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A Heredity Property of Sufficiency

Jürgen HILLE

University of Münster
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Abstract. If $(X, \mathscr{A})$ is a measurable space, $(\mathscr{P}_n)_{n \in \mathbf{N}}$ is an increasing sequence of nonempty sets $\mathscr{P}_n$ of probability measures and $\mathscr{B}_n$ is a sub-σ-field of $\mathscr{A}$ which is sufficient for the statistical experiment $(X, \mathscr{A}, \mathscr{P}_n)$, $n \in \mathbf{N}$, then the terminal σ-field of the sequence $(\mathscr{B}_n)_{n \in \mathbf{N}}$ contains a σ-field which is sufficient for $\bigcup_{n \in \mathbf{N}} \mathscr{P}_n$.

1. Introduction.

D. Landers has shown in [4], that the terminal σ-field $\mathscr{B}_\infty$ of a sequence $(\mathscr{B}_n)_{n \in \mathbf{N}}$ of sub-σ-fields of a given σ-field $\mathscr{A}$ is minimal sufficient for a class $\mathscr{P}$ of probability measures on $\mathscr{A}$, if $\mathscr{B}_n$ is minimal sufficient for $\mathscr{P}_n$, where $\mathscr{P}_n \subset \mathscr{P}$, $n \in \mathbf{N}$, are subsets such that $\mathscr{P}_n \subset \mathscr{P}_{n+1}$, $n \in \mathbf{N}$, and $\mathscr{P} = \bigcup_{n \in \mathbf{N}} \mathscr{P}_n$. If the assumption of minimality is dropped and $\mathscr{A}$ is countably generated, the terminal σ-field is sufficient ([4], p. 204, Proposition 9). If $\mathscr{B}_n$ is sufficient for $\mathscr{P}_n$ but not necessarily minimal sufficient and $\mathscr{A}$ is not countably generated, $\mathscr{B}_\infty$ is not sufficient in general ([4], p. 203, Example 8). We show in this paper, that in any case $\mathscr{B}_\infty$ contains a sub-σ-field which is sufficient for $\mathscr{P}$, if $\mathscr{B}_n$ is sufficient for $\mathscr{P}_n$.

2. Notations.

A statistical experiment is a triple $(X, \mathscr{A}, \mathscr{P})$, where $(X, \mathscr{A})$ is a measurable space and $\mathscr{P}$ is a nonempty set of probability measures on $(X, \mathscr{A})$. The system of all $\mathscr{A}$-measurable subsets of X, which are P-null sets, for all $P \in \mathscr{P}$, is denoted by $\mathscr{N}(\mathscr{P})$. If $\mathscr{E}$ is a system of subsets of X, the σ-field generated by $\mathscr{E}$ is denoted by $S(\mathscr{E})$. If $g, h \colon X \to \mathbf{R}$ are $\mathscr{A}$-measurable functions we write $g = h$ $[\mathscr{P}]$ if $P\{x \in X \mid g(x) \neq h(x)\} = 0$, for every $P \in \mathscr{P}$. For $\mathscr{A}$-measurable subsets A_1, A_2 of X we write $A_1 = A_2$ $[\mathscr{P}]$ instead of $1_{A_1} = 1_{A_2}$ $[\mathscr{P}]$. If $\mathscr{B}$ and $\mathscr{C}$ are sub-σ-fields of $\mathscr{A}$, $\mathscr{B} \subset \mathscr{C}$ $[\mathscr{P}]$ means that for every $B \in \mathscr{B}$ there is a set $C \in \mathscr{C}$ such that $B = C$ $[\mathscr{P}]$. It is easy to see that $S(\mathscr{C} \cup \mathscr{N}(\mathscr{P}))$ consists exactly of those sets $A \in \mathscr{A}$ for which there is a set $C \in \mathscr{C}$ such that $A = C$ $[\mathscr{P}]$. Hence $\mathscr{B} \subset \mathscr{C}$ $[\mathscr{P}]$ is equivalent to $\mathscr{B} \subset S(\mathscr{C} \cup \mathscr{N}(\mathscr{P}))$. A sub-$\sigma$-field $\mathscr{B} \subset \mathscr{A}$ is called sufficient

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for $(X, \mathscr{A}, \mathscr{P})$, if for every bounded $\mathscr{A}$-measurable function $f\colon X \to \mathbf{R}$ there is a $\mathscr{B}$-measurable function $g\colon X \to \mathbf{R}$ such that g is a version of the conditional expectation $E_P(g \mid \mathscr{B})$, for all $P \in \mathscr{P}$.

3. Results.

We need three auxiliary results:

LEMMA 1. *Let $(X, \mathscr{A})$ be a measurable space, $\mathscr{B} \subset \mathscr{A}$ be a σ-field and $(\mathscr{P}_n)_{n \in \mathbf{N}}$ be an increasing sequence of nonempty classes of probability measures on $(X, \mathscr{A})$. If $\mathscr{B}$ is sufficient for every statistical experiment $(X, \mathscr{A}, \mathscr{P}_n)$ then it is sufficient for $(X, \mathscr{A}, \mathscr{P})$, where $\mathscr{P} = \bigcup_{n=1}^{\infty} \mathscr{P}_n$.*

PROOF. [4], p. 206, Lemma 4. □

LEMMA 2. *Let $(X, \mathscr{A}, \mathscr{P})$ be a statistical experiment and let $(\mathscr{B}_n)_{n \in \mathbf{N}}$ be a sequence of sufficient sub-σ-fields of $\mathscr{A}$ such that*

$$\mathscr{B}_{n+1} \subset \mathscr{B}_n \quad [\mathscr{P}] \qquad \forall n \in \mathbf{N}.$$

Then the terminal σ-field

$$\mathscr{B}_\infty = \bigcap_{n=1}^{\infty} S\left(\bigcup_{m=n}^{\infty} \mathscr{B}_m\right)$$

is sufficient.

PROOF. Let $\tilde{\mathscr{B}}_n := S(\mathscr{B}_n \cup \mathscr{N}(\mathscr{P}))$, for all $n \in \mathbf{N}$. Then $\tilde{\mathscr{B}}_{n+1} \subset \tilde{\mathscr{B}}_n$, for all $n \in \mathbf{N}$, and

$$\mathscr{B}_\infty \subset \tilde{\mathscr{B}}_\infty := \bigcap_{n=1}^{\infty} \tilde{\mathscr{B}}_n .$$

A result of Burkholder ([2], p. 1197, Corollary 2) yields the sufficiency of $\tilde{\mathscr{B}}_\infty$. Hence, if f is a bounded real valued measurable function on X, there exists a $\tilde{\mathscr{B}}_\infty$-measurable and bounded function g on X which is a version of the conditional expectation $E_P(f \mid \tilde{\mathscr{B}}_\infty)$ simultaneously for all $P \in \mathscr{P}$. By definition of $\tilde{\mathscr{B}}_\infty$, for every $n \in \mathbf{N}$ there is a $\mathscr{B}_n$-measurable real valued function g_n satisfying $g_n = g$ $[\mathscr{P}]$ (comp. [5], p. 56, Lemma 1.10.3). Since

$$g = g_* := \limsup_{n \to \infty} g_n \quad [\mathscr{P}] ,$$

g_* is a $\mathscr{B}_\infty$-measurable version of $E_P(f \mid \tilde{\mathscr{B}}_\infty)$, $P \in \mathscr{P}$, and consequently $g_* \in \bigcap_{P \in \mathscr{P}} E_P(f \mid \mathscr{B}_\infty)$. □

LEMMA 3. *Let $(X, \mathscr{A}, \mathscr{P})$ be a statistical experiment and let $\mathscr{B}, \mathscr{C}$ be sub-σ-fields of $\mathscr{A}$ such that $\mathscr{N}(\mathscr{P}) \subset \mathscr{C}$. Then*

$$\mathscr{C} \cap S(\mathscr{B} \cup \mathscr{N}(\mathscr{P})) = S((\mathscr{C} \cap \mathscr{B}) \cup \mathscr{N}(\mathscr{P})) .$$

PROOF. $\mathscr{C} \cap S(\mathscr{B} \cup \mathscr{N}(\mathscr{P}))$ is a σ-field which contains $\mathscr{C} \cap \mathscr{B}$ and $\mathscr{N}(\mathscr{P})$. Hence $\mathscr{C} \cap S(\mathscr{B} \cup \mathscr{N}(\mathscr{P})) \supset S((\mathscr{C} \cap \mathscr{B}) \cup \mathscr{N}(\mathscr{P}))$. Conversely let C be an element of $\mathscr{C} \cap S(\mathscr{B} \cup \mathscr{N}(\mathscr{P}))$. Then there is a set $B \in \mathscr{B}$ such that $C = B$ $[\mathscr{P}]$. Hence $C \Delta B \in \mathscr{N}(\mathscr{P}) \subset \mathscr{C}$, Δ denoting the symmetric difference, and so $B = C\Delta(C\Delta B)$ is an element of $\mathscr{C} \cap \mathscr{B}$. This shows $C \in S((\mathscr{C} \cap \mathscr{B}) \cup \mathscr{N}(\mathscr{P}))$. □

THEOREM 1. *Let $(X, \mathscr{A})$ be a measurable space, let, for all $n \in \mathbf{N}$, $\mathscr{B}_n$ be a sub-σ-field of $\mathscr{A}$ which is sufficient for the statistical experiment $(X, \mathscr{A}, \mathscr{P}_n)$, where $(\mathscr{P}_n)_{n \in \mathbf{N}}$ is an increasing sequence of nonempty classes of probability measures on $(X, \mathscr{A})$. Let $\mathscr{P} := \bigcup_{n=1}^{\infty} \mathscr{P}_n$. Then the terminal σ-field $\mathscr{B}_\infty := \bigcap_{n=1}^{\infty} S(\bigcup_{m \geq n} \mathscr{B}_m)$ contains a sub-σ-field $\mathscr{B}_*$ which is sufficient for $(X, \mathscr{A}, \mathscr{P})$.*

PROOF. For all $n \in \mathbf{N}$, $k \geq n$ and $m \geq k$ let

$$\mathscr{B}(n,k,m) := \bigcap_{i=k}^{m-1} S(\mathscr{B}_i \cup \mathscr{N}(\mathscr{P}_n)) \cap \mathscr{B}_m$$

(the empty intersection is understood to be the σ-field $\mathscr{A}$, hence $\mathscr{B}(n,k,k) = \mathscr{B}_k$). If $i \geq k \geq n$ then $\mathscr{B}_i$ is sufficient for $\mathscr{P}_n$, since $\mathscr{P}_n \subset \mathscr{P}_i$. Burkholder (comp. [2], p. 1196, Theorem 4) has proved, that $\mathscr{B}(n,k,m)$ is therefore sufficient for $\mathscr{P}_n$. We have

$$\mathscr{B}(n,k,m+1) \subset \bigcap_{i=k}^{m} S(\mathscr{B}_i \cup \mathscr{N}(\mathscr{P}_n)) = S(\mathscr{B}(n,k,m) \cup \mathscr{N}(\mathscr{P}_n)),$$

for all $m \geq k$, where the last equality follows from Lemma 3, applied to the σ-fields $\mathscr{C} := \bigcap_{i=k}^{m-1} S(\mathscr{B}_i \cup \mathscr{N}(\mathscr{P}_n))$ and $\mathscr{B} = \mathscr{B}_m$. Hence $(\mathscr{B}(n,k,m))_{m \geq k}$ is a $\mathscr{P}_n$-essentially decreasing sequence of sufficient subfields. By Lemma 2 the terminal σ-field

$$\mathscr{B}(n,k) := \bigcap_{i=k}^{\infty} S\left(\bigcup_{j \geq i} \mathscr{B}(n,k,j)\right)$$

is sufficient for $\mathscr{P}_n$, for every $k \geq n$. Next observe that $(\mathscr{B}(n,k))_{k \geq n}$ is increasing: By definition one has

$$\mathscr{B}(n,k,m) \subset \mathscr{B}(n,k+1,m) \qquad \forall n \in \mathbf{N}, \ k \geq n, \ m \geq k+1,$$

and consequently

$$\begin{aligned} \mathscr{B}(n,k) &= \bigcap_{i=k+1}^{\infty} S\left(\bigcup_{j \geq i} \mathscr{B}(n,k,j)\right) \\ &\subset \bigcap_{i=k+1}^{\infty} S\left(\bigcup_{j \geq i} \mathscr{B}(n,k+1,j)\right) = \mathscr{B}(n,k+1), \end{aligned}$$

for all $n \in \mathbf{N}$, $k \geq n$ (note that $\mathscr{B}(n,k) = \bigcap_{i=l}^{\infty} S(\bigcup_{j \geq i} \mathscr{B}(n,k,j))$ for any $l \geq k$, since $S(\bigcup_{j \geq i} \mathscr{B}(n,k,j))_{i \geq k}$ is decreasing). Again by one of Burkholder's theorems ([2], p. 1196, Theorem 3) the upper envelope

$$\mathscr{B}(n) := S\left(\bigcup_{k=n}^{\infty} \mathscr{B}(n,k)\right)$$

is sufficient for $\mathscr{P}_n$. Finally observe that $(\mathscr{B}(n))_{n\in\mathbf{N}}$ is a decreasing sequence. To prove this let $n\in\mathbf{N}$. $\mathscr{P}_n \subset \mathscr{P}_{n+1}$ implies $\mathscr{N}(\mathscr{P}_n) \supset \mathscr{N}(\mathscr{P}_{n+1})$. This yields

$$\mathscr{B}(n,k,m) \supset \mathscr{B}(n+1,k,m) \qquad \forall k\geq n+1\,,\ m\geq k\,,$$

and it follows immediately that

$$\mathscr{B}(n,k) \supset \mathscr{B}(n+1,k) \qquad \forall k\geq n+1\,.$$

This implies

$$\mathscr{B}(n) = S\left(\bigcup_{k=n+1}^{\infty} \mathscr{B}(n,k)\right) \supset S\left(\bigcup_{k=n+1}^{\infty} \mathscr{B}(n+1,k)\right) = \mathscr{B}(n+1)\,,$$

where the first equality is valid since $\mathscr{B}(n,k)_{k\geq n}$ is increasing. Now $\mathscr{B}_* = \bigcap_{n=1}^{\infty} \mathscr{B}(n)$ is sufficient for $\mathscr{P}_n$ and every $n\in\mathbf{N}$ (again [2], p. 1196, Theorem 3). Lemma 1 yields the sufficiency of $\mathscr{B}_*$ for $(X,\mathscr{A},\mathscr{P})$.

$\mathscr{B}_\infty$ contains $\mathscr{B}_*$, since $\mathscr{B}_j$ contains $\mathscr{B}(n,k,j)$ if $j\geq k\geq n$ and hence

$$\mathscr{B}(n,k) \subset \bigcap_{i=k}^{\infty} S\left(\bigcup_{j=i}^{\infty} \mathscr{B}_j\right) = \mathscr{B}_\infty \qquad \forall k\geq n\,.$$

It follows immediately that $\mathscr{B}(n) \subset \mathscr{B}_\infty$ is valid, for all $n\in\mathbf{N}$, i.e. $\mathscr{B}_* \subset \mathscr{B}_\infty$. □

REMARK 1. If the assumptions of Theorem 1 hold and in addition all sub-σ-fields $\mathscr{B}_n$ are necessary for $(X,\mathscr{A},\mathscr{P}_n)$ (i.e. are $\mathscr{P}_n$-essentially included in any sub-σ-field which is sufficient for $(X,\mathscr{A},\mathscr{P}_n)$) then the terminal σ-field $\mathscr{B}_\infty$ of the sequence $(\mathscr{B}_n)_{n\in\mathbf{N}}$ is necessary for $(X,\mathscr{A},\mathscr{P})$. This follows immediately from

$$S\left(\bigcup_{n=m}^{\infty} \mathscr{B}_m\right) \subset \mathscr{C} \quad [\mathscr{P}_n]\,,$$

which is valid for any sufficient sub-σ-field $\mathscr{C}$ of $\mathscr{A}$ and any $n\in\mathbf{N}$. Hence it may be deduced from Theorem 1 that minimalsufficiency of $\mathscr{B}_n$ for $(X,\mathscr{A},\mathscr{P}_n)$, $n\in\mathbf{N}$, implies minimalsufficiency of $\mathscr{B}_\infty$. This result is due to D. Landers ([4], p. 202, Theorem 7).

We now give a reformulation of Theorem 1 in terms of weak Blackwell-sufficiency. This is a weak version of a notion of sufficiency which has been introduced by D. Blackwell in [1].

DEFINITION 1. Let $(X,\mathscr{A},\mathscr{P})$ be a statistical experiment.

1. Let $(\Delta,\mathscr{D})$ be a measurable space. A *$\mathscr{P}$-weak Markov kernel* from $(X,\mathscr{A})$ into $(\Delta,\mathscr{D})$ is a function $K\colon X\times\mathscr{D}\to\mathbf{R}$ such that
(a) $\forall D\in\mathscr{D}\colon K(x,D)\geq 0$ $[\mathscr{P}]$,

(b) $K(x,\Delta)=1\ [\mathscr{P}]$,

(c) for all sequences $(D_n)_{n\in\mathbf{N}}$ of pairwise disjoint $\mathscr{D}$-measurable subsets of Δ

$$K\left(x,\sum_{n=1}^{\infty} D_n\right)=\sum_{n=1}^{\infty} K(x,D_n)\quad [\mathscr{P}]$$

holds true,

(d) $\forall D\in\mathscr{D}: x\mapsto K(x,D)$ is $\mathscr{A}$-measurable.

2. A sub-σ-field $\mathscr{B}$ of $\mathscr{A}$ is called *weakly Blackwell sufficient* if there is a $\mathscr{P}$-weak Markov kernel K from $(X,\mathscr{B})$ into $(X,\mathscr{A})$ such that

$$\int P(dx)K(x,A)=P(A)\qquad \forall A\in\mathscr{A}\ ,\ P\in\mathscr{P}\ .$$

THEOREM 2. *Let $(X,\mathscr{A},\mathscr{P})$ be a statistical experiment and $\mathscr{B}$ be a sub-σ-field of $\mathscr{A}$. Then*

$\mathscr{B}$ is weakly Blackwell sufficient

$\Leftrightarrow \mathscr{B}$ contains a sub-σ-field which is sufficient for $(X,\mathscr{A},\mathscr{P})$.

PROOF. This is shown in [3]. □

COROLLARY 1. *Let $(X,\mathscr{A})$ be a measurable space, $(\mathscr{P}_n)_{n\in\mathbf{N}}$ an increasing sequence of nonempty classes of probability measures. For every $n\in\mathbf{N}$ let $\mathscr{B}_n$ be a sub-σ-field which is weakly Blackwell sufficient for $\mathscr{P}_n$. Then the terminal σ-field $\mathscr{B}_\infty:=\bigcap_{n=1}^{\infty}S(\bigcup_{m=n}^{\infty}\mathscr{B}_m)$ is weakly Blackwell sufficient for $\mathscr{P}:=\bigcup_{n=1}^{\infty}\mathscr{P}_n$.*

PROOF. This follows immediately from Theorem 1 and Theorem 2. □

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Present Address:
INSTITUT FÜR MATHEMATISCHE STATISTIK, UNIVERSITÄT MÜNSTER,
EINSTEINSTRAßE 62, D-48149 MÜNSTER.