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A Generalized Frame Bundle for Certain Fréchet Vector Bundles and Linear Connections

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Abstract. Let $(E_i)_{i\in\mathbf{N}}$ be a projective system of Banach vector bundles whose limit is a Fréchet bundle of fibre type $\mathbf{F}$. We construct a generalized bundle of frames $\mathbf{P}(E)$ of E by revising entirely the classical notion and by substituting $GL(\mathbf{F})$ with an appropriate enlarged structure group. This is imposed by the pathology of $GL(\mathbf{F})$, which renders meaningless the ordinary frame bundle. As a result, we prove that E is associated with $\mathbf{P}(E)$ and linear connections of E correspond to (principal) connections of $\mathbf{P}(E)$. In particular, the former are necessarily projective limits of connections on the bundles E_i.

0. Introduction.

The study of many geometrical entities of a vector bundle, such as connections, is reduced to the study of their counterparts on the corresponding principal bundle of frames. The idea works well for finite-dimensional and Banach bundles. However it fails if we move one step further and consider vector bundles of fibre type a Fréchet space $\mathbf{F}$, due to the topological pathology of $GL(\mathbf{F})$. Therefore, it seems to be meaningless to think of a Fréchet vector bundle as associated to its bundle of frames $P(E)$ (after all, how could the latter be defined as a principal bundle ?) and to reduce linear connections on E to connections on $P(E)$.

The aim of the present paper is to overcome the previous impasse by a radical change of the classical notion of the bundle of frames, for the category of (Fréchet) vector bundles obtained as the limit of a projective system of Banach vector bundles. Such bundles occur quite naturally in many instances (e.g. projective limits of tangent bundles of manifolds and Lie groups [8], [9]; projective limits of jet bundles [11]). Outside this category, that is for arbitrary Fréchet vector bundles, the problem of defining an appropriate frame bundle remains open.

To be a little more specific, we start with a projective system $(E_i)_{i\in\mathbf{N}}$ of Banach vector bundles, with respective fibres $\mathbf{F}_i$ also forming a projective system $\{\mathbf{F}_i, \rho_{ji}\}$. Under reasonable conditions, the projective limit $E=\varprojlim E_i$ is a vector bundle of fibre type the Fréchet space $\mathbf{F}=\varprojlim \mathbf{F}_i$. Then we replace the ordinary bundle of frames $P(E)$

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with a new principal bundle $\mathbf{P}(E)$ whose structure group is

$$H^{\circ}(\mathbf{F})=\{(f_i)_{i\in\mathbf{N}} \mid f_i\in \mathcal{L}is(\mathbf{F}_i): \rho_{ji}\circ f_j=f_i\circ \rho_{ji}, j\geq i\}\ .$$

We call $\mathbf{P}(E)$ the generalized frame bundle of E and we show that $\mathbf{P}(E)$ is the limit of a projective system of Banach principal bundles (Proposition 2.2), each one being, so to speak, an enlargement of an ordinary frame bundle. As a consequence, the first main result of the paper (Theorem 2.5) shows that, analogously to the classical case, E is associated with $\mathbf{P}(E)$.

The above association allows us to establish a correspondence between linear connections of E and principal connections of $\mathbf{P}(E)$. The crucial step here is the construction of a (generalized) connection on $\mathbf{P}(E)$, obtained from a particular system of connections on the Banach principal bundles producing $\mathbf{P}(E)$ (Theorem 3.2). As a byproduct of the previous situation we obtain the following characterization concluding the paper: a linear connection ∇ on E corresponds to a unique principal connection on $\mathbf{P}(E)$ if and only if $\nabla=\varprojlim \nabla_i$, where ∇_i is a linear connection on E_i.

1. Preliminaries.

Some particular cases of projective systems of vector bundles and their limits have been considered, among other authors, by [8], [9], [11] and [13]. However, they do not study in depth the vector bundle structure of the limit, since they rather focus to various algebraic and/or topological properties.

Here we are mainly interested in the mere vector bundle structure of the limit of a projective system (over $\mathbf{N}$) of Banach vector bundles and its geometry viz. connections.

Since the previous projective limit is not always endowed with a vector bundle structure, in the sense of [1], [5], G. Galanis ([3]) proposed the following modified version of projective systems which are of interest to us.

1.1. Definition. Let $\{(E_i, B, \pi_i); f_{ji}\}_{i,j\in\mathbf{N}}$ be a projective system of Banach vector bundles, over the same base B, with corresponding fibres of type $\mathbf{F}_i$. The system is said to be *strong* if the following conditions are satisfied:

(i) $\mathbf{F}_i$ $(i\in\mathbf{N})$ form a projective system with corresponding connecting morphisms ρ_{ji} $(j\geq i)$.

(ii) For any $b\in B$, there exist local trivializations (U, τ_i) of E_i respectively, such that the following diagram is commutative:

$$\begin{array}{ccc} \pi_j^{-1}(U) & \xrightarrow{\tau_j} & U\times\mathbf{F}_j \\ {\scriptstyle f_{ji}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_U\times\rho_{ji}} \\ \pi_i^{-1}(U) & \xrightarrow{\tau_i} & U\times\mathbf{F}_i \end{array} \qquad (j\geq i)$$

Under the above conditions it has been proved (cf. [3]) that the limit $E := \varprojlim E_i$ is a locally trivial fibre bundle over B, whose fibres are of type $\mathbf{F} := \varprojlim \mathbf{F}_i$.

As discussed in the Introduction, E cannot be considered as a vector bundle associated with its frame bundle (in the sense of [1]). In order to obtain a generalized frame bundle from which we fully recover E, we replace the structure group $GL(\mathbf{F})$ with an appropriate "enlarged" group $H^\circ(\mathbf{F})$, explained below.

More explicitly, we start with the following general situation (needed also in Section 3). Let $\mathbf{E}$, $\mathbf{F}$ be two fixed Fréchet spaces, obtained as the limits of the corresponding $\mathbf{N}$-projective systems $\{\mathbf{E}_i; \sigma_{ji}\}$, $\{\mathbf{F}_i; \rho_{ji}\}$. We denote by

$$H_i(\mathbf{E},\mathbf{F}) := \{(f_1, \cdots, f_i) \mid f_k \in \mathscr{L}(\mathbf{E}_k, \mathbf{F}_k) : \rho_{jk} \circ f_j = f_k \circ \sigma_{jk},\ i \geq j \geq k\},$$

$$H(\mathbf{E},\mathbf{F}) := \{(f_i)_{i\in\mathbf{N}} \mid f_i \in \mathscr{L}(\mathbf{E}_i, \mathbf{F}_i) : \rho_{ji} \circ f_j = f_i \circ \sigma_{ji},\ j \geq i\}.$$

It follows that $H_i(\mathbf{E},\mathbf{F})$ (resp. $H(\mathbf{E},\mathbf{F})$) is a Banach (resp. Fréchet) space as a closed subspace of $\prod_{j=1}^{i} \mathscr{L}(\mathbf{E}_j, \mathbf{F}_j)$ (resp. $\prod_{j=1}^{\infty} \mathscr{L}(\mathbf{E}_j, \mathbf{F}_j)$). Moreover, $\{H_i(\mathbf{E},\mathbf{F}); h_{ji}\}_{i,j\in\mathbf{N}}$ is a projective system, where

$$h_{ji} : H_j(\mathbf{E},\mathbf{F}) \to H_i(\mathbf{E},\mathbf{F}) : (f_1, \cdots, f_j) \mapsto (f_1, \cdots, f_i), \qquad j \geq i.$$

1.2. Proposition ([3]). *$H(\mathbf{E},\mathbf{F}) = \varprojlim H_i(\mathbf{E},\mathbf{F})$, within the isomorphism*

$$(f_1, f_2, \cdots) \overset{\cong}{\longmapsto} ((f_1), (f_1, f_2), \cdots).$$

Restricting now to the case $\mathbf{E} = \mathbf{F}$, we obtain the groups

$$H_i^\circ(\mathbf{F}) := H_i(\mathbf{F},\mathbf{F}) \cap \prod_{j=1}^{i} \mathscr{L}is(\mathbf{F}_j),$$

$$H^\circ(\mathbf{F}) := H(\mathbf{F},\mathbf{F}) \cap \prod_{j=1}^{\infty} \mathscr{L}is(\mathbf{F}_j),$$

where $\mathscr{L}is(\mathbf{F}_j)$ is the group of all invertible elements of $\mathscr{L}(\mathbf{F}_j)$. As a result we have

1.3. Corollary. (i) *Each $H_i^\circ(\mathbf{F})$, $i \in \mathbf{N}$, is a Banach-Lie group modelled on $H_i(\mathbf{F}) := H_i(\mathbf{F},\mathbf{F})$, and $H^\circ(\mathbf{F})$ is a topological group with the relative topology of $H(\mathbf{F}) := H(\mathbf{F},\mathbf{F})$.*

(ii) *The limit $\varprojlim H_i^\circ(\mathbf{F})$ exists and $H^\circ(\mathbf{F}) \equiv \varprojlim H_i^\circ(\mathbf{F})$.*

Under the previous notations the following holds.

1.4. Theorem ([3]). *Let $\{E_i; f_{ji}\}_{i,j\in\mathbf{N}}$ be a strong projective system of Banach vector bundles, as in Definition 1.1. Then $E := \varprojlim E_i$ is a Fréchet vector bundle.*

In particular, Theorem 1.4 implies that the structure of E is fully determined by a generalized cocycle of the form

$$T_{UV}^{*} : U \cap V \to H^\circ(\mathbf{F})$$

(U, V are in the open cover of the basis defined by Condition (ii) of Definition 1.1), which also determines the ordinary transition functions

$$T_{UV} : U \cap V \to GL(\mathbf{F}) \subseteq \mathscr{L}(\mathbf{F})$$

by $T_{UV} = \varepsilon \circ T^*_{UV}$, where

$$\varepsilon : H^\circ(\mathbf{F}) \to GL(\mathbf{F}) : \quad (f_i) \mapsto \varprojlim f_i \,.$$

Each T^*_{UV} is thought of as a smooth map since it can be considered as taking values in the Fréchet space $H(\mathbf{F}) \supseteq H^\circ(\mathbf{F})$. For later use, we also note that, in virtue of Definition 1.1, the local trivializations of $\mathbf{E}$ have the form $(U, \varprojlim \tau_i)$.

2. A generalized type of frame bundle.

As alluded to in the Introduction, the ordinary definition of the frame bundle of the Fréchet vector bundle $E = \varprojlim E_i$ has no meaning at all. Therefore, beside the replacement of the structure group $GL(\mathbf{F})$ by $H^\circ(\mathbf{F})$ discussed in the preceding section, we need to revise also the very notion of the frame bundle. To this end we proceed as follows:

We fix a strong projective system $\{E_i, f_{ij}\}_{i,j \in \mathbf{N}}$ as in Definition 1.1. For each Banach vector bundle E_i we define the space

$$\mathbf{P}(E_i) := \bigcup_{b \in B} H^\circ_i(\mathbf{F}, E_b)$$

if E_b denotes the fibre of E over $b \in B$. Here we use the bold character $\mathbf{P}$ in order to distinguish $\mathbf{P}(E_i)$ from the ordinary bundle of frames $P(E_i)$ in N. Bourbaki's [1] notation.

2.1. LEMMA. *$\mathbf{P}(E_i)$ is a principal fibre bundle over B, with structure group $H^\circ_i(\mathbf{F})$ and projection $\mathbf{p}_i : \mathbf{P}(E_i) \to B$, where*

$$\mathbf{p}_i(g_1, \cdots, g_i) := b \,; \qquad (g_1, \cdots, g_i) \in H^\circ_i(\mathbf{F}, E_b) \,.$$

PROOF. First we determine a smooth structure on $\mathbf{P}(E_i)$: for any $u = (g_1, \cdots, g_i) \in \mathbf{P}(E_i)$ with $\mathbf{p}_i(u) = b$, we choose the local trivialization $(U, \varprojlim \tau_i)$ of E (cf. Definition 1.1) with $b \in U$ and define the bijection $\Phi_i : \mathbf{p}_i^{-1}(U) \to U \times H^\circ_i(\mathbf{F})$ given by

$$\Phi_i(u) := (b; \tau_{1b} \circ g_1, \cdots, \tau_{ib} \circ g_i) \,; \qquad \tau_{kb} := \tau_k \big| \pi_k^{-1}(b) \,. \tag{2.1}$$

Now considering another bijection Ψ_i with respect to $(V, \varprojlim \sigma_i)$, $U \cap V \neq \varnothing$, we check that $\Psi_i \circ \Phi_i^{-1}$ is a diffeomorphism. Thus, by the gluing lemma (cf. e.g. [1; N° 5.2.4]), $\mathbf{P}(E_i)$ is indeed a Banach manifold. This structure turns the quadruple $(\mathbf{P}(E_i), H^\circ_i(\mathbf{F}), B, \mathbf{p}_i)$ into a Banach principal fibre bundle with $H^\circ_i(\mathbf{F})$ acting on $\mathbf{P}(E_i)$ in the obvious way. □

Inducing the connecting morphisms

$$r_{ji} : \mathbf{P}(E_j) \to \mathbf{P}(E_i) : (g_1, \cdots, g_i, \cdots, g_j) \mapsto (g_1, \cdots, g_i) \tag{2.2}$$

$$h_{ji} \equiv h_{ji}|_{H_j^\circ(\mathbf{F})} : H_j^\circ(\mathbf{F}) \to H_i^\circ(\mathbf{F}) , \tag{2.3}$$

for any $j \geq i$, we obtain

2.2. PROPOSITION. *The following conditions are true*:

(i) $\{(\mathbf{P}(E_i), H_i^\circ(\mathbf{F}), B, \mathbf{p}_i); (r_{ji}, h_{ji}, id_B)\}_{i,j\in\mathbf{N}}$ *is a projective system of Banach principal fibre bundles.*

(ii) $\mathbf{P}(E) := \varprojlim \mathbf{P}(E_i)$ *is a locally trivial topological principal fibre bundle with structure group* $H^\circ(\mathbf{F})$.

PROOF. The first condition is immediate, therefore $\mathbf{P}(E)$ exists. Now taking any $b \in B$ and considering the family $\{\Phi_i; i \in \mathbf{N}\}$, we check that the diagram

$$\begin{array}{ccc} \mathbf{p}_j^{-1}(U) & \xrightarrow{\Phi_j} & U \times H_j^\circ(\mathbf{F}) \\ {\scriptstyle r_{ij}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_U \times h_{ji}} \\ \mathbf{p}_i^{-1}(U) & \xrightarrow{\Phi_i} & U \times H_i^\circ(\mathbf{F}) \end{array}$$

is commutative. As a result, the morphism

$$\Phi := \varprojlim \Phi_i : \varprojlim \mathbf{p}_i^{-1}(U) \to U \times H^\circ(\mathbf{F}) \tag{2.4}$$

exists and determines a topological trivialization of $\mathbf{P}(E)$ over U. □

2.3. REMARKS. 1) The elements of $\mathbf{P}(E)$ are of the form $(g_i)_{i\in\mathbf{N}}$, where $g_i \in \mathbf{P}(E_i)$, since $\varprojlim g_i$ exists.

2) The homomorphism Φ defined by (2.4) is not smooth in the ordinary sense, since $H^\circ(\mathbf{F})$ is not a Lie group. However, following the customary procedure, Φ is called a (*generalized*) *diffeomorphism*, as being a projective limit of diffeomorphisms. Besides, if Φ is thought of as taking values in (the Fréchet manifold) $H(\mathbf{F})$, then we can show that it is smooth in the sense of J. Leslie ([6], [7]).

3) With the previous terminology, $\mathbf{P}(E)$ is a generalized smooth principal Fréchet bundle.

2.4. DEFINITION. Under the considerations of Remark 2.3 (3) above, $\mathbf{P}(E)$ is said to be the *generalized frame bundle* of E.

The significance of $\mathbf{P}(E)$ lies in the fact that E is associated with $\mathbf{P}(E)$, as it is shown in the next main result. Before the statement, we introduce a natural action of $H^\circ(\mathbf{F})$ on (the right of) $\mathbf{P}(E) \times \mathbf{F}$, given by

$$((g_i), (u_i)) \cdot (f_i) := ((g_i \circ f_i), (f_i^{-1}(u_i))) .$$

Note that the family $(g_i \circ f_i)$ belongs to $\mathbf{P}(E)$, since $\varprojlim g_i$ and $\varprojlim f_i$ already exist.

2.5. THEOREM. *$\bar{E} := \mathbf{P}(E) \times \mathbf{F}/H^\circ(\mathbf{F})$ is a Fréchet vector bundle, isomorphic to E.*

PROOF. We define the projection

$$\bar{\pi} : \bar{E} \to B : [(g_i), (u_i)] \mapsto \mathbf{p}((g_i))$$

and we consider the trivializations $(U, \varprojlim \tau_i)$ as well as the corresponding pairs (U, Φ) of $\mathbf{P}(E)$ (in this respect cf. also (2.4)). Then we see that the mappings

$$\bar{\Phi} : \bar{\pi}^{-1}(U) \to U \times \mathbf{F} : [(g_i), (u_i)] \mapsto (\mathbf{p}((g_i)), \Phi_2((g_i))((u_i))) ,$$

where $\Phi_2 := \mathrm{pr}_2 \circ \Phi$ and pr_2 is the projection to the second factor, determine a differential structure on $\bar{E}$. This is again a consequence of the gluing lemma (cf. also the proof of Lemma 2.1).

Finally, we check that the mapping $h: \bar{E} \to E$, given by

$$h([(g_i), (u_i)]) := (g_i(u_i)) ,$$

is a bijection identifying each Φ with $\varprojlim \tau_i$. This concludes the proof. □

2.6. REMARK. Thinking of Φ as a (generalized) smooth morphism in the sense of Remark 2.3(2), h is also smooth and (h, id_B) may be considered as an isomorphism of Fréchet vector bundles.

3. Linear connections.

In this section we fix again a strong projective system of vector bundles, each bundle E_i of which is endowed with a linear connection ∇_i: $TE_i \to E_i$ (in the sense of [14] and [2]). We further assume that, for any $j \geq i$,

$$f_{ji} \circ \nabla_j = \nabla_i \circ Tf_{ji} , \tag{3.1}$$

that is ∇_j and ∇_i are *f_{ji}-conjugate* (cf. also [12]). If we denote by Γ^i_U: $\phi(U) \to \mathscr{L}(\mathbf{F}_i, \mathscr{L}(\mathbf{B}, \mathbf{F}_i))$ the Christoffel symbols of ∇_i, with respect to the open cover of Definition 1.1 (here ϕ is the coordinate map of U and $\mathbf{B}$ the ambient space of the chart), then (3.1) implies that

$$\bar{\rho}_{ji} \circ \Gamma^j_U(x) = \Gamma^i_U(x) \circ \rho_{ji} \tag{3.2}$$

where $\bar{\rho}_{ji}(f) := \rho_{ji} \circ f$, $f \in \mathscr{L}(\mathbf{B}, \mathbf{F}_j)$ (for details we also refer to the general case of [12; Prop. 3.7]).

3.1. PROPOSITION ([3]). *$\nabla := \varprojlim \nabla_i$ is a linear connection on E with Christoffel symbols Γ_U given by*

$$\Gamma_U(x) = \varprojlim \Gamma^i_U(x) ; \qquad x \in U .$$

Motivated by the classical description of linear connections as connection forms on the corresponding bundle of frames, we construct a family of principal connections θ_i on the bundles $\mathbf{P}(E_i)$ (cf. Lemma 2.1) from which we obtain a generalized principal connection θ on $\mathbf{P}(E)$, which ultimately is related with ∇.

First we construct the family θ_i using *local* connection forms. This is convenient in the context of projective limits since all the bundles E_i have the same basis.

We fix again an open cover $\mathscr{C}=(U_\alpha)_{\alpha\in I}$, as in Definition 1.1, (the need for indices will be apparent in the use of local connection forms in the next result). Recalling the notations (2.2) and (2.3), we are in a position to prove the following main

3.2. THEOREM. *Each linear connection ∇_i gives rise to a principal connection θ_i on $\mathbf{P}(E_i)$. Moreover, for $j\geq i$, θ_j and θ_i are $(r_{ji}, h_{ji}, \mathrm{id}_{\mathbf{B}})$-conjugate, i.e.*

$$r_{ji}^*\theta_i=\bar{h}_{ji}\cdot\theta_j\,, \tag{3.3}$$

where $\bar{h}_{ji}$ is the Lie algebra homomorphism induced by h_{ji}.

PROOF. Each linear connection ∇_i determines a connection form

$$\omega_i\in\bigwedge{}^1(P(E_i),\mathscr{G}\ell(\mathbf{F}_i))\,,\qquad \mathscr{G}\ell(\mathbf{F}_i)\equiv\mathscr{L}(\mathbf{F}_i)$$

on the ordinary frame bundle $P(E_i)$ of E_i. The corresponding Christoffel symbols $\Gamma_\alpha^i\colon \phi_\alpha(U_\alpha)\to\mathscr{L}(\mathbf{F}_i,\mathscr{L}(\mathbf{B},\mathbf{F}_i))$ of ∇_i and the local connection forms $\omega_\alpha^i\in\bigwedge^1(U_\alpha,\mathscr{G}\ell(\mathbf{F}_i))$ of ω_i, with respect to $\mathscr{C}$, are related by

$$\Gamma_\alpha^i(x)(w,y)=[((\phi_\alpha^{-1})^*\omega_\alpha^i)_x\cdot y](w)\,, \tag{3.4}$$

for every $x\in\phi_\alpha(U_\alpha)$, $y\in\mathbf{B}$, $w\in\mathbf{F}_i$ (for relevant details cf. [12; Corollary 2.3]). Hence, for any $i\in\mathbf{N}$ and $\alpha\in I$, we define the (local) differential 1-forms $\theta_\alpha^i\in\bigwedge^1(U_\alpha,H_i(\mathbf{F}))$ given by

$$(\theta_\alpha^i)_b(v):=((\omega_\alpha^1)_b(v),\,\cdots,(\omega_\alpha^i)_b(v))\,;\qquad b\in U_\alpha\,,\ v\in T_bB. \tag{3.5}$$

After some tedious calculations, we check that (3.2) and (3.4) ensure that θ_α^i are indeed $H_i(\mathbf{F})$-valued forms.

Moreover, the ordinary compatibility conditions of the local connection forms $(\omega_\alpha^i)_{\alpha\in I}$, for each $i\in\mathbf{N}$, imply the analogous condition

$$\theta_\beta^i=\mathrm{Ad}_i(\mathbf{g}_{\alpha\beta}^{-1})\cdot\theta_\alpha^i+\mathbf{g}_{\alpha\beta}^{-1}\cdot d\mathbf{g}_{\alpha\beta}. \tag{3.6}$$

Here, $\mathbf{g}_{\alpha\beta}\colon U_{\alpha\beta}:=U_\alpha\cap U_\beta\to H_i^\circ(\mathbf{F})$ are the transition functions of $\mathbf{P}(E_i)$ and Ad_i is the adjoint representation of $H_i^\circ(\mathbf{F})$. The proof of this equality is based on (3.5) and the fact that

$$\mathbf{g}_{\alpha\beta}(x)=(g_{\alpha\beta}^1(x),\,\cdots,g_{\alpha\beta}^i(x))\,;\qquad x\in U_{\alpha\beta}\,,$$

where $(g_{\alpha\beta}^k)_{\alpha,\beta\in I}$ are the transition functions of $P(E_k)$, $k\in\mathbf{N}$. Therefore, each family $(\theta_\alpha^i)_{\alpha\in I}$ determines a unique principal connection (form) θ_i (with local connection forms given precisely by the previous family).

Finally, for the proof of (3.3) we distinguish the following cases (cf. also e.g. [10]):

i) Let $u \in T_q(\mathbf{P}(E_j))$ be any non-vertical vector at $q \in \mathbf{P}(E_j)$ with $\mathbf{p}_j(q) = b \in U_\alpha$. Then there exists a smooth local section $s_j\colon U \to \mathbf{P}(E_j)$ ($U \subseteq U_\alpha$ some open neighborhood of b) with $s_j(b) = q$ and $T_q(s_j \circ \mathbf{p}_j)(u) = u$. If $\mathbf{g}^j_\alpha\colon U \to H^\circ_j(\mathbf{F})$ is the smooth map connecting the natural local section s^j_α of $\mathbf{P}(E_j)$ with s_j, i.e. $s_j = s^j_\alpha \cdot \mathbf{g}^j_\alpha$, then

$$\theta_j(u) = \mathrm{Ad}_j(\mathbf{g}^j_\alpha(b)^{-1}) \cdot (\mathbf{p}^*_j \theta^j_\alpha)(u) + (\mathbf{g}^j_\alpha)^{-1} \cdot d\mathbf{g}^j_\alpha.$$

The last equality along with its counterpart for θ_i and the vector $Tr_{ji}(u)$ (obtained by considering the local section $s_i = r_{ji} \circ s_j$ and the morphism $\mathbf{g}^i_\alpha = r_{ji} \circ \mathbf{g}^j_\alpha$) prove (3.3) in the present case.

ii) Let u be any vertical vector at q, i.e. $u \in V_q(\mathbf{P}(E_j))$. In this case there exists a left invariant vector field

$$A_j \in \mathscr{L}(H^\circ_j(\mathbf{F})) \equiv H_j(\mathbf{F})$$

such that $u = A^*_j(q)$. Hence $\theta_j(u) = A^*_j$. On the other hand,

$$\theta_i(Tr_{ji}(u)) = (h_{ji} \circ A_j)^*(r_{ji}(q))$$

from which we get (3.3) and complete the proof. □

Condition (3.3) of Theorem 3.2 now allows one to determine the following 1-form $\theta \in \bigwedge^1(\mathbf{P}(E), H(\mathbf{F}))$, with

$$\theta((g_i)) := \varprojlim (\theta_i(g_1, \cdots, g_i))$$

(cf. also Remark 2.3(1)). Using the generalized smooth structure of $\mathbf{P}(E)$, we may consider θ as a generalized smooth connection form. Hence, we have

3.3. COROLLARY. *If θ_α are the local connection forms of θ, over the open cover $\mathscr{C}$, then*

$$\Gamma_\alpha(x) \cdot (w, y) = [((\phi_\alpha^{-1})^* \theta_\alpha)_x(y)](w) \tag{3.7}$$

for any $x \in \phi_\alpha(U_\alpha)$, $y \in \mathbf{B}$, $w = (w_i) \in \mathbf{F}$.

PROOF. By the very construction of θ and the fact that $\theta_i \equiv (\theta^i_\alpha)_{\alpha \in I}$, we conclude that $\theta_\alpha = \varprojlim \theta^i_\alpha$. Therefore,

$$[((\phi_\alpha^{-1})^* \theta_\alpha)_x(y)](w) = ([((\phi_\alpha^{-1})^* \theta^i_\alpha)_x(y)](w_1, \cdots, w_i))_{i \in \mathbf{N}}$$

[(3.5)] $\qquad = ([((\phi_\alpha^{-1})^* \omega^i_\alpha)_x(y)](w_i))_{i \in \mathbf{N}}$

[(3.4)] $\qquad = (\Gamma^i_\alpha(x) \cdot (w_i, y))_{i \in \mathbf{N}}$

[Prop. 3.1] $\qquad = \Gamma_\alpha(x) \cdot (w, y)$. □

Corollary 3.3 along with the definition of θ and the comments following Proposition 3.1, imply also

3.4. COROLLARY. *There is a bijective correspondence between linear connections* $\nabla = \varprojlim \nabla_i$ *on* E *and generalized connection forms* θ *on* $\mathbf{P}(E)$.

Furthermore, for arbitrary linear connections on E we prove

3.5. PROPOSITION. *Let* ∇ *be any linear connection on* E. *If we assume that* ∇ *corresponds to a generalized connection form* θ *of* $\mathbf{P}(E)$, *via* (3.7), *then necessarily* $\nabla = \varprojlim \nabla_i$, *where* ∇_i *is a linear connection on* E_i.

PROOF. Since θ is an $H(\mathbf{F})$-valued form and $H(\mathbf{F}) = \varprojlim H_i(\mathbf{F})$, it is proved in [4] that $\theta = \varprojlim \theta_i$, where θ_i are connection forms of $\mathbf{P}(E_i)$. Using (3.7) as well as equality $\theta_\alpha = \varprojlim \theta^i_\alpha$, we check that $\Gamma_\alpha(x) = \varprojlim \Gamma^i_\alpha(x)$, which concludes the proof. □

Summarizing the last results we obtain the following characterization.

3.6. THEOREM. *A linear connection* ∇ *on* E *corresponds to a generalized connection form* θ *on* $\mathbf{P}(E)$ *if and only if* $\nabla = \varprojlim \nabla_i$.

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