp\lambda(v)\end{pmatrix}.$$

Moreover, we see that both characteristic fields are neither genuinely nonlinear nor linearly degenerate in the neighborhood of $v=0$. In fact, the quantity

$$\nabla\lambda(v)\cdot r_{\pm}(v)=\lambda'(v)=\sigma''(v)/2\sqrt{\sigma'(v)}$$

changes its sign at $v=0$, where ∇ denotes the gradient with respect to (v, u).

The travelling wave solutions are solutions of the form

$$(v, u)(t, x)=(V, U)(\xi)\,,\qquad \xi=x-st\,, \tag{1.6}$$

$$(V, U)(\xi)\to(v_{\pm}, u_{\pm})\,,\qquad \xi\to\pm\infty\,, \tag{1.7}$$

where s is the shock speed and $(v_{\pm}, u_{\pm})$ are constant states at $\pm\infty$. Let the system (1.1), (1.2) admit the travelling wave solutions, then $(v_{\pm}, u_{\pm})$ and s satisfy the Rankine-Hugoniot condition

$$\begin{cases}-s(v_{+}-v_{-})-(u_{+}-u_{-})=0\,,\\ -s(u_{+}-u_{-})-(\sigma(v_{+})-\sigma(v_{-}))=0\,,\end{cases} \tag{1.8}$$

and the generalized shock condition

$$\frac{1}{s}h(v)\equiv\frac{1}{s}[-s^2(v-v_{\pm})+\sigma(v)-\sigma(v_{\pm})]\begin{cases}<0\,, & \text{if}\quad v_{+}<v<v_{-}\\ >0\,, & \text{if}\quad v_{-}<v<v_{+}\,.\end{cases} \tag{1.9}$$

We note that the condition (1.9) with (1.4) and (1.5) implies

$$\lambda(v_{+})\le s<\lambda(v_{-})\quad\text{or}\quad -\lambda(v)\le s<-\lambda(v_{-})\,, \tag{1.10}$$

and that, especially when $\sigma''(v)>0$, the condition (1.9) is equivalent to

$$\lambda(v_{+})<s<\lambda(v_{-})\quad\text{or}\quad -\lambda(v_{+})<s<-\lambda(v_{-})\,, \tag{1.11}$$

which is well-known as Lax's shock condition (Lax [5]). We call the condition (1.10) with $s=\lambda(v_{+})$ (or $s=-\lambda(v_{+})$) and the condition (1.11) as the degenerate and non-degenerate shock condition, respectively.

Throughout this paper, without loss of generality, let us suppose $\sigma(0)=0$. In fact, if $\sigma(0)\neq0$, setting $\sigma_1(v)=\sigma(v)-\sigma(0)$, then $\sigma_1(0)=0$ and $\sigma_1(v)$ satisfies equations (1.1), (1.2) and (1.4), (1.5), (1.8), (1.9) corresponding to $\sigma(v)$. Thus, we may denote $\sigma_1(v)$ by $\sigma(v)$ again.

The stability problem of travelling wave solutions for systems has been one of hot spots and interested many mathematicians (see [2, 3, 4, 6, 7, 8, 10, 11]). In genuinely nonlinear cases, the stability theorems have been studied by many authors [3, 6, 7, 10] etc. See also the references in [8] for the single equation. Recently, the stability in the non-degenerate shock case of (1.11) without convexity of $\sigma(v)$ was investigated by Kawashima-Matsumura [4] for the first time. Although it seemed hardly to solve the stability in the case of the degenerate shock, Mei [9] and Matsumura-Nishihara [8] proved the stability of degenerate shock profile for a single conservation law and then Nishihara [10] successfully showed the stability for the system (1.1), (1.2) provided that the integral of the initial disturbance over $(-\infty, x]$, say $(\phi_0, \psi_0)(x)$, have a polynomial decay $O(|x|^{-(1+\alpha)/2})$ $(0<\alpha<1)$ as $x\to+\infty$. In the papers [4, 10], the authors supposed as sufficient conditions that the third derivative of the stress function $\sigma'''(v)$ is positive and the shock strength $|(v_+-v_-, u_+-u_-)|$ is suitably small.

In this paper, we have two purposes. One is to show the stability of travelling wave solutions without the condition $\sigma'''(v)>0$. Another is to improve the weight in [10] in the degenerate shock case. The stability theorems are shown even in the degenerate shock case with the improved weight. Here, the smallness of both the shock strength and the initial disturbance is assumed. In the degenerate case, the initial disturbances have the decay order $O(|x|^{-1/2})$ as $x\to+\infty$. Thus, we improve the results in both [4] and [10]. Throughout [4, 10] and the present paper, the integrals over R of the initial disturbances are assumed to be zero. When they are not zero, the stability problem is open in the case of non-convex nonlinearity. In the genuinely nonlinear case, see the interesting papers [6, 11].

Proofs are due to an elementary weighted energy method. However, the weight functions are suitably selected, which play a key role in our procedure. Our plan of this paper is as follows. After stating the notations and an embedding theorem in the next section, we will state the properties of the travelling waves in Section 3. In Section 4, the stability theorems and their proofs will be given. Finally, we will complete the proofs of *a priori* estimates which are key steps for our stability theorems in Section 5.

2. Notations and an embedding theorem.

L^2 denotes the space of measurable functions on R which are square integrable, with the norm

$$\|f\|=\left(\int |f(x)|^2\,dx\right)^{1/2}.$$

H^l $(l\geq 0)$ denotes the Sobolev space of L^2-functions f on R whose derivatives $\partial_x^j f$, $j=1,\cdots,l$, are also L^2-functions, with the norm

$$\|f\|_l=\left(\sum_{j=0}^{l}\|\partial_x^j f\|^2\right)^{1/2}.$$

L_w^2 denotes the space of measurable functions on R which satisfy $w(x)^{1/2}f \in L^2$, where $w(x)>0$ is a weight function, with the norm

$$|f|_w = \left(\int w(x)|f(x)|^2 dx\right)^{1/2}.$$

H_w^l $(l \geq 0)$ denotes the weighted Sobolev space of L_w^2-functions f on R whose derivatives $\partial_x^j f$, $j=1, \cdots, l$, are also L_w^2-functions, with the norm

$$|f|_{l,w} = \left(\sum_{j=0}^{l} |\partial_x^j f|_w^2\right)^{1/2}.$$

C_w^l $(l \geq 0)$ denotes the weighted l-times continuously differentiable space whose functions $f(x)$ satisfy $w(x)\partial_x^j f \in C^0$, $j=0, 1, \cdots, l$, with the norm

$$\|f\|_{C_w^l} = \sup_{x \in R} \sum_{j=0}^{l} w(x)|\partial_x^j f(x)|.$$

Denoting

$$\langle x \rangle_+ = \begin{cases} \sqrt{1+x^2}, & \text{if } x \geq 0 \\ 1, & \text{if } x < 0, \end{cases} \tag{2.1}$$

we will make use of the space $L_{\langle x \rangle_+}^2$ and $H_{\langle x \rangle_+}^l$ $(l=1, 2)$. We also denote $f(x) \sim g(x)$ as $x \to a$ when $C^{-1}g \leq f \leq Cg$ in a neighborhood of a. Here and after here, we denote generic positive constant by C, without confusions. When $C^{-1} \leq w(x) \leq C$ for $x \in R$, we note that $L^2 = H^0 = L_w^2 = H_w^0$ and $\|\cdot\| = \|\cdot\|_0 \sim |\cdot|_w = |\cdot|_{0,w}$. Especially, when $w(x) = \langle x \rangle_+$, we have the following embedding theorem.

EMBEDDING THEOREM. *There exists the embedding relation $H_{\langle x \rangle_+}^l \hookrightarrow H^l$, i.e., if $f \in H_{\langle x \rangle_+}^l$, then $f \in H^l$ and the following inequality holds*:

$$\|f\|_l \leq C|f|_{l,\langle x \rangle_+}. \tag{2.2}$$

Moreover, if $f \in H_{\langle x \rangle_+}^l$ for $l \leq 2$, then $\langle x \rangle_+^{1/2} f \in H^l$ and it holds

$$\|\langle x \rangle_+^{1/2} f\|_l \leq C|f|_{l,\langle x \rangle_+}. \tag{2.3}$$

When $l \geq 1$, then $H_{\langle x \rangle_+}^l \hookrightarrow C_{\langle x \rangle_+^{1/2}}^0 \hookrightarrow C^0$, i.e., if $f \in H_{\langle x \rangle_+}^l$, then $\langle x \rangle_+^{1/2} f \in C^0$ and it holds

$$\sup_{x \in R} |f(x)| \leq C \sup_{x \in R} \langle x \rangle_+^{1/2} |f(x)| \leq C|f|_{2,\langle x \rangle_+}. \tag{2.4}$$

Furthermore, the embedding relation $H^l \cap L_{\langle x \rangle_+}^2 \hookrightarrow C_{\langle x \rangle_+^{1/4}}^0 \hookrightarrow C^0$, and

$$\sup_{x \in R} |f(x)| \leq C \sup_{x \in R} \langle x \rangle_+^{1/4} |f(x)| \leq C(\|f\|_l + |f|_{\langle x \rangle_+}) \tag{2.5}$$

hold for any $l \geq 1$.

PROOF. Noting that $\langle x\rangle_+ \geq C$ for all $x\in R$, and $|(d^k/dx^k)\langle x\rangle_+^{1/2}|\leq C$ for $x\in(-\infty, -0]$ and $x\in[+0, +\infty)$, and using Sobolev's embedding theorem $H^l \hookrightarrow C^0$ ($l\geq 1$), we can prove the embedding results (2.2)–(2.4) by simple but tedious calculations. By Hölder inequality, we have

$$\begin{aligned}\langle x\rangle_+^{1/2} f(x)^2 &= \int_{-\infty}^{x} \frac{d}{dy}(\langle y\rangle_+^{1/2} f(y)^2)dy \\ &= \int_{-\infty}^{x} \frac{1}{2}(1+y^2)^{-3/4} y f(y)^2 dy + \int_{-\infty}^{x} \langle y\rangle_+^{1/2} 2 f(y) f'(y) dy \\ &\leq C(\|f\|^2 + |f|^2_{\langle x\rangle_+} + \|f_x\|^2),\end{aligned}$$

which proves (2.5).

Let T and B be a positive constant and a Banach space, respectively. We denote $C^k(0, T; B)$ ($k\geq 0$) as the space of B-valued k-times continuously differentiable functions on $[0, T]$, and $L^2(0, T; B)$ as the space of B-valued L^2-functions on $[0, T]$. The corresponding spaces of B-valued functions on $[0, \infty)$ are defined similarly.

3. Properties of travelling wave solution with shock profile.

In this section, we state the properties of travelling wave solution with shock profile. If $(v, u)(t, x)=(V, U)(\xi)$ ($\xi=x-st$) is the travelling wave solution with shock profile connecting (v_-, u_-) and (v_+, u_+), then $(V, U)(\xi)$ must satisfy

$$\begin{cases} -sV' - U' = 0 \\ -sU' - \sigma(V)' = \mu U'' . \end{cases} \tag{3.1}$$

Integrating (3.1) over $(-\infty, +\infty)$, we have Rankine-Hugoniot condition (1.8). We integrate (3.1) and eliminate U, then we obtain a single ordinary differential equation for $V(\xi)$

$$\mu s V' = -s^2 V + \sigma(V) - a \equiv h(V), \tag{3.2}$$

where

$$a = -s^2 v_\pm + \sigma(v_\pm). \tag{3.3}$$

Letting $(v_+, u_+)\neq(v_-, u_-)$ and $s>0$, we are now ready to summarize a characterization of the generalized shock condition (1.9) and the results on the existence of shock profile studied in [4]:

PROPOSITION 3.1 ([4]). *Suppose that* (1.4) *and* (1.5) *hold. Then the following statements are equivalent to each other.*

(i) *The generalized shock condition* (1.9) *holds.*

(ii) $\sigma'(v_+)\leq s^2$, *i.e.*, $\lambda(v_+)\leq s$.

(iii) $\sigma'(v_+)\leq s^2<\sigma'(v_-)$, *i.e.*, $\lambda(v_+)\leq s<\lambda(v_-)$.

(iv) *There exists uniquely a* $v_*\in(v_+, v_-)$ *such that* $\sigma'(v_*)=s^2$ *and it holds*

$$\sigma'(v)<s^2 \text{ for } v\in(v_+, v_*)\,, \qquad s^2<\sigma'(v) \text{ for } v\in(v_*, v_-)\,, \tag{3.4}$$

i.e.,

$$h'(v_*)=0\,, \quad h'(v)<0 \text{ for } v\in(v_+, v_*)\,, \quad h'(v)>0 \text{ for } v\in(v_*, v_-)\,. \tag{3.5}$$

Moreover, if one of the above four conditions holds, then we must have $v_-\neq 0$. *In addition,* $v_+\lesseqgtr v_-$ *and* $v_*\gtreqless 0$ *hold when* $v_-\gtreqless 0$.

PROPOSITION 3.2 ([4]). *Suppose that* (1.4) *and* (1.5) *hold.*

(i) *If* (1.1), (1.2) *admits a travelling wave solution with shock profile* $(V(x-st), U(x-st))$ *connecting* $(v_\pm, u_\pm)$, *then* $(v_\pm, u_\pm)$ *and* s *must satisfy the Rankine-Hugoniot condition* (1.8) *and the generalized shock condition* (1.9).

(ii) *Conversely, suppose that* (1.8) *and* (1.9) *hold, then there exists a shock profile* $(V, U)(x-st)$ *of* (1.1), (1.2) *which connects* $(v_\pm, u_\pm)$. *The* $(V, U)(\xi)$ $(\xi=x-st)$ *is unique up to a shift in* ξ *and is a monotone function of* ξ. *In particular, when* $v_+\lesseqgtr v_-$ *(and hence* $u_+\gtreqless u_-$*) we have*

$$u_+\gtreqless U(\xi)\gtreqless u_-\,, \qquad U_\xi(\xi)\gtreqless 0\,, \tag{3.6}$$

$$v_+\lesseqgtr V(\xi)\lesseqgtr v_-\,, \qquad V_\xi(\xi)\lesseqgtr 0\,, \tag{3.7}$$

for all $\xi\in R$. *Moreover,* $(V, U)(\xi)\to(v_\pm, u_\pm)$ *exponentially as* $\xi\to\pm\infty$, *with the following exceptional case: when* $\lambda(v_+)=s$, $(V, U)(\xi)\to(v_+, u_+)$ *at the rate* $|\xi|^{-1}$ *as* $\xi\to+\infty$, *and* $|h(V)|=|\mu s V_\xi|=O(|\xi|^{-2})$ *as* $\xi\to\infty$.

For the graphes of $\sigma(v)$ and $h(v)$, see Figures 3.1 and 3.2.

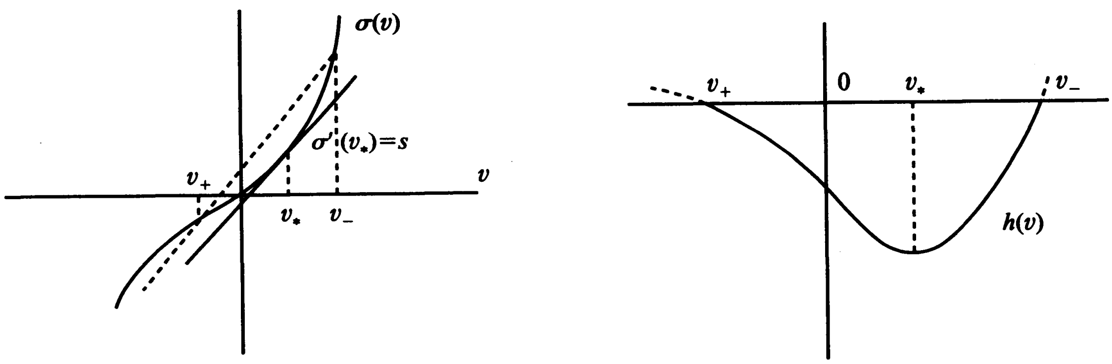

FIGURE 3.1. Non-degenerate case

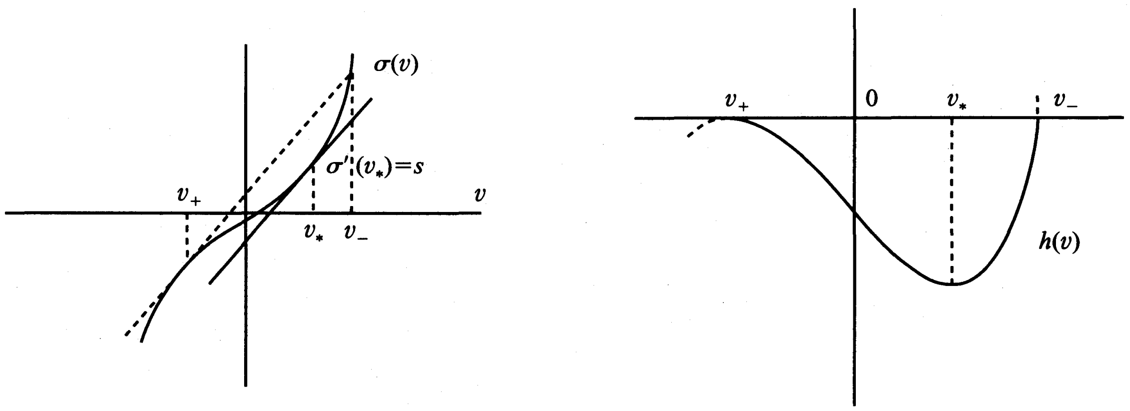

FIGURE 3.2. Degenerate case

Now we give a function of the form

$$G(v) \equiv h(v)\sigma''(v) - h'(v)\sigma'(v) = h(v)^2\left(\frac{\sigma'(v)}{h(v)}\right)', \qquad v \in [0, v_*], \tag{3.8}$$

which plays an important role in our proof. We know that $G(v)$ is continuous, and $G(v)$ satisfies, by virtue of (3.5),

$$G(0) = -\sigma'(0)h'(0) > 0, \qquad G(v_*) = h(v_*)\sigma''(v_*) < 0. \tag{3.9}$$

According to these facts, we know that there exist some finite or infinite points in $(0, v_*)$ such that $G(v)=0$. These points divide $[0, v_*]$ into sub-intervals such that $G(v)>0$ or $\equiv 0$ or <0 on these sub-intervals.

We now only pay our attention to the case in which there are finite number of the points $v_i \in (0, v_*)$ defined as follows:

$$\begin{cases} v_1 = \sup\{v \mid G(v) \geq 0 \text{ on } [0, v]\}, \\ v_2 = \sup\{v \mid G(v) \leq 0 \text{ on } [v_1, v]\}, \\ v_{2i-1} = \sup\{v \mid G(v) \geq 0 \text{ on } [v_{2i-2}, v]\}, \\ v_{2i} = \sup\{v \mid G(v) \leq 0 \text{ on } [v_{2i-1}, v]\}. \end{cases} \tag{3.10}$$

In this case, the graph of $G(v)$ looks like the following Figure 3.3.

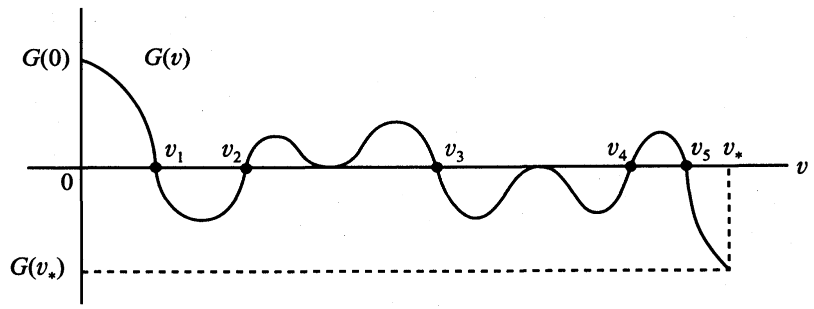

FIGURE 3.3

The function $G(v)$ may have infinitely many v_i's, whose case will be remarked later. See Remark 5.6. By our choice (3.10), we get the properties of $G(v)$ as follows:

PROPOSITION 3.3. *There exist the odd number points, without loss of generality, say n points (n is an odd number), $v_i \in (0, v_*)$, $i=1, \cdots, n$, such that $G(v_i)=0$ and*

$$\begin{cases} G(v) \geq 0 & \text{on} \quad I_{2j-1} \equiv [v_{2j-2}, v_{2j-1}] \,, \quad j=1,2,\cdots,(n+1)/2 \,, \\ G(v) \leq 0 & \text{on} \quad I_{2j} \equiv [v_{2j-1}, v_{2j}] \,, \quad j=1,2,\cdots,(n+1)/2 \,, \end{cases} \tag{3.11}$$

where v_0 and v_{n+1} denote 0 and v_*, respectively. Especially, $G(v)<0$ on $(v_n, v_*]$.

PROOF. By the continuity of $G(v)$ and (3.9), and our choice (3.10), we see easily that Proposition 3.3 is true.

We also denote I_0 and I_{n+2} as the following intervals

$$I_0 \equiv (v, 0] \,, \qquad I_{n+2} \equiv [v_*, v_-] \,. \tag{3.12}$$

Since $V(\xi)$ is monotonic on $[v_+, v_-]$ (see (3.7)), there exist the unique numbers ξ_0, ξ_i ($i=1,2,\cdots,n$), and ξ_* such that $V(\xi_0)=v_0\equiv 0$, $V(\xi_i)=v_i$, $i=1,\cdots,n$ and $V(\xi_*)=v_*\equiv v_{n+1}$ (see Proposition 3.3). Here, we also denote by R_i ($i=0,1,\cdots,n+1,n+2$) the following sub-intervals of $(-\infty,+\infty)$: $R_0=[\xi_0,+\infty)$, $R_i=[\xi_i,\xi_{i-1}]$ $i=1,\cdots,n$, $R_{n+1}=[\xi_{n+1},\xi_n]=[\xi_*,\xi_n]$, $R_{n+2}=(-\infty,\xi_*]$, respectively. It is clear that $R=\bigcup_{i=0}^{n+2} R_i$.

4. Stability theorems.

In this section, we shall state the stability theorems of travelling wave solutions with shock profiles for (1.1)–(1.3) without the condition $\sigma'''(v)>0$. To state our result in the degenerate case, we set

$$\bar{\sigma}(v) \equiv \sigma(v) - \sigma'(0)v \,. \tag{4.1}$$

Then we have $\bar{\sigma}(0)=\bar{\sigma}'(0)=\bar{\sigma}''(0)$, $\bar{\sigma}''(v) \lesseqgtr 0$ for $v \lesseqgtr 0$ and

$$0 < -\bar{\sigma}(v_+) < -v_+\bar{\sigma}'(v_+) \qquad \text{for } v_+ < 0 < v_- \tag{4.2}$$

by an observation of the graph of the function $\bar{\sigma}'(v)$, $v_+ \leq v \leq 0$ (see Figure 4.1).

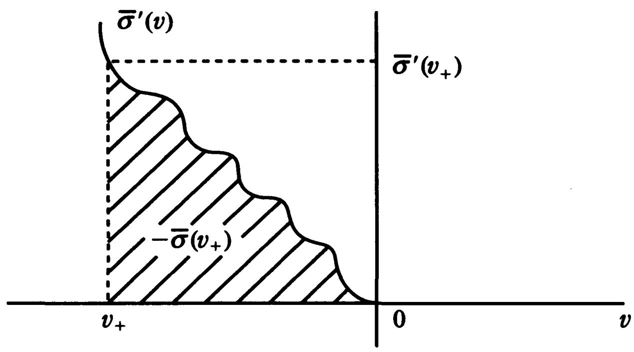

FIGURE 4.1

We assume that there is a constant $\bar{\delta}$ $(0<\bar{\delta}<1)$ such that

$$-\bar{\sigma}(v_+)<-\bar{\delta}v_+\bar{\sigma}'(v_+) \qquad \text{as } v_+\to -0\,. \tag{4.3}$$

We note that $\bar{\delta}$ can be taken as $\bar{\delta}=1/4+\varepsilon$ if $\bar{\sigma}'''(0)$ $(=\sigma'''(0))>0$, where $\varepsilon>0$ is any given constant. In fact, we have

$$-\bar{\sigma}(v_+)=-\frac{\sigma'''(0)}{3!}v_+^3+o(v_+^3)\,, \qquad -v_+\bar{\sigma}'(v_+)=-\frac{\sigma'''(0)}{2!}v_+^3+o(v_+^3)\,,$$

which mean that (4.3) holds for $\bar{\delta}>1/3$ as $v_+\to 0$.

Now, without loss of generality, we restrict our attention to the case

$$s>0 \text{ and } v_+<0<v_-\,, \quad \text{i.e.,} \quad \mu s V_\xi=h(V)<0\,. \tag{4.4}$$

Let $(V, U)(x-st)$ be a pair of travelling wave solutions connecting $(v_\pm, u_\pm)$. We assume the integrability of $(v_0-V, u_0-U)(x)$ over R and express that integral in the form

$$\int_{-\infty}^{\infty}(v_0-V, u_0-U)(x)dx=x_0(v_+-v_-, u_+-u_-)+\beta r_-(v_-)\,, \tag{4.5}$$

where $r_-(v_-)$ is the right eigenvector evaluated at $v=v_-$. We note that the coefficients β and x_0 are uniquely determined by (4.5) provided that (v_0-V, u_0-U) is integrable over R. Throughout this paper, we assume that $\beta=0$. Then the shifted function $(V, U)(x-st+x_0)$ is also a pair travelling wave solution with shock profile connecting $(v_\pm, u_\pm)$ such that

$$\int_{-\infty}^{\infty}(v_0(x)-V(x+x_0), u_0(x)-U(x+x_0))dx=0\,. \tag{4.6}$$

We also suppose $x_0=0$ for simplicity.

Let us define (ϕ_0, ψ_0) by

$$(\phi_0, \psi_0)(x)=\int_{-\infty}^{x}(v_0-V, u_0-U)(y)dy\,. \tag{4.7}$$

Our main theorems are the following.

THEOREM 4.1 (Non-degenerate case: $\lambda(v_+)<s<\lambda(v_-)$). *Suppose* (1.4), (1.5), (1.8), (4.4), (4.6), *and* $(\phi_0, \psi_0)\in H^2$. *Then there exists a positive constant* δ_1 *such that if* $|(v_+-v_-, u_+-u_-)|+\|(\phi_0, \psi_0)\|_2<\delta_1$, *then* (1.1)–(1.3) *has a unique global solution* $(v, u)(t, x)$ *satisfying*

$$v-V\in C^0([0, \infty)\,;\, H^1)\cap L^2([0, \infty)\,;\, H^1)\,,$$

$$u-U\in C^0([0, \infty)\,;\, H^1)\cap L^2([0, \infty)\,;\, H^2)\,.$$

Furthermore, the solution verifies

$$\sup_{x\in R} |(v, u)(t, x)-(V, U)(x-st)| \to 0 \qquad as\ t\to\infty . \tag{4.8}$$

THEOREM 4.2 (Degenerate case: $\lambda(v_+)=s<\lambda(v_-)$). *Suppose* (1.4), (1.5), (1.8), (4.4) *and* (4.6). *Assume* $|(v_+-v_-, u_+-u_-)|\ll 1$ *and* (4.3), *then the following holds*:

(i) *Suppose that* $(\phi_0, \psi_0)\in H^2_{\langle x\rangle_+}$, *then there exists a positive constant* δ_2 *such that if* $|(\phi_0, \psi_0)|_{2,\langle x\rangle_+}<\delta_2$, *then* (1.1)–(1.3) *has a unique global solution* $(v, u)(t, x)$ *satisfying*

$$v-V\in C^0([0, \infty)\,;\, H^1_{\langle x\rangle_+})\cap L^2([0, \infty)\,;\, H^1_{\langle x\rangle_+^{1/2}}),$$

$$u-U\in C^0([0, \infty)\,;\, H^1_{\langle x\rangle_+})\cap L^2([0, \infty)\,;\, H^2_{\langle x\rangle_+}).$$

Furthermore, the solution verifies the asymptotic stability (4.8).

(ii) *Suppose that* $(\phi_0, \psi_0)\in H^2\cap L^2_{\langle x\rangle_+}$ *and* $\phi_{0,x}\in L^2_{\langle x\rangle_+^{3/4}}$. *Then there exists a positive constant* δ_3 *such that if* $\|(\phi_0, \psi_0)\|_2+|(\phi_0, \psi_0)|_{\langle x\rangle_+}+|\phi_{0,x}|_{\langle x\rangle_+^{3/4}}<\delta_3$, *then* (1.1)–(1.3) *has a unique global solution* $(v, u)(t, x)$ *satisfying*

$$v-V\in C^0([0, \infty)\,;\, H^1\cap L^2_{\langle x\rangle_+})\cap L^2([0, \infty)\,;\, H^1\cap L^2_{\langle x\rangle_+^{3/4}}),$$

$$u-U\in C^0([0, \infty)\,;\, H^1\cap L^2_{\langle x\rangle_+})\cap L^2([0, \infty)\,;\, H^2\cap L^2_{\langle x\rangle_+}).$$

Furthermore, the solution verifies the asymptotic stability (4.8).

REMARK 4.3. In the stability results in [4, 10] both $\sigma'''(v)>0$ and smallness of shock strength $|(v_+-v_-, u_+-u_-)|$ are assumed as sufficient conditions. In the non-degenerate shock case, Theorem 4.1 deletes the condition $\sigma'''(v)>0$. In the degenerate shock condition, $\lambda(v_+)=s<\lambda(v_-)$, the condition (4.3) in Theorem 4.2 seems to be much weaker than the condition $\sigma'''(v)>0$, and also the weight is improved compared to that in Nishihara [10]. As an example of $\sigma(v)$, we have

$$\sigma(v)=bv+\int_0^v\int_0^x y^k\left(\sin\frac{1}{y}+2\right)dydx, \qquad k=1, 3, 5, \cdots,$$

where b is a constant satisfying

$$b>\max_{v_+\le v\le 0}\left|\int_0^v x^k\left(\sin\frac{1}{x}+2\right)dx\right|.$$

Then, note that $\sigma'''(v)$ does not exist for $k=1$ and $\sigma'''(v)$ changes the sign on $[v_+, v_-]$ for $k\ge 3$.

In order to show the stability, we make a reformulation for the problem (1.1)–(1.3) as in [3, 4, 6, 7, 10] by changing the unknown variables as

$$(v, u)(t, x)=(V, U)(\xi)+(\phi_\xi, \psi_\xi)(t, \xi), \qquad \xi=x-st. \tag{4.9}$$

Then the problem (1.1)–(1.3) is reduced to the following "integrated" system

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0 \\ \psi_t - s\psi_\xi - \sigma'(V)\phi_\xi - \mu\psi_{\xi\xi} = F \\ (\phi, \psi)(0, \xi) = (\phi_0, \psi_0)(\xi) \end{cases} \tag{4.10}$$

with

$$F = \sigma(V + \phi_\xi) - \sigma(V) - \sigma'(V)\phi_\xi .$$

For any fixed $T \in (0, \infty)$, let us define the solution spaces of (4.10) by

$$X_0(0, T) = \{(\phi, \psi) \in C^0([0, T]; H^2),\ \phi_\xi \in L^2([0, T]; H^1),\ \psi_\xi \in L^2([0, T]; H^2)\} ,$$

$$X_1(0, T) = \{(\phi, \psi) \in C^0([0, T]; H^2_{\langle\xi\rangle_+}),\ \phi_\xi \in L^2([0, T]; H^1_{\langle\xi\rangle_+^{1/2}}),$$
$$\psi_\xi \in L^2([0, \infty); H^2_{\langle\xi\rangle_+})\} ,$$

$$X_2(0, T) = \{(\phi, \psi) \in C^0([0, T]; H^2 \cap L^2_{\langle\xi\rangle_+}),\ \phi_\xi \in L^2([0, T]; H^1 \cap L^2_{\langle\xi\rangle_+^{3/4}}),$$
$$\psi_\xi \in L^2([0, T]; H^2 \cap L^2_{\langle\xi\rangle_+})\} .$$

Setting

$$N_0(t) = \sup_{0 \le \tau \le t} \|(\phi, \psi)(\tau)\|_2 , \qquad N_1(t) = \sup_{0 \le \tau \le t} |(\phi, \psi)(\tau)|_{2, \langle\xi\rangle_+} ,$$

$$N_2(t) = \sup_{0 \le \tau \le t} (\|(\phi, \psi)(\tau)\|_2 + |(\phi, \psi)(\tau)|_{\langle\xi\rangle_+} + |\phi_\xi(\tau)|_{\langle\xi\rangle_+^{3/4}}) ,$$

we have, by the embedding theorem in Section 2,

$$\begin{cases} \sup_{\xi \in R} |(\phi, \psi)(t, \xi)| \le C N_0(t) , \\ \sup_{\xi \in R} |(\phi, \psi)(t, \xi)| \le C \sup_{\xi \in R} |\langle\xi\rangle_+^{1/2} (\phi, \psi)(t, \xi)| \le C N_1(t) , \\ \sup_{\xi \in R} |\psi(t, \xi)| \le C \sup_{\xi \in R} |\langle\xi\rangle_+^{3/4} \psi(t, \xi)| \le C N_2(t) , \\ \sup_{\xi \in R} |(\phi, \psi)(t, \xi)| \le C N_2(t) . \end{cases} \tag{4.11}$$

Theorem 4.1 and Theorem 4.2 can be regarded as the direct consequences from the following theorem.

THEOREM 4.4. (A) (Non-degenerate case): *Suppose the assumptions in Theorem 4.1. Then there exists a positive constant δ_4 such that if $\|(\phi_0, \psi_0)\|_2 < \delta_4$, then* (4.10) *has a unique global solution $(\phi, \psi) \in X_0(0, \infty)$ satisfying*

$$\|(\phi, \psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\} d\tau \le C \|(\phi_0, \psi_0)\|_2^2 \tag{4.12$_0$}$$

for any $t \ge 0$. Moreover, the stability holds in the following sense:

$$\sup_{\xi \in R} |(\phi_\xi, \psi_\xi)(t, \xi)| \to 0 \qquad as\ t \to \infty . \tag{4.13}$$

(B) (Degenerate case): *Suppose the assumptions in Theorem* 4.2, *then we have the following*:

(i) *There exists a positive constant* δ_5 *such that if* $|(\phi_0, \psi_0)|_{2,\langle\xi\rangle_+} < \delta_5$, *then* (4.10) *has a unique global solution* $(\phi, \psi) \in X_1(0, \infty)$ *satisfying*

$$|(\phi, \psi)(t)|^2_{2,\langle\xi\rangle_+} + \int_0^t \{|\phi_\xi(\tau)|^2_{1,\langle\xi\rangle_+^{1/2}} + |\psi_\xi(\tau)|^2_{2,\langle\xi\rangle_+}\} d\tau$$
$$\leq C|(\phi_0, \psi_0)|^2_{2,\langle\xi\rangle_+} \tag{4.12$_1$}$$

for any $t \geq 0$. *Moreover, the stability* (4.13) *holds*.

(ii) *There exists a positive constant* δ_6 *such that if* $\|(\phi_0, \psi_0)\|_2 + |(\phi_0, \psi_0)|_{\langle\xi\rangle_+} + |\phi_{0,\xi}|_{\langle\xi\rangle_+^{3/4}} < \delta_6$, *then* (4.10) *has a unique global solution* $(\phi, \psi) \in X_2(0, \infty)$ *satisfying*

$$\|(\phi, \psi)(t)\|^2_2 + |(\phi, \psi)(t)|^2_{\langle\xi\rangle_+} + |\phi_\xi(t)|^2_{\langle\xi\rangle_+^{3/4}}$$
$$+ \int_0^t \{\|\phi_\xi(\tau)\|^2_1 + |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}} + \|\psi_\xi(\tau)\|^2_2 + |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+}\} d\tau$$
$$\leq C(\|(\phi_0, \psi_0)\|^2_2 + |(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{3/4}}) \tag{4.12$_2$}$$

for any $t \geq 0$. *Moreover, the stability* (4.13) *also holds*.

Theorem 4.4 is proved by a weighted energy method combining the local existence with a priori estimates.

PROPOSITION 4.5 (Local existence). *For any* $\delta_0 > 0$, *there exists a positive constant* T_0 *depending on* δ_0 *which satisfies the following*.

(A) (Non-degenerate case): *If* $(\phi_0, \psi_0) \in H^2$ *and* $\|(\phi_0, \psi_0)\|_2 \leq \delta_0$, *then the problem* (4.10) *has a unique solution* $(\phi, \psi) \in X_0(0, T_0)$ *satisfying* $\|(\phi, \psi)(t)\|_2 \leq 2\delta_0$ *for* $0 \leq t \leq T_0$.

(B) (Degenerate case): (i) *If* $(\phi_0, \psi_0) \in H^2_{\langle\xi\rangle_+}$ *and* $|(\phi_0, \psi_0)|_{2,\langle\xi\rangle_+} \leq \delta_0$, *then the problem* (4.10) *has a unique solution* $(\phi, \psi) \in X_1(0, T_0)$ *satisfying* $|(\phi, \psi)(t)|_{2,\langle\xi\rangle_+} \leq 2\delta_0$ *for* $0 \leq t \leq T_0$.

(ii) *If* $(\phi_0, \psi_0) \in H^2 \cap L^2_{\langle\xi\rangle_+}$, *and* $\phi_{0,\xi} \in L^2_{\langle\xi\rangle_+^{3/4}}$, $\|(\phi_0, \psi_0)\|_2 + |(\phi_0, \psi_0)|_{\langle\xi\rangle_+} + |\phi_{0,\xi}|_{\langle\xi\rangle_+^{3/4}} \leq \delta_0$, *then the problem* (4.10) *has a unique solution* $(\phi, \psi) \in X_2(0, T_0)$ *satisfying* $\|(\phi, \psi)(t)\|_2 + |(\phi, \psi)(t)|_{\langle\xi\rangle_+} + |\phi_\xi|_{\langle\xi\rangle_+^{3/4}} \leq 2\delta_0$ *for* $0 \leq t \leq T_0$.

PROPOSITION 4.6 (A priori estimates). (A) (Non-degenerate case): *Let* $(\phi, \psi) \in X_0(0, T)$ *be a solution for a positive* T. *Then there exists a positive constant* δ_7 *independent of* T *such that if* $N_0(T) < \delta_7$, *then* (ϕ, ψ) *satisfies the a priori estimate* (4.12)$_0$ *for* $0 \leq t \leq T$.

(B) (Degenerate case): (i) *Let* $(\phi, \psi) \in X_1(0, T)$ *be a solution for a positive* T. *Then there exists a positive constant* δ_8 *independent of* T *such that if* $N_1(T) < \delta_8$, *then* (ϕ, ψ) *satisfies the a priori estimate* (4.12)$_1$ *for* $0 \leq t \leq T$.

(ii) *Let* $(\phi, \psi) \in X_2(0, T)$ *be a solution for a positive* T. *Then there exists a positive constant* δ_9 *independent of* T *such that if* $N_2(T) < \delta_9$, *then* (ϕ, ψ) *satisfies the a priori* *estimate* $(4.12)_2$ *for* $0 \le t \le T$.

Proposition 4.5 can be proved in the standard way. So we omit the proof. We shall prove Proposition 4.6 in the next section.

5. The proofs of a priori estimates.

The section is a key step to complete the proofs of the stability theorems. At first, we introduce our desired weight functions which play a key role for our a priori estimates. Let us define a weight function $w(v)$ by

$$w(v)=\begin{cases} w_0(v)=(v^2-v_+^2)/h(v)\,, & v\in I_0\,, \\ w_{2j-1}(v)=k_{2j-1}\cdot(-1)/h(v)\,, & v\in I_{2j-1}\,, \\ w_{2j}(v)=k_{2j}\cdot 1/\sigma'(v)\,, & v\in I_{2j}\,, \\ w_{n+2}(v)=k_{n+1}\cdot 1/\sigma'(v)\,, & v\in I_{n+2}\,, \end{cases} \tag{5.1}$$

where $j=1,\cdots,(n+1)/2$, $I_i(i=0,1,\cdots,n+1,n+2)$ are as in (3.10)–(3.12) and $k_1=v_+^2$, $k_2=-k_1\sigma'(v_1)/h(v_1)$, $k_{2j-1}=-k_{2j-2}h(v_{2j-2})/\sigma'(v_{2j-2})$, $k_{2j}=-k_{2j-1}\sigma'(v_{2j-1})/h(v_{2j-1})$, $j=2,\cdots,(n+1)/2$. Note $k_i>0$ $(i=1,2,\cdots,n+1)$. We also denote by $r(\xi)$ another weight function in the form

$$r(\xi)=\begin{cases} 1+\xi-\xi_0\,, & \text{as } \xi\ge\xi_0\,, \\ 1\,, & \text{as } \xi\le\xi_0\,, \end{cases} \tag{5.2}$$

where ξ_0 is defined as such number that $V(\xi_0)=0$ in the section 3. Then we know that $w(V)\in C^0(v_+,v_-]$, $w(V)\notin C^1(v_+,v_-]$, but $w_i(V)\in C^2(I_i)$, $i=0,1,\cdots,n+1,n+2$, and $r(\xi)$ has the same property as $w(V)$. Moreover, we find

$$\text{non-degenerate case:}\quad w(V(\xi))\sim\text{Const.},\quad L_w^2=L^2\,, \tag{5.3$_1$}$$

$$\text{degenerate case:}\quad w(V(\xi))\sim r(\xi)\sim\langle\xi\rangle_+\,,\quad L_w^2=L_r^2=L_{\langle\xi\rangle_+}^2\,. \tag{5.3$_2$}$$

Now we are going to prove part (B) of Proposition 4.6 by the following two subsections. Since part (A) of Proposition 4.6 can be proved in the same procedure as (B), we omit its details and only give a remark in the following sub-section.

5.1. The proof of Proposition 4.6 B(i). Let $(\phi,\psi)\in X_1(0,T)$ be a solution of (4.10). On the every interval R_i $(i=0,1,\cdots,n+2)$, multiplying the first equation of (4.10) by $(w_i\sigma')(V)\phi$ and the second equation of (4.10) by $w_i(V)\psi$ and adding these equations, we have

$$\frac{1}{2}\{(w_i\sigma')(V)\phi^2+w_i(V)\psi^2\}_t-\{(w_i\sigma')(V)\phi\psi+\mu w_i(V)\psi\psi_\xi\}_\xi$$

$$-\frac{s}{2}\{(w_i\sigma')(V)\phi^2+w_i(V)\psi^2\}_\xi+\mu w_i(V)\psi_\xi^2+A_i(t,\xi)=Fw_i(V)\psi\,, \tag{5.4}$$

where

$$A_i(t,\xi)=\frac{s}{2}(w_i\sigma')'(V)V_\xi\phi^2+\mu w_i'(V)V_\xi\psi\psi_\xi$$
$$+(w_i\sigma')'(V)V_\xi\phi\psi+\frac{s}{2}w_i'(V)V_\xi\psi^2\,,\qquad i=0,1,\cdots,n+2\,. \tag{5.5}$$

Integrating (5.4) over R_i and adding these integrated equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_R((w\sigma')(V)\phi^2+w(V)\psi^2)d\xi+\mu\sum_{i=0}^{n+2}\int_{R_i}w(V)\psi_\xi^2d\xi$$
$$+\sum_{i=0}^{n+2}\int_{R_i}A_i(t,\xi)d\xi=\int_R Fw(V)\psi d\xi\,. \tag{5.6}$$

Step 1. When $\xi\in R_0$, i.e., $v\in I_0=(v_+,0]$, we can check the facts $w_0'(v)<0$, $(w_0\sigma')'(v)\le 0$ and $(w_0h)''(v)=2$, similar to Nishihara [11]. By (5.5), (4.4) and Cauchy's inequality, and noting

$$-\frac{1}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2=-\left(\frac{1-\alpha}{2s}+\frac{\alpha}{2s}\right)w_0'(0)h(0)\psi(t,\xi_0)^2\,,$$

where α is a constant which will be suitably chosen as $0<\alpha<1$, we obtain

$$\int_{R_0}A_0(t,\xi)d\xi=-\frac{1}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2$$
$$+\int_{R_0}\frac{sV_\xi}{2}\left[(w_0\sigma')'(V)\left(\phi+\frac{1}{s}\psi\right)^2-\frac{(w_0h)''(V)}{s^2}\psi^2\right]d\xi$$
$$\ge-\frac{1-\alpha}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2+\frac{\alpha}{2s}w_0'(0)\int_{R_0}\frac{\partial}{\partial\xi}(h(V)\psi(t,\xi)^2)d\xi+\int_{R_0}-\frac{V_\xi}{s}\psi^2d\xi$$
$$\ge-\frac{1-\alpha}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2-\frac{\mu}{2}\int_{R_0}w_0(V)\psi_\xi^2d\xi+\int_{R_0}-\frac{V_\xi}{2s}p_\alpha(V)\psi^2d\xi\,, \tag{5.7}$$

where

$$p_\alpha(V)=2-\alpha w_0'(0)h'(V)+\alpha^2w_0'(0)^2h(V)/w_0(V)\,. \tag{5.8}$$

LEMMA 5.1. *Suppose that* (4.3) *holds. Let* $\alpha=(1-\delta)^2$, *then* $p_\alpha(V)\ge\delta(2-\delta)$.

PROOF. Since $\sigma''(V)<0(V<0)$, and $\sigma'(V)>0$ (i.e., $\sigma'(V)$ is decreasing on I_0 and $\sigma(V)$ is increasing on $[v_+,v_-]$), we have $0<(s^2-\sigma'(V))/(s^2-\sigma'(0))\le 1$, $0<-(V-v_+)/(V+v_+)<1$, and $0\le(s^2-(\sigma(V)-\sigma(v_+))/(V-v_+))/(s^2-(\sigma(0)-\sigma(v_+))/(-v_+))\le 1$. Noting $w_0'(0)=v_+^2h'(0)/h(0)^2$ and (5.8), we obtain

$$
\begin{aligned}
p_\alpha(V) = 2 - \alpha\left(\frac{v_+ h'(0)}{h(0)}\right)^2 &\left\{\frac{s^2-\sigma'(V)}{s^2-\sigma'(0)}\right. \\
&\left. -\alpha\frac{V-v_+}{V+v_+}\left(s^2-\frac{\sigma(V)-\sigma(v_+)}{V-v_+}\right)^2 \Big/ \left(s^2-\frac{\sigma(0)-\sigma(v_+)}{-v_+}\right)^2\right\} \\
\geq 2-\alpha(1+\alpha)&\left(\frac{v_+ h'(0)}{h(0)}\right)^2 .
\end{aligned} \tag{5.9}
$$

By (4.3), we find

$$
\left|\frac{v_+ h'(0)}{h(0)}\right| = \frac{-v_+\bar{\sigma}'(v_+)}{\bar{\sigma}(v_+)-v_+\bar{\sigma}'(v_+)} = \frac{1}{1-\bar{\delta}} .
$$

Due to (5.9) and $\alpha=(1-\bar{\delta})^2$, we obtain $p_\alpha \geq \bar{\delta}(2-\bar{\delta})$. This completes the proof of Lemma 5.1.

LEMMA 5.2. *Consider the non-degenerate case. For any fixed* α $(0<\alpha<1)$, *if* $|v_+ - v_-|$ *is suitably small, then there exits a positive constant* C_α *depending on* α, *such that* $p_\alpha(V(\xi)) \geq C_\alpha$ *for any* $\xi \in R$.

PROOF. Let $H(v_+) \equiv -v_+ h'(0)/h(0) > 0$ be a function on v_+. Due to (1.9), (5.10) and $s^2 = (\sigma(v_+)-\sigma(v_-))/(v_+ - v_-)$, where, v_+ and v_- are independent, we know $\lim_{v_+\to 0} H(v_+)$ exists with the type of "0/0" as follows:

$$
\lim_{v_+\to 0} H(v_+) = \lim_{v_+\to 0} \frac{-v_+(\sigma'(0)-s^2)}{\sigma(0)-\sigma(v_+)+s^2 v_+} = 1 . \tag{5.11}
$$

For an arbitrary given constant $\varepsilon_1 > 0$, by (5.11), there exists a $\delta=\delta(\varepsilon_1)>0$ such that if $|v_+|<\delta$, then $0<H(v_+)<1+\varepsilon_1$ holds. Thus, by (5.9), we get

$$
p_\alpha(V(\xi)) \geq 2-\alpha(1+\alpha)(1+\varepsilon_1)^2 \equiv C_\alpha , \qquad \text{for } \xi \in R
$$

by choosing such positive constant ε_1 that $\varepsilon_1 < \sqrt{2/(\alpha(1+\alpha))} - 1$ for any $\alpha \in (0, 1)$.

By Lemma 5.1, substituting (5.7) into (5.6), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R \{(w\sigma')(V)\phi^2 + w(V)\psi^2\}\,d\xi + \frac{\mu}{2}\int_{R_0} w_0(V)\psi_\xi^2\,d\xi \\
&\quad + \mu\sum_{i=1}^{n+2}\int_{R_i} w_i(V)\psi_\xi^2\,d\xi - \frac{1-\alpha}{2s} w_0'(0)h(0)\psi(t,\xi_0)^2 \\
&+ \int_{R_0} -\frac{V_\xi}{2s}\bar{\delta}(2-\bar{\delta})\psi(t,\xi)^2\,d\xi + \sum_{i=1}^{n+2}\int_{R_i} A_i(t,\xi)\,d\xi \leq \int_R Fw(V)\psi\,d\xi .
\end{aligned} \tag{5.12}
$$

Step 2. Due to the continuity of $w(V)$, i.e., $w_i(v_i) = w_{i+1}(v_i)$, and $w_0'(0) = -w_1(0)h'(0)/h(0)$, $h'(v_*)=0$ (see (3.5)), we have

$$
\begin{aligned}
-\frac{1-\alpha}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2 &= \frac{1-\alpha}{2s}w_1(0)h'(0)\psi(t,\xi_0)^2 \\
&= \frac{1-\alpha}{2s}\sum_{i=1}^{n+1}\{w_i(v_{i-1})h'(v_{i-1})\psi(t,\xi_{i-1})^2 - w_i(v_i)h'(v_i)\psi(t,\xi_i)^2\} \\
&= \frac{1-\alpha}{2s}\sum_{i=1}^{n+1}\int_{R_i}\frac{\partial}{\partial\xi}(w_i(V)h'(V)\psi(t,\xi)^2)d\xi = \frac{1-\alpha}{2s}\sum_{i=1}^{n+1}\int_{R_i}B_i(t,\xi)d\xi\,, \qquad (5.13)
\end{aligned}
$$

where

$$
\begin{aligned}
B_i(t,\xi) = {}& [w_i'(V)h'(V) + w_i(V)h''(V)]V_\xi\psi^2 \\
& + 2w_i(V)h'(V)\psi\psi_\xi\,, \qquad i=1,\cdots,n+1\,. \qquad (5.14)
\end{aligned}
$$

Substituting (5.13) into (5.12), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R\{(w\sigma')(V)\phi^2 + w(V)\psi^2\}d\xi + \frac{\mu}{2}\int_{R_0}w_0(V)\psi_\xi^2 d\xi \\
&\quad + \mu\sum_{i=1}^{n+2}\int_{R_i}w_i(V)\psi_\xi^2 d\xi + \int_{R_0} -\frac{V_\xi}{2s}\delta(2-\delta)\psi(t,\xi)^2 d\xi \\
&+\sum_{i=1}^{n+1}\int_{R_i}\Big(A_i(t,\xi) + \frac{1-\alpha}{2s}B_i(t,\xi)\Big)d\xi + \int_{R_i}A_{n+2}(t,\xi)d\xi \le \int_R Fw(V)\psi d\xi\,. \qquad (5.15)
\end{aligned}
$$

Using Cauchy's inequality and $\mu s V_\xi = h(V)$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n+1}\int_{R_i}\Big(A_i(t,\xi) + \frac{1-\alpha}{2s}B_i(t,\xi)\Big)d\xi \\
&= \int_{R_i}\left\{\frac{1}{2s}(w_i\sigma')'(V)V_\xi(s\phi+\psi)^2 - \frac{1-\alpha}{2s}w_i(V)V_\xi\left[\frac{w_i'(V)}{w_i(V)}h'(V) + h''(V)\right]\psi^2\right. \\
&\qquad \left. + \mu w_i(V)\left[\frac{w_i'(V)}{w_i(V)}V_\xi + \frac{1-\alpha}{s\mu}h'(V)\right]\psi\psi_\xi\right\}d\xi \\
&\ge \int_{R_i} -\frac{V_\xi}{2s}\{\alpha w_i(V)z_i(V)\psi^2 - y_i(V)(s\phi+\psi)^2\}d\xi - \frac{\mu}{2}\int_{R_i}w_i(V)\psi_\xi^2 d\xi\,, \\
&\qquad\qquad i=1,\cdots,n+1\,, \qquad (5.16)
\end{aligned}
$$

where

$$
z_i(V) = h''(V) + h'(V)\frac{w_i'(V)}{w_i(V)} + \frac{h(V)}{\alpha}\left[\frac{w_i'(V)}{w_i(V)} + (1-\alpha)\frac{h'(V)}{h(V)}\right]^2\,, \qquad (5.17)
$$

$$
y_i(V) = (w_i\sigma')'(V)\,. \qquad (5.18)
$$

We can prove the following

LEMMA 5.3. *It holds*

$$z_i(V)\geq 0\,,\qquad y_i(V)\leq 0\qquad \textit{for } V\in I_i\,,\ i=1,\cdots,n+1\,, \tag{5.19}$$

provided the shock strength is suitably small.

PROOF. Since the weight functions $w_{2j}(V)$ on I_{2j} are different from $w_{2j-1}(V)$ on $I_{2j-1}, j=1,\cdots,(n+1)/2$, we have to divide the arguments into two parts to discuss (5.19) as follows.

Part 1. When $V\in I_{2j}$, i.e., $\xi\in R_{2j}$, $j=1,\cdots,(n+1)/2$, noting $w_{2j}=k_{2j}\sigma'(V)^{-1}$, $\sigma''(V)=h''(V)\geq 0$ and $G(V)\leq 0$, i.e., $0<h'(V)/h(V)\leq\sigma''(V)/\sigma'(V)$, we have $y_{2j}(V)\equiv 0$ and

$$\begin{aligned}z_{2j}(V)&=s^2\frac{\sigma''(V)}{\sigma'(V)}+\frac{h(V)}{\alpha}\left[-\frac{\sigma''(V)}{\sigma'(V)}+(1-\alpha)\frac{h'(V)}{h(V)}\right]^2\\&\geq s^2\frac{\sigma''(V)}{\sigma'(V)}+\frac{h(V)}{\alpha}\left(\frac{\sigma''(V)}{\sigma'(V)}\right)^2=s^2\frac{\sigma''(V)}{\sigma'(V)}(1-q_{2j,\alpha}(V))\geq 0\,,\end{aligned}$$

where

$$q_{2j,\alpha}(V)=-\frac{h(V)\sigma''(V)}{s^2\alpha\sigma'(V)}\geq 0\,,$$

and $\max_{V\in I_{2j}}q_{2j,\alpha}(V)<1$ if $|v_+-v_-|\ll 1$.

Part 2. When $V\in I_{2j-1}$, i.e., $\xi\in R_{2j-1}$, $j=1,\cdots,(n+1)/2$, since $w_{2j-1}=-k_{2j-1}/h(V)$, $\sigma''(V)=h''(V)\geq 0$, $h(V)<0$ and $G(V)\geq 0$, we have

$$\begin{aligned}y_{2j-1}(V)&=(w_{2j-1}\sigma')'(V)=w'_{2j-1}(V)\sigma'(V)+w_{2j-1}(V)\sigma''(V)\\&=k_{2j-1}\left[\frac{h'(V)\sigma'(V)}{h(V)^2}-\frac{\sigma''(V)}{h(V)}\right]=-k_{2j-1}G(V)/h(V)^2\leq 0\,,\end{aligned}$$

$$\begin{aligned}z_{2j-1}(V)&=h''(V)+h'(V)\frac{w'_{2j-1}(V)}{w_{2j-1}(V)}+\frac{h(V)}{\alpha}\left[\frac{w'_{2j-1}(V)}{w_{2j-1}(V)}+(1-\alpha)\frac{h'(V)}{h(V)}\right]^2\\&=h''(V)-(1-\alpha)\frac{h'(V)^2}{h(V)}\geq 0\,.\end{aligned}$$

Thus, we have proved Lemma 5.3.

By (5.15), (5.16) and (5.19), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_R((w\sigma')(V)\phi^2+w(V)\psi^2)d\xi+\frac{\mu}{2}\int_{R_0}w_0(V)\psi_\xi^2d\xi\\&+\frac{\mu}{2}\sum_{i=1}^{n+1}\int_{R_i}w_i(V)\psi_\xi^2d\xi+\mu\int_{R_{n+2}}w_{n+2}(V)\psi_\xi^2d\xi\\&+\int_{R_0}-\frac{V_\xi}{2s}\delta(2-\delta)\psi(t,\xi)^2d\xi+\int_{R_{n+2}}A_{n+2}(t,\xi)d\xi\end{aligned}$$

$$\leq \int_R Fw(V)\psi\, d\xi\,. \tag{5.20}$$

Step 3. Now we consider the last term in the left hand side of (5.20). When $V \in I_{n+2}$, i.e., $\xi \in R_{n+2} = (-\infty, \xi_*]$, due to Kawashima and Matsumura [4], we have

$$\begin{aligned}
\int_{R_{n+2}} A_{n+2}(t,\xi)\,d\xi &= \int_{R_{n+2}} -\frac{w_{n+2}(V)}{2s} V_\xi \left[\frac{w'_{n+2}(V)}{w_{n+2}(V)} h'(V) + h''(V)\right]\psi^2\, d\xi \\
&\quad + \int_{R_{n+2}} \frac{V_\xi}{2s}(w_{n+2}\sigma')'(V)(s\phi+\psi)^2\, d\xi + \mu \int_{R_{n+2}} w'_{n+2}(V) V_\xi \psi\psi_\xi\, d\xi \\
&\geq -\frac{\mu}{2}\int_{R_{n+2}} w_{n+2}(V)\psi_\xi^2\, d\xi + \int_{R_{n+2}} \frac{V_\xi}{2s} y_{n+2}(V)(s\phi+\psi)^2\, d\xi \\
&\quad - \int_{R_{n+2}} \frac{V_\xi}{2s} w_{n+2}(V) z_{n+2}(V)\psi^2\, d\xi\,,
\end{aligned} \tag{5.21}$$

where $y_{n+2}(V) = (w_{n+2}\sigma')(V)$ and

$$z_{n+2}(V) = h''(V) + h'(V)\frac{w'_{n+2}(V)}{w_{n+2}(V)} + 2h(V)\left[\frac{w'_{n+2}(V)}{w_{n+2}(V)}\right]^2.$$

As in [4], it is easily checked that

$$y_{n+2}(V) \equiv 0 \quad \text{and} \quad z_{n+2}(V) \geq 0 \tag{5.22}$$

provided $|v_+ - v_-| \ll 1$.

Substituting (5.21)–(5.22) into (5.20) and integrating the resultant inequality over $[0, t]$, we have the following first Key Lemma.

KEY LEMMA 5.4. *It holds that*

$$\begin{aligned}
&|(\phi,\psi)(t)|^2_{\langle\xi\rangle_+} + \int_0^t |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+}\, d\tau + \int_0^t\int_{R_0} |V_\xi|\psi(\tau,\xi)^2\, d\xi\, d\tau \\
&\qquad \leq C\left(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + N_1(t)\int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{1/2}}\, d\tau\right).
\end{aligned} \tag{5.23}$$

REMARK 5.5. In the non-degenerate case, noting Lemma 5.2 and $(5.3)_1$, we get

$$\begin{aligned}
&\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2\, d\tau + \int_0^t\int_{R_0} |V_\xi|\psi(\tau,\xi)^2\, d\xi\, d\tau \\
&\qquad \leq C\left(\|(\phi_0,\psi_0)\|^2 + N_0(t)\int_0^t \|\phi_\xi(\tau)\|^2\, d\tau\right).
\end{aligned}$$

Thus, we can prove part (A) of Proposition 4.6 corresponding to the procedure in [4, 10].

REMARK 5.6. $G(v)$ may have infinitely many v_i's defined in (3.10), so that there are cluster points. But, both endpoints 0 and v_* are not cluster points by (3.9). Denote a cluster point by $\bar{v}^i$ with $v_j^i \to \bar{v}^i$ as $j \to \infty$. Then, we have $\lim_{j\to\infty} k_j^i \equiv k^i < +\infty$ and also $\lim_{j\to\infty} k_{2j-1}^i / k_{2j}^i = -h(\bar{v}^i)/\sigma'(\bar{v}^i)$. Due to this, at each cluster point, by changing $\sum_{j=0}^{n+2}$ to $\sum_{ij}$, the procedures in steps 2–3 are still available.

The next Key Lemma is to estimate the last term in (5.23) for suitably small $N_1(t)$.

KEY LEMMA 5.7. *It holds that*

$$|\phi_\xi(t)|^2_{\langle\xi\rangle_+^{1/2}} + (1 - CN_1(t)) \int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{1/2}} d\tau \le C(\|(\phi_0, \psi_0)\|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{1/2}}) . \quad (5.24)$$

PROOF. From equations (4.10), we have

$$\mu\phi_{\xi t} - s\mu\phi_{\xi\xi} + \sigma'(V)\phi_\xi + s\psi_\xi - \psi_t = -F . \quad (5.25)$$

Since $L^2_{w(V)} = L^2_{r(\xi)} = L^2_{\langle\xi\rangle_+}$, consider our problem in the weighted space $L^2_{r(\xi)}$ at first.

(i) On the interval $[\xi_0, +\infty) = R_0$, i.e., $v \in (v_+, 0]$, multiplying (5.25) by $r(\xi)^{1/2}\phi_\xi$ (here $r(\xi) = 1 + \xi - \xi_0$, see (5.2)), we get

$$\begin{aligned} &\frac{\mu}{2}\{r(\xi)^{1/2}\phi_\xi^2\}_t - \frac{s\mu}{2}\{r(\xi)^{1/2}\phi_\xi^2\}_\xi + \frac{s\mu}{4} r(\xi)^{-1/2}\phi_\xi^2 \\ &+ r(\xi)^{1/2}\sigma'(V)\phi_\xi^2 + sr(\xi)^{1/2}\phi_\xi\psi_\xi - r(\xi)^{1/2}\psi_t\phi_\xi = -Fr(\xi)^{1/2}\phi_\xi . \end{aligned} \quad (5.26)$$

By the first equation in (4.10), we note that

$$\begin{aligned} -r(\xi)^{1/2}\psi_t\phi_\xi &= -\{r(\xi)^{1/2}\psi\phi_\xi\}_t + r(\xi)^{1/2}\psi\phi_{\xi t} = -\{r(\xi)^{1/2}\psi\phi_\xi\}_t + r(\xi)^{1/2}\psi(s\phi_\xi + \psi_\xi)_\xi \\ &= -\{r(\xi)^{1/2}\psi\phi_\xi\}_t + \{r(\xi)^{1/2}\psi(s\phi_\xi + \psi_\xi)\}_\xi - sr(\xi)^{1/2}\psi_\xi\phi_\xi - r(\xi)^{1/2}\psi_\xi^2 \\ &\quad - \frac{s}{2} r(\xi)^{-1/2}\psi\phi_\xi - \left\{\frac{1}{4} r(\xi)^{-1/2}\psi^2\right\}_\xi - \frac{1}{8} r(\xi)^{-3/2}\psi^2 , \end{aligned} \quad (5.27)$$

and

$$\left| -\frac{s}{2} r(\xi)^{-1/2}\psi\phi_\xi \right| \le \varepsilon_2 r(\xi)^{1/2}\sigma'(V)\phi_\xi^2 + (16\sigma'(0)\varepsilon_2)^{-1} s^2 r(\xi)^{-3/2}\psi^2 \quad (5.28)$$

where $0 < \varepsilon_2 < 1$.

Substituting (5.28), (5.27) into (5.26), and integrating it over R_0, we have

$$\frac{\mu}{2}\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi^2 d\xi + \{\cdots\}|_{\xi=\xi_0} - \frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi\psi\, d\xi + \frac{s\mu}{4}\int_{\xi_0}^{+\infty} r(\xi)^{-1/2}\phi_\xi^2 d\xi$$

$$+(1-\varepsilon_2)\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\sigma'(V)\phi_\xi^2 d\xi-\left(\frac{s^2}{16\sigma'(0)\varepsilon_2}+\frac{1}{8}\right)\int_{\xi_0}^{+\infty} r(\xi)^{-3/2}\psi^2 d\xi$$

$$-\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\psi_\xi^2 d\xi+\frac{1}{4}r(\xi_0)^{-1/2}\psi(t,\xi_0)^2\le-\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi F d\xi\,, \tag{5.29}$$

where $\{\cdots\}=-(s\mu/2)r(\xi)^{1/2}\phi_\xi^2+r(\xi)^{1/2}\psi(s\phi_\xi+\psi_\xi)$. Moreover, by Cauchy's inequality, we find the fact

$$\begin{aligned}\frac{1}{4}r(\xi_0)^{-1/2}\psi(t,\xi_0)^2&=-\frac{1}{4}r(\xi_0)^{-1/4}\int_{\xi_0}^{+\infty}\frac{\partial}{\partial\xi}(r(\xi)^{-1/4}\psi(t,\xi)^2)d\xi\\&=-\frac{1}{4}r(\xi_0)^{-1/4}\int_{\xi_0}^{+\infty}\left\{-\frac{1}{4}r(\xi)^{-5/4}\psi^2+r(\xi)^{-1/4}2\psi\psi_\xi\right\}d\xi\\&\ge\frac{1}{4}\int_{\xi_0}^{+\infty}\left\{\frac{1}{4}r(\xi)^{-5/4}\psi^2-r(\xi)^{-3/2}\psi^2-r(\xi)\psi_\xi^2\right\}d\xi\,,\end{aligned}\tag{5.30}$$

where $r(\xi_0)=1$.

Substituting (5.30) into (5.29), we have

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi^2 d\xi+\{\cdots\}|_{\xi=\xi_0}-\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi\psi\, d\xi\\&+\frac{s\mu}{4}\int_{\xi_0}^{+\infty} r(\xi)^{-1/2}\phi_\xi^2+(1-\varepsilon_2)\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\sigma'(V)\phi_\xi^2 d\xi+\int_{\xi_0}^{+\infty} C_{\varepsilon_2}(\xi)\psi^2 d\xi\\&\le CN_1(t)\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi^2 d\xi+C\int_{\xi_0}^{+\infty} r(\xi)\psi_\xi^2 d\xi\,,\end{aligned}\tag{5.31}$$

where

$$C_{\varepsilon_2}(\xi)=\frac{r(\xi)^{-3/2}}{16}\left\{r(\xi)^{1/4}-\frac{s^2}{\varepsilon_4\sigma'(0)}-6\right\}. \tag{5.32}$$

Since $r(\xi)=O(|\xi|)$ as $\xi\to\infty$, we know there exists a larger number ξ_{**} ($>\xi_0$) such that

$$C_{\varepsilon_2}(\xi)\ge 0 \quad \text{on } [\xi_{**},+\infty)\,,\qquad |C_{\varepsilon_2}(\xi)|\le C \quad \text{on } [\xi_0,\xi_{**}]\,. \tag{5.33}$$

Due to (5.23) in Key Lemma 5.4 and the boundedness of $|V_\xi|$ on $[\xi_0,\xi_{**}]$, noting (5.33) and $w(V)\sim\langle\xi\rangle_+\sim r(\xi)$, we obtain

$$\int_0^t\int_{\xi_0}^{\xi_{**}}|C_{\varepsilon_2}(\xi)|\psi(\tau,\xi)^2 d\xi d\tau\le C\left(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+}+N_1(t)\int_0^t|\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{1/2}}d\tau\right), \tag{5.34}$$

$$\int_{\xi_{**}}^{+\infty} C_{\varepsilon_2}(\xi)\psi(\tau,\xi)^2 d\xi\ge 0\,. \tag{5.35}$$

Then we can rewrite (5.31) as

$$\frac{\mu}{2}\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi^2\,d\xi+\{\cdots\}|_{\xi=\xi_0}-\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi\psi\,d\xi$$
$$+(1-\varepsilon_2)\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\sigma'(V)\phi_\xi^2\,d\xi+\int_{\xi_{**}}^{+\infty} C_{\varepsilon_2}(\xi)\psi^2\,d\xi$$
$$\leq\int_{\xi_0}^{\xi_{**}} |C_{\varepsilon_2}(\xi)|\psi^2\,d\xi+CN_1(t)\int_{\xi_0}^{+\infty} r(\xi)^{1/2}\phi_\xi^2\,d\xi+C\int_{\xi_0}^{+\infty} r(\xi)\psi_\xi^2\,d\xi\,. \tag{5.36}$$

(ii) On the another interval $(-\infty, \xi_0]$, i.e., $v\in[0, v_-]$, multiplying (5.25) by ϕ_ξ, and integrating it over $(-\infty, \xi_0]$ (here $r(\xi)=1$), we have

$$\frac{\mu}{2}\frac{d}{dt}\int_{-\infty}^{\xi_0}\phi_\xi^2\,d\xi-\{\cdots\}|_{\xi=\xi_0}-\frac{d}{dt}\int_{-\infty}^{\xi_0}\phi_\xi\psi\,d\xi+\int_{-\infty}^{\xi_0}\sigma'(V)\phi_\xi^2\,d\xi$$
$$\leq CN_1(t)\int_{-\infty}^{\xi_0}\phi_\xi^2\,d\xi+C\int_{-\infty}^{\xi_0}\psi_\xi^2\,d\xi\,. \tag{5.37}$$

The continuity of $r(\xi)$ at ξ_0 admits the addition of (5.36) and (5.37). Noting

$$\int_{-\infty}^{+\infty} r(\xi)^{1/2}|\phi_\xi\psi|\,d\xi\leq\frac{\mu}{4}|\phi_\xi|^2_{r\langle\xi\rangle^{1/2}}+\frac{1}{\mu}|\psi|^2_{r(\xi)}\,,$$

and (5.34)–(5.35), Key Lemma 5.4, and $\langle\xi\rangle_+\sim r(\xi)$, we obtain (5.24).

By Key Lemma 5.4 and Key Lemma 5.7, we have the following

LEMMA 5.8. *It holds that*

$$|(\phi,\psi)(t)|^2_{\langle\xi\rangle_+}+\int_0^t|\psi_\xi(\tau)|^2_{\langle\xi\rangle_+}d\tau\leq C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+}+|\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{1/2}})\,, \tag{5.39}$$

$$|\phi_\xi(t)|^2_{\langle\xi\rangle_+^{1/2}}+\int_0^t|\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{1/2}}d\tau\leq C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+}+|\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{1/2}})\,, \tag{5.40}$$

provided $N_1(T)$ is suitably small.

Next, we shall derive the higher order estimates on the solution (ϕ, ψ). Let us differentiate equations (4.10) in ξ, and multiply the first equation by $w(V)\sigma(V)\phi_\xi$ and the second one by $w(V)\psi_\xi$, respectively. We add them and integrate the resultant equation over $[0, t]\times R$, similar to the procedures in Lemma 5.1–Lemma 5.8. Then, by the fact $w(V)\sim\langle\xi\rangle_+\sim r(\xi)$ and $L^2_{w(V)}=L^2_{\langle\xi\rangle_+}=L^2_{r(\xi)}$, we have

LEMMA 5.9. *It holds that*

$$|(\phi_\xi,\psi_\xi)(t)|^2_{w(V)}+\int_0^t|\psi_{\xi\xi}(\tau)|^2_{w(V)}d\tau\leq C(|(\phi_0,\psi_0)|^2_{1,w(V)}+|\phi_{0,\xi\xi}|^2_{w(V)^{1/2}})\,, \tag{5.41}$$

$$|\phi_{\xi\xi}(t)|^2_{w(V)^{1/2}}+\int_0^t |\phi_{\xi\xi}(\tau)|^2_{w(V)^{1/2}}d\tau \le C(|(\phi_0, \psi_0)|^2_{1,w(V)}+|\phi_{0,\xi\xi}|^2_{w(V)^{1/2}}) \tag{5.42}$$

for suitably small $N_1(T)$.

Similarly, the second order estimate of the solutions can be proved as follows.

LEMMA 5.10. *It holds that*

$$|(\phi_{\xi\xi}, \psi_{\xi\xi})(t)|^2_{\langle\xi\rangle_+}+\int_0^t |\psi_{\xi\xi\xi}(\tau)|^2_{\langle\xi\rangle_+}d\tau \le C|(\phi_0, \psi_0)|^2_{2,\langle\xi\rangle_+} \tag{5.43}$$

for suitably small $N_1(T)$.

THE PROOF OF PROPOSITION 4.6 B(i). Combining Lemmas 5.8–5.10, we have

$$|(\phi, \psi)(t)|^2_{2,\langle\xi\rangle_+}+\int_0^t \{|\phi_\xi(\tau)|^2_{1,\langle\xi\rangle_+^{1/2}}+|\psi_\xi(\tau)|^2_{2,\langle\xi\rangle_+}\}d\tau \le C|(\phi_0, \psi_0)|^2_{2,\langle\xi\rangle_+}$$

for suitably small $N_1(T)$.

Thus, we have completed the proof of (i) of part (B) in Proposition 4.6.

5.2. The proof of Proposition 4.6 B(ii). Let $(\phi, \psi)\in X_2(0, T)$ be a solution of (4.10). By the same procedures as in the last sub-section, we establish the key estimates corresponding to Lemmas 5.4–5.7 by the weight functions $w(V)$ and $r(\xi)$. Noting $w(V)\sim \langle\xi\rangle_+\sim r(\xi)$, and $L^2_{w(V)}=L^2_{\langle\xi\rangle_+}=L^2_{r(\xi)}$, we obtain the following lemma.

LEMMA 5.11. *It holds that*

$$|(\phi, \psi)(t)|^2_{\langle\xi\rangle_+}+\int_0^t |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+}d\tau \le C(|(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+}+|\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{3/4}}), \tag{5.44}$$

$$|\phi_\xi(t)|^2_{\langle\xi\rangle_+^{3/4}}+\int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}}d\tau \le C(|(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+}+|\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{3/4}}). \tag{5.45}$$

Next, we shall derive the higher order estimates on the solution (ϕ, ψ) without weight function. This procedure is simpler than the previous one. According to Lemma 5.11, we can prove the following Lemmas by the same way as in [4, 10]. So, we only give the sketch of the proofs.

Multiplying the second equation of (4.10) by $-\psi_{\xi\xi}$, and integrating it over $[0, t]\times R$, we have by Lemma 5.11

LEMMA 5.12. *It holds that*

$$\|\psi_\xi(t)\|^2+\int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau \le C(\|(\phi_{0,\xi}, \psi_{0,\xi})\|^2+|(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+}+|\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{3/4}}) \tag{5.46}$$

for suitably small $N_2(T)$.

When we differentiate (5.25) in ξ and multiply it by $\phi_{\xi\xi}$ and integrate the resultant equation over $[0, t] \times R$, we get

LEMMA 5.13. *It holds that*

$$\|\phi_{\xi\xi}(t)\|^2 + \int_0^t \|\phi_{\xi\xi}(\tau)\|^2 d\tau$$
$$\leq C(\|\phi_{0,\xi}\|_1^2 + \|\psi_{0,\xi}\|^2 + |(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{3/4}}) \tag{5.47}$$

for suitably small $N_2(T)$.

Differentiating the second equation of (4.10) in ξ and multiplying it by $-\psi_{\xi\xi\xi}$, and integrating the resultant equation over $[0, t] \times R$, we obtain

LEMMA 5.14. *It holds that*

$$\|\psi_{\xi\xi}(t)\|^2 + \int_0^t \|\phi_{\xi\xi\xi}(\tau)\|^2 d\tau$$
$$\leq C(\|\phi_{0,\xi}, \psi_{0,\xi}\|_1^2 + |(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{1/2}}) \tag{5.48}$$

for suitably small $N_2(T)$.

Finally, combining Lemmas 5.11–5.14, we complete the proof of Proposition 4.6 B(ii).

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