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Volumes of Compact Symmetric Spaces

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H. Freudenthal [4] defined the natural volume of a semi-simple compact Lie group G induced from the Killing form and gave a formula of the natural volume of G. S. A. Broughton [3] calculated the volume in the case of G a classical Lie group. Another volume formula of the semi-simple compact Lie groups has been studied by H. Urakawa [7] and I. G. Macdonald [6] in different ways, respectively.

If G/K is a compact symmetric space, then the Killing form of G also induces a natural volume of G/K. The volumes of the projective spaces are obtained by using Jacobi fields (cf. [2], [5]). In the previous paper [1] we calculated the volumes of the Hermitian exceptional symmetric spaces $EIII$, $EVII$ and the twister space $Z(EIX)$ of the exceptional symmetric space EIX by using the computations of the 1st Chern classes. From those results we can calculate the natural volumes of the compact symmetric spaces as follows:

symbol	space	volume
A_n	$SU(n+1)$	$\dfrac{2^{n(2n+5)/2}(n+1)^{(n+1)^2/2}}{1!\,2!\cdots n!}\pi^{n(n+3)/2}$
B_n	$Spin(2n+1)$	$\dfrac{2^{n(4n+5)/2+1}(2n-1)^{n(2n+1)/2}}{1!\,3!\cdots(2n-1)!}\pi^{n(n+1)}$
C_n	$Sp(n)$	$\dfrac{2^{n(3n+1)}(n+1)^{n(2n+1)/2}}{1!\,3!\cdots(2n-1)!}\pi^{n(n+1)}$
D_n	$Spin(2n)$	$\dfrac{2^{3n^2}(n-1)^{n(2n-1)/2}}{2!\,4!\cdots(2n-2)!}\pi^{n^2}$
G_2	G_2	$\dfrac{2^{26}3^2\sqrt{3}}{5}\pi^8$
F_4	F_4	$\dfrac{2^{52}\,3^{45}}{5^4\,7^2\,11}\pi^{28}$

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E_6	E_6	$\dfrac{2^{134}\, 3^{29} \sqrt{3}}{5^5\, 7^3\, 11}\pi^{42}$
E_7	E_7	$\dfrac{2^{156} \sqrt{2}\, 3^{111}}{5^{10}\, 7^6\, 11^3\, 13^2\, 17}\pi^{70}$
E_8	E_8	$\dfrac{2^{279}\, 3^{77}\, 5^{103}}{7^{14}\, 11^8\, 13^6\, 17^4\, 19^3\, 23^2\, 29}\pi^{128}$
AI	$SU(n)/SO(n)$	$\dfrac{2^{(n-1)(n+3)/2}\, n^{n(n+1)/4}}{1!\,2!\cdots(n-1)!\, S_1 S_2 \cdots S_{n-1}}\pi^{(n-1)(n+2)/2}$
AII	$SU(2n)/Sp(n)$	$\dfrac{2^{(n-1)(4n+3)/2}\, n^{n(2n-1)/2}}{2!\,4!\cdots(2n-2)!}\pi^{n^2-1}$
$AIII$	$SU(m+n)/S(U(m)\times U(n))$	$\dfrac{1!\,2!\cdots(m-1)!\,1!\,2!\cdots(n-1)!}{1!\,2!\cdots(m+n-1)!} \times 2^{2mn}(m+n)^{mn}\pi^{mn}$
BDI	$SO(m+n)/(SO(m)\times SO(n))$	$\dfrac{(2(m+n-2))^{mn/2} S_1 S_2 \cdots S_{m+n-1}}{S_1 S_2 \cdots S_{m-1} S_1 S_2 \cdots S_{n-1}}$
$DIII$	$SO(2n)/U(n)$	$2^{3n(n-1)/2}\dfrac{1!\,2!\cdots(n-2)!}{1!\,3!\cdots(2n-3)!} \times (n-1)^{n(n-1)/2}\pi^{n(n-1)/2}$
CI	$Sp(n)/U(n)$	$2^{n(3n+1)/2}\dfrac{1!\,2!\cdots(n-1)!}{1!\,3!\cdots(2n-1)!} \times (n+1)^{n(n+1)/2}\pi^{n(n+2)/2}$
CII	$Sp(m+n)/(Sp(m)\times Sp(n))$	$\dfrac{1!\,3!\cdots(2m-1)!\,1!\,3!\cdots(2n-1)!}{1!\,3!\cdots(2m+2n-1)!} \times 2^{6mn}(m+n+1)^{2mn}\pi^{2mn}$

Here $S_{n-1} = (2/\Gamma(n/2))\pi^{n/2}$ is the usual volume of the unit sphere S^{n-1}.

G	$G_2/(Sp(1)\times Sp(1))/\mathbf{Z}_2$	$\dfrac{2^{13}\, 3}{5}\pi^4$
FI	$F_4/(Sp(1)\times Sp(3))/\mathbf{Z}_2$	$\dfrac{2^{23}\, 3^{23}}{5^3\, 7^2\, 11}\pi^{14}$
FII	$F_4/Spin(9)$	$\dfrac{2^{17}\, 3^{13}}{5^2\, 7\, 11}\pi^8$
EI	$E_6/Sp(4)/\mathbf{Z}_2$	$\dfrac{2^{55}\, 3^{15}\sqrt{3}}{5^3\, 7^2\, 11}\pi^{22}$

EII	$E_6/(Sp(1)\times SU(6))/\mathbf{Z}_2$	$\dfrac{2^{63}\,3^{13}}{5^4\,7^3\,11}\pi^{20}$
$EIII$	$E_6/(U(1)\times Spin(10))/\mathbf{Z}_4$	$\dfrac{2^{50}\,3^{11}}{5^3\,7^2\,11}\pi^{16}$
EIV	E_6/F_4	$\dfrac{2^{30}\,3^{10}\sqrt{3}}{5\cdot 7}\pi^{14}$
EV	$E_7/SU(8)/\mathbf{Z}_2$	$\dfrac{2^{74}\,3^{55}}{5^7\,7^5\,11^3\,13^2\,17}\pi^{35}$
EVI	$E_7/(SU(2)\times Spin(12))/\mathbf{Z}_2$	$\dfrac{2^{67}\,3^{51}}{5^6\,7^4\,11^3\,13^2\,17}\pi^{32}$
$EVII$	$E_7/(U(1)\times E_6)/\mathbf{Z}_3$	$\dfrac{2^{59}\,3^{42}}{5^5\,7^3\,11^2\,13^2\,17}\pi^{27}$
$EVIII$	$E_8/Ss(16)$	$\dfrac{2^{132}\,3^{36}\,5^{51}}{7^9\,11^6\,13^6\,17^4\,19^3\,23^2\,29}\pi^{64}$
EIX	$E_8/(SU(2)\times E_7)/\mathbf{Z}_2$	$\dfrac{2^{118}\,3^{31}\,5^{45}}{7^8\,11^5\,13^4\,17^3\,19^3\,23^2\,29}\pi^{56}$

1. Preliminaries.

LEMMA 1. *Let (M, g) be an n-dimensional Riemannian manifold. Then, for a positive number a, we have that*

$$\mu(M, ag)=\sqrt{a}^{\,n}\mu(M, g)\,.$$

If G is a Lie group, then we shall denote the small German letter $\mathfrak{g}$ as the Lie algebra of G. If G is a compact semi-simple Lie group, then the negative of the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$ defines a G-invariant metric g_G on G. Let K be a closed subgroup of G and $\mu_G(K)$ denote the volume of K with respect to the metric induced from g_G. Here we assume that $M=G/K$ is a compact symmetric space. Then the Lie algebra $\mathfrak{g}$ of G has a canonical decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ such that $\mathfrak{m}$ is identified with the tangent space $T_o(M)$ of M at $o=K$. Since $\mathfrak{m}$ is K-invariant, the metric g_G induces a G-invariant metric $\bar{g}_G$ on M. Let $\mu_G(M)=\mu(M)$ denote the volume of M with respect to $\bar{g}_G$ on M.

LEMMA 2 (Broughton [3]). *Let G be a compact semi-simple Lie group and $G_1,\cdots, G_m$ be closed subgroups of G such that $\phi: G_1\times\cdots\times G_m\to G$ is a covering group. Then we have that*

$$\mu(G)=\frac{1}{|\,ker\phi\,|}\mu_G(G_1)\cdots\mu_G(G_m)\,.$$

LEMMA 3 (Broughton [3]). *Let $M=G/K$ be a compact symmetric space, then we have that*

$$\mu(G/K)=\mu(G)/\mu_G(K)\,.$$

Let $B_{\mathfrak{g}}$ denote the Killing form of a compact simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $(\ ,\ )_{\mathfrak{g}}$ be a symmetric bilinear form of $\mathfrak{h}^*$ (which is the dual space of $\mathfrak{h}$) defined by $(\alpha, \beta)_{\mathfrak{g}}=B_{\mathfrak{g}}(H_\alpha, H_\beta)$, $\alpha, \beta\in\mathfrak{h}^*$, where H_α is the element of $\mathfrak{h}$ such that $B_{\mathfrak{g}}(H_\alpha, H)=\alpha(H)$ for all $H\in\mathfrak{h}$.

LEMMA 4. *Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{k}$ be a simple Lie subalgebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{k}$ such that $\mathfrak{h}'$ is contained in $\mathfrak{h}$. Assume that α is a root of $\mathfrak{k}$ with respect to $\mathfrak{h}'$ which is also a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then we have that*

$$B_{\mathfrak{g}}(X, Y)=\frac{(\alpha, \alpha)_{\mathfrak{k}}}{(\alpha, \alpha)_{\mathfrak{g}}}B_{\mathfrak{k}}(X, Y)\,, \qquad X, Y\in\mathfrak{k}\,.$$

PROOF. Let H_α and H'_α be elements of $\mathfrak{h}$ such that $B_{\mathfrak{g}}(H_\alpha, H)=\alpha(H)$ for $H\in\mathfrak{h}$ and $B_{\mathfrak{k}}(H'_\alpha, H')=\alpha(H')$ for $H'\in\mathfrak{h}'$. Note that there exists $c\in\mathbf{R}$ such that $B_{\mathfrak{g}}(H'_1, H'_2)=cB_{\mathfrak{k}}(H'_1, H'_2)$ for $H'_1, H'_2\in\mathfrak{h}'$. Then we have $H'_\alpha=cH_\alpha$ and Lemma 4 follows.

2. Volumes of compact classical groups.

Let $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ denote the fields of real, complex and quaternionic numbers, respectively.

2.1. $SU(n+1)$. Let $SU(n+1)$ be the special unitary group given by

$$SU(n+1)=\{A\in M(n+1, \mathbf{C})\,|\,A^*A=E, \det A=1\}$$

and $\mathbf{C}P_n$ be the complex projective space defined by

$$\mathbf{C}P_n=\{X\in M(n+1, \mathbf{C})\,|\,X^*=X, X^2=X, \mathrm{tr}(X)=1\}\,.$$

The group $SU(n+1)$ acts naturally and transitively on $\mathbf{C}P_n$ and the isotropy subgroup $SU(n+1)_{E_1}$ at $E_1=\mathrm{diag}(1, 0, \cdots, 0)\in\mathbf{C}P_n$ is $S(U(1)\times U(n))$. Let $\varphi : U(1)\times SU(n)\to S(U(1)\times U(n))$ be a map defined by $\varphi(z, A)=\begin{pmatrix} z^{-n} & 0 \\ 0 & zA \end{pmatrix}$, then φ induces an isomorphism $(U(1)\times SU(n))/\mathbf{Z}_n\cong S(U(1)\times U(n))$. Hence we have that

$$SU(n+1)/(U(1)\times SU(n))/\mathbf{Z}_n\cong\mathbf{C}P_n\,.$$

The Lie algebra $\mathfrak{su}(n+1)=\{X\in M(n+1, \mathbf{C})\,|\,X^*=-X, \mathrm{tr}(X)=0\}$ of $SU(n+1)$ has a canonical decomposition $\mathfrak{su}(n+1)=\mathfrak{k}\oplus\mathfrak{m}$ such that $\mathfrak{k}\cong\mathfrak{u}(1)\oplus\mathfrak{su}(n)$ and

$$\mathfrak{m}=\left\{\begin{pmatrix} 0 & -\mathbf{x}^* \\ \mathbf{x} & 0 \end{pmatrix} \middle| \mathbf{x}\in\mathbf{C}^n\right\}$$

which can be identified with the tangent space of $\mathbf{C}P_n$ at E_1. The Killing form $B_{\mathfrak{su}(n+1)}$ of $\mathfrak{su}(n+1)$ is given by

$$B_{\mathfrak{su}(n+1)}(XY)=2(n+1)\,\mathrm{tr}(XY)\,,\qquad X,\ Y\in\mathfrak{su}(n+1)\,.$$

Let g be an invariant metric on $\mathbf{C}P_n$ given by $g(X, Y)=-\frac{1}{2}\,\mathrm{tr}(XY)$, X, $Y\in\mathfrak{m}$. Then, by [2], the volume of $\mathbf{C}P_n$ with respect to g is given by

$$\mu(\mathbf{C}P_n, g)=\pi^n/n!\,.$$

By Lemma 1 we have that

$$\mu_{SU(n+1)}(\mathbf{C}P_n)=\sqrt{4(n+1)}^{\,2n}\mu(\mathbf{C}P_n, g)=\frac{2^{2n}(n+1)^n}{n!}\pi^n\,.$$

Let $g_{\mathbf{C}}$ be a metric on $U(1)$ induced from the usual metric on $\mathbf{C}$. Then $\mu(U(1), g_{\mathbf{C}})=2\pi$. By Lemma 1 we have that

$$\mu_{SU(n+1)}(U(1))=\sqrt{-2(n+1)\,\mathrm{tr}(D^2)}\,\mu(U(1), g_{\mathbf{C}})=\sqrt{2n}(n+1)2\pi\,,$$

where D is a diagonal matrix $D=\mathrm{diag}(n\mathbf{i}, -\mathbf{i},\cdots, -\mathbf{i})$. Comparing the Killing forms of $\mathfrak{su}(n)$ and $\mathfrak{su}(n+1)$, we have that

$$\mu_{SU(n+1)}(SU(n))=\sqrt{\frac{2(n+1)}{2n}}^{\,n^2-1}\mu(SU(n))\,,\qquad n\geq 2\,.$$

From Lemma 2 and Lemma 3 we have that

$$\begin{aligned}\mu(SU(n+1))&=\tfrac{1}{n}(\mu_{SU(n+1)}(U(1))\mu_{SU(n+1)}(SU(n)))\mu_{SU(n+1)}(\mathbf{C}P_n)\\ &=\frac{2^{(4n+3)/2}(n+1)^{(n+1)^2/2}}{n^{n^2/2}n!}\pi^n\mu(SU(n))\,.\end{aligned}$$

Also we have that

$$\mu(SU(2))=\mu_{SU(2)}(\mathbf{C}P_1)\mu_{SU(2)}(U(1))=2^5\sqrt{2}\,\pi^2\,.$$

Thus it follows by induction that

$$\mu(SU(n+1))=\frac{2^{n(2n+5)/2}(n+1)^{(n+1)^2/2}}{1!\,2!\cdots n!}\pi^{n(n+3)/2}\,.$$

2.2. $SO(n)$ $(n\geq 3)$. Let $SO(n)$ be the special orthogonal group given by

$$SO(n)=\{A\in M(n, \mathbf{R})\,|\,{}^tAA=E, \det A=1\}$$

and S^{n-1} be the unit sphere in $\mathbf{R}^n$. The group $SO(n)$ acts naturally and transitively on S^{n-1} and the isotropy subgroup $SO(n)_{e_1}$ at $\mathbf{e}_1={}^t(1,0,\cdots,0)\in S^{n-1}$ is isomorphic to $SO(n-1)$. Hence we have that

$$SO(n)/SO(n-1)\cong S^{n-1}.$$

The Lie algebra $\mathfrak{so}(n)=\{X\in M(n,\mathbf{R})\,|\,{}^tX=-X\}$ of $SO(n)$ has a canonical decomposition $\mathfrak{so}(n)=\mathfrak{k}\oplus\mathfrak{m}$ such that $\mathfrak{k}\cong\mathfrak{so}(n-1)$ and

$$\mathfrak{m}=\left\{\begin{pmatrix}0 & -{}^t\mathbf{x}\\ \mathbf{x} & 0\end{pmatrix}\,\middle|\,\mathbf{x}\in\mathbf{R}^{n-1}\right\}$$

which can be identified with the tangent space of S^{n-1} at $\mathbf{e}_1$. The Killing form $B_{\mathfrak{so}(n)}$ of $\mathfrak{so}(n)$ is given by

$$B_{\mathfrak{so}(n)}(X,Y)=(n-2)\,\mathrm{tr}(XY)\,,\qquad X,Y\in\mathfrak{so}(n)\,.$$

Now we denote the volume of S^{n-1} with respect to the usual metric g by S_{n-1}. Then it is known that

$$S_{n-1}=\frac{2}{\Gamma(n/2)}\pi^{n/2}\,.$$

By Lemma 1 we have that

$$\mu_{SO(m)}(S^{n-1})=\sqrt{2(n-2)}^{\,n-1}S_{n-1}\,.$$

Comparing the Killing forms of $\mathfrak{so}(n-1)$ and $\mathfrak{so}(n)$, we have that

$$\mu_{SO(n)}(SO(n-1))=\sqrt{(n-2)/(n-3)}^{\,(n-1)(n-2)/2}\mu(SO(n-1))\,.$$

From Lemma 2 and Lemma 3 we have that

$$\begin{aligned}\mu(SO(n))&=\mu_{SO(n)}(SO(n-1))\,\mu_{SO(n)}(S^{n-1})\\&=\frac{2^{(n-1)/2}(n-2)^{n(n-1)/2}}{(n-3)^{(n-1)(n-2)/2}}\mu(SO(n-1))S_{n-1}\,.\end{aligned}$$

Since $SO(3)\cong SU(2)/\mathbf{Z}_2$, we have that

$$\mu(SO(3))=\tfrac{1}{2}\mu(SU(2))=\tfrac{1}{2}2^5\sqrt{2}\,\pi^2=\sqrt{2}^{\,3}S_1S_2\,.$$

Thus it follows by induction that

$$\mu(SO(n))=(2(n-2))^{n(n-1)/4}S_1S_2\cdots S_{n-1}\,.$$

Since $Spin(n)$ is a double covering group of $SO(n)$, we have that

$$\mu(Spin(n))=2\mu(SO(n))=2(2(n-2))^{n(n-1)/4}S_1S_2\cdots S_{n-1}\,.$$

2.3. *Sp(n).* Let $Sp(n)$ be the symplectic group given by

$$Sp(n)=\{A\in M(n,\mathbf{H})\mid A^*A=E\}$$

and $\mathbf{H}P_{n-1}$ be the symplectic projective space defined by

$$\mathbf{H}P_{n-1}=\{X\in M(n,\mathbf{H})\mid X^*=X,\ X^2=X,\ \mathrm{tr}(X)=1\}\ .$$

The group $Sp(n)$ acts naturally and transitively on $\mathbf{H}P_{n-1}$ and the isotropy subgroup $Sp(n)_{E_1}$ at $E_1\in\mathbf{H}P_{n-1}$ is $Sp(1)\times Sp(n-1)$. Hence we have that

$$Sp(n)/(Sp(1)\times Sp(n-1))\cong\mathbf{H}P_{n-1}\ .$$

The Lie algebra $\mathfrak{sp}(n)=\{X\in M(n,\mathbf{H})\mid X^*=-X\}$ of $Sp(n)$ has a canonical decomposition $\mathfrak{sp}(n)=\mathfrak{k}\oplus\mathfrak{m}$ such that $\mathfrak{k}\cong\mathfrak{sp}(1)\oplus\mathfrak{sp}(n-1)$ and

$$\mathfrak{m}=\left\{\begin{pmatrix}0 & -\mathbf{x}^*\\ \mathbf{x} & 0\end{pmatrix}\middle|\ \mathbf{x}\in\mathbf{H}^{n-1}\right\}$$

which can be identified with the tangent space of $\mathbf{H}P_{n-1}$ at E_1. The Killing form $B_{\mathfrak{sp}(n)}$ of $\mathfrak{sp}(n)$ is given by

$$B_{\mathfrak{sp}(n)}(X,Y)=4(n+1)\,\mathrm{tr}(XY)\ ,\qquad X,\ Y\in\mathfrak{sp}(n)\ .$$

Let g be an invariant metric on $\mathbf{H}P_{n-1}$ given by $g(X,Y)=-\frac{1}{2}\mathrm{tr}(XY)$, $X,\ Y\in\mathfrak{m}$. Then, by [2], the volume of $\mathbf{H}P_{n-1}$ with respect to g is given by

$$\mu(\mathbf{H}P_{n-1})=\pi^{2(n-1)}/(2n-1)!\ .$$

By Lemma 1 we have that

$$\begin{aligned}\mu_{Sp(n)}(\mathbf{H}P_{n-1})&=\sqrt{2\cdot 4(n+1)}^{\,4(n-1)}\mu(\mathbf{H}P_{n-1},g)\\&=\frac{2^{6(n-1)}(n+1)^{2(n-1)}}{(2n-1)!}\pi^{2(n-1)}\ .\end{aligned}$$

Since $Sp(1)\cong SU(2)$, we have that $\mu(Sp(1))=\mu(SU(2))=2^5\sqrt{2}\,\pi^2$. Comparing the Killing forms of $\mathfrak{sp}(1)$, $\mathfrak{sp}(n-1)$ and $\mathfrak{sp}(n)$, we have that

$$\begin{aligned}\mu(Sp(n))&=\left(\sqrt{\frac{4(n+1)}{4\cdot 2}}^{\,3}\mu(Sp(1))\sqrt{\frac{4(n+1)}{4n}}^{\,(n-1)(2n-1)}\mu(Sp(n-1))\right)\mu_{Sp(n)}(\mathbf{H}P_{n-1})\\&=\frac{2^{6n-2}(n+1)^{n(2n+1)/2}}{n^{(n-1)(2n-1)/2}(2n-1)!}\pi^{2n}\mu(Sp(n-1))\ .\end{aligned}$$

Thus it follows by induction that

$$\mu(Sp(n))=\frac{2^{n(3n+1)}(n+1)^{n(2n+1)/2}}{1!\,3!\cdots(2n-1)!}\pi^{n(n+1)}\ .$$

3. Volumes of compact exceptional groups.

In this section we use the notations in [1], [9].

3.1. G_2. Let $\mathfrak{C}$ be the Cayley algebra with a canonical basis $\{e_0, e_1, \cdots, e_7\}$. Let G_2 be the exceptional Lie group given by

$$G_2 = \{\alpha \in \mathrm{Iso}_{\mathbf{R}}(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$$

and S^6 be a unit sphere given by $S^6 = \{x \in \mathfrak{C} \mid \bar{x} = -x, \bar{x}x = 1\}$. Then the group G_2 acts naturally and transitively on S^6 and the isotropy subgroup $(G_2)_{e_4}$ at $e_4 \in S^6$ is isomorphic to $SU(3)$. Hence we have that

$$G_2/SU(3) \cong S^6 .$$

Put $\omega = -1/2 + (\sqrt{3}/2)e_4 \in \mathfrak{C}$ and we define $w \in G_2$ by

$$w(x) = \bar{\omega} x \omega , \qquad x \in \mathfrak{C} .$$

Then w induces an automorphim $\tilde{w}$ of order 3 of G_2 by

$$\tilde{w}(\alpha) = w \alpha w^{-1} , \qquad \alpha \in G_2 .$$

The fixed subgroup $(G_2)^w = \{\alpha \in G_2 \mid \tilde{w}(\alpha) = \alpha\}$ of $\tilde{w}$ coincides with $(G_2)_{e_4} \cong SU(3)$. Note that $G_2/SU(3) \cong S^6$ is not any symmetric space but is a symmetric space of order 3 (see [8]). The Lie algebra $\mathfrak{g}_2 = \{D \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{C}, \mathfrak{C}) \mid D(xy) = (Dx)y + x(Dy)\}$ of G_2 has a canonical decomposition $\mathfrak{g}_2 = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{k} = (D \in g_2 \mid \tilde{w}_*(D) = D\} \cong \mathfrak{su}(3)$ and

$$\mathfrak{m} = \{D \in \mathfrak{g}_2 \mid (\tilde{w}_*^2 + \tilde{w}_* + 1)D = 0\}$$

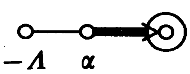
which can be identified with the tangent space of S^6 at e_4. The Killing form B_2 of $\mathfrak{g}_2$ is given by

$$B_2(D_1, D_2) = 4\,\mathrm{tr}(D_1 D_2) , \qquad D_1, D_2 \in \mathfrak{g}_2 .$$

For $a, b \in \mathfrak{C}$, $D_{a,b} \in \mathfrak{g}_2$ is defined by $D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b]$, where $L_a, R_a : \mathfrak{C} \to \mathfrak{C}$ are $\mathbf{R}$-linear mappings defined by $L_a x = ax$, $R_a x = xa$, for $x \in \mathfrak{C}$, respectively. Put $D = \frac{1}{2} D_{e_1, e_4}$. Then we can verify that $D \in \mathfrak{m}$. Since $B_2(D, D) = 4\,\mathrm{tr}(DD) = -48$ and $(De_4, De_4) = (-2e_1, -2e_1) = 4$, it follows from Lemma 1 and Lemma 4 that

$$\mu_{G_2}(S^6) = \sqrt{48/4}^{\,6} S_6 = 12^3 \frac{2^4}{3 \cdot 5} \pi^3 .$$

The extended Dynkin diagram of $\mathfrak{g}_2$ is given as follows:



Here Λ is the maximal root and the Dynkin diagram of the subalgebra $\mathfrak{su}(3)$ coincides with the subdiagram removing the circled vertex from the extended diagram, and α is a simple root of $\mathfrak{su}(2)$ which is a long root of $\mathfrak{g}_2$. Since $(\alpha, \alpha)_{\mathfrak{su}(3)} = 1/3$ and $(\alpha, \alpha)_{\mathfrak{g}_2} = 1/4$, from Lemma 1 and 3 we have that

$$\mu_{G_2}(SU(3)) = \sqrt{4/3}^{\,8} \mu(SU(3)) .$$

Note that Lemma 3 is valid in the case that $M = G/K$ is a symmetric space of order 3. Thus it follows from Lemma 3 and 3.1 that

$$\mu(G_2) = \mu_{G_2}(SU(3)) \mu_{G_2}(S^6) = \frac{2^{26}\, 3^2 \sqrt{3}}{5} \pi^8 .$$

3.2. F_4. Let F_4 be the exceptional Lie group given by

$$F_4 = \{\alpha \in \mathrm{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$$

and $FII = \mathfrak{C}P_2$ be the symmetric space, called the Cayley projective plane, defined by

$$FII = \{X \in \mathfrak{J} \mid X^2 = X, \operatorname{tr}(X) = 1\} .$$

The group F_4 acts naturally and transitively on FII and the isotropy subgroup $(F_4)_{E_1}$ at E_1 is isomorphic to $Spin(9)$. Hence we have that

$$F_4/Spin(9) \cong FII .$$

The Killing form B_4 of $\mathfrak{f}_4$ is given by

$$B_4(\delta_1, \delta_2) = 3 \operatorname{tr}(\delta_1 \delta_2) , \qquad \delta_1, \delta_2 \in \mathfrak{f}_4 .$$

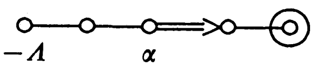
Let g be an invariant metric on FII defined by $g(X, Y) = \frac{1}{2}(X, Y)$, $X, Y \in T_{E_1}(FII)$. Then, by [2], the volume of FII with respect to g is given by

$$\mu(FII, g) = \frac{6}{11!} \pi^8 .$$

Since $B_4(\tilde{A}_2(1), \tilde{A}_2(1)) = 3 \operatorname{tr}(\tilde{A}_2(1)^2) = -72$ and $g(\tilde{A}_2(1)(E_1), \tilde{A}_2(1)(E_1)) = g(F_2(1), F_2(1)) = \frac{1}{2}(F_2(1), F_2(1)) = 1$, by Lemma 1 we have that

$$\mu_{F_4}(FII) = \sqrt{72}^{\,16} \mu(FII, g) = 72^8 \frac{6}{11!} \pi^8 .$$

Since $(\alpha, \alpha)_{\mathfrak{spin}(9)} = 1/7$ and $(\alpha, \alpha)_{\mathfrak{f}_4} = 1/9$ in the extended Dynkin diagram of $\mathfrak{f}_4$ below,



by Lemma 4 we have that

$$\mu_{F_4}(Spin(9)) = \sqrt{9/7}^{\,36} \mu(Spin(9)) .$$

Thus it follows from Lemma 3 and 2.3 that

$$\mu(F_4)=\mu_{F_4}(Spin(9))\mu_{F_4}(FII)=\frac{2^{52}\,3^{45}}{5^4\,7^2\,11}\pi^{28}.$$

3.3. E_6. Let E_6 be the exceptional Lie group given by

$$E_6=\{\alpha\in \mathrm{Iso}_C(\mathfrak{J}^C)\,|\,\tau\alpha\tau(X\times Y)=\alpha X\times\alpha Y,\ \langle\alpha X, \alpha Y\rangle=\langle X, Y\rangle\}$$

and $EIII$ be the symmetric space defined by

$$EIII=\{X\in\mathfrak{J}^C\,|\,X\times X=0,\ X\neq 0\}/C^*.$$

The group E_6 acts naturally and transitively on $EIII$ and the isotropy subgroup $(E_6)_{[E_1]}$ at $[E_1]\in EIII$ is isomorphic to $(U(1)\times Spin(10))/\mathbf{Z}_4$. Hence we have that

$$E_6/(U(1)\times Spin(10))/\mathbf{Z}_4\cong EIII.$$

The Killing form B_6 of $\mathfrak{e}_6$ is given by

$$B_6(\phi_1, \phi_2)=4\,\mathrm{tr}(\phi_1\phi_2), \qquad \phi_1, \phi_2\in\mathfrak{e}_6.$$

Let g be an invariant metric on $EIII$ given by $g(X, Y)=\mathrm{Re}\langle X, Y\rangle$, $X, Y\in T_o(EIII)$. Then, by [1], the volume of $EIII$ with repect to g is given by

$$\mu(EIII, g)=\frac{78}{16!}\pi^{16}.$$

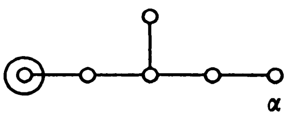
Since $B_6(\tilde{A}_2(1), \tilde{A}_2(1))=4\,\mathrm{tr}(\tilde{A}_2(1)^2)=-48$ and $g(\tilde{A}_2(1)(E_1), \tilde{A}_2(1)(E_1))=g(F_2(1), F_2(1))=(F_2(1), F_2(1))=1$, by Lemma 1 we have that

$$\mu_{E_6}(EIII)=\sqrt{48}^{\,32}\mu(EIII, g)=48^{16}\frac{78}{16!}\pi^{16}.$$

Let $\phi_*: \mathfrak{u}(1)\to\mathfrak{e}_6$ be a Lie algebra homomorphism induced by the inclusion $\phi: U(1)\to E_6$ (see [1]). Since $B_6(\phi_*(i), \phi(i))=4\,\mathrm{tr}(\phi_*(i)^2)=-288$, by Lemma 1 we have that

$$\mu_{E_6}(U(1))=\sqrt{288}\,2\pi.$$

Since $(\alpha, \alpha)_{\mathfrak{spin}(10)}=1/8$ and $(\alpha, \alpha)_{\mathfrak{e}_6}=1/12$ in the Dynkin diagram of $\mathfrak{e}_6$ below,



by Lemma 4 we have that

$$\mu_{E_6}(Spin(10))=\sqrt{12/8}^{\,45}\mu(Spin(10)).$$

Thus it follows from Lemma 3 and 3.2 that

$$\mu(E_6)=\tfrac{1}{4}(\mu_{E_6}(U(1))\,\mu_{E_6}(Spin(10)))\,\mu_{E_6}(EIII)=\frac{2^{134}\,3^{29}\sqrt{3}}{5^5\,7^3\,11}\pi^{42}\,.$$

3.4. E_7. Let E_7 be the exceptional Lie group given by

$$E_7=\{\alpha\in \mathrm{Iso}_C(\mathfrak{P}^C)\,|\,\alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q,\ \langle\alpha P,\alpha Q\rangle=\langle P,Q\rangle\}$$

and $EVII$ be the symmetric space defined by

$$EVII=\{P\in\mathfrak{P}^C\,|\,P\times P=0,\ P\neq 0\}/C^*\,.$$

The group E_7 acts naturally and transitively on $EVII$ and the isotropy subgroup $(E_7)_{[\dot{1}]}$ at $[\dot{1}]\in EVII$ is isomorphic to $(U(1)\times E_6)/\mathbf{Z}_3$. Hence we have that

$$E_7/(U(1)\times E_6)/\mathbf{Z}_3\cong EVII\,.$$

The Killing form B_7 of $\mathfrak{e}_7$ is given by

$$B_7(\Phi_1,\Phi_2)=3\,\mathrm{tr}(\Phi_1\Phi_2)\,,\qquad \Phi_1,\Phi_2\in\mathfrak{e}_7\,.$$

Let g be an invariant metric on $EVII$ such that $g(P,Q)=\mathrm{Re}\langle P,Q\rangle$, $P,Q\in T_{[\dot{1}]}(EVII)$. Then, by [1], the volume of $EVII$ with respect to g is given by

$$\mu(EVII,g)=\frac{13110}{27!}\pi^{27}\,.$$

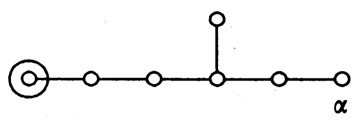
For $\Phi=\Phi(0,-E_1,E_1,0)$, $B_7(\Phi,\Phi)=3\mathrm{tr}(\Phi^2)=-72$ and $g(\Phi(\dot{1}),\Phi(\dot{1}))=(E_1,E_1)=1$. Hence by Lemma 1 we have that

$$\mu_{E_7}(EVII)=\sqrt{72}^{\,54}\mu(EVII,g)=72^{27}\frac{13110}{27!}\pi^{27}\,.$$

Let $\phi_*:\mathfrak{u}(1)\to\mathfrak{e}_7$ be a Lie algebra homomorphism induced by the inclusion $\phi: U(1)\to E_7$ (see [1]). Since $B_6(\phi_*(i),\phi_*(i))=3\,\mathrm{tr}(\phi_*(i)^2)=-216$, by Lemma 1 we have that

$$\mu_{E_7}(U(1))=\sqrt{216}\,2\pi\,.$$

Since $(\alpha,\alpha)_{\mathfrak{e}_6}=1/12$ and $(\alpha,\alpha)_{\mathfrak{e}_7}=1/18$ in the Dynkin diagram of $\mathfrak{e}_7$ below,



by Lemma 1 and Lemma 4 we have that

$$\mu_{E_7}(E_6)=\sqrt{18/12}^{\,78}\mu(E_6)\,.$$

Thus it follows from Lemma 3 and 3.3 that

$$\mu(E_7)=\tfrac{1}{3}(\mu_{E_7}(U(1)\mu_{E_7}(E_6)))\mu_{E_7}(EVII)=\frac{2^{156}\sqrt{2}\,3^{111}}{5^{10}\,7^6\,11^3\,13^2\,17}\pi^{70}\,.$$

3.5. E_8. Let E_8 be the exceptional Lie group defined by

$$E_8=\{\alpha\in \mathrm{Iso}_C\,\mathfrak{e}_8^C \mid \alpha[R_1, R_2]=[\alpha R_1, \alpha R_2], \langle\alpha R_1, \alpha R_2\rangle=\langle R_1, R_2\rangle\}$$

and $Z(EIX)$ be the twister space of the symmetric quaterinon-Kähler manifold EIX defined by

$$Z(EIX)=\{R\in\mathfrak{e}_8^C \mid R\times R=0, R\neq 0\}/C^*\,.$$

The group E_8 acts naturally and transitively on $Z(EIX)$ and the isotropy subgroup $(E_8)_{[1^-]}$ at $[1^-]\in Z(EIX)$ is isomorphic to $(U(1)\times E_7)/\mathbf{Z}_2$. Hence we have

$$E_8/(U(1)\times E_7)/\mathbf{Z}_2\cong Z(EIX)\,.$$

By [1], the tangent space $T_{[1^-]}(Z(EIX))$ at $[1^-]$ is isomorphic to $\mathfrak{P}^C\oplus C$. Let g be an invariant Kähler-Einstein metric on $Z(EIX)$ given by $g((P_1, s_1), (P_2, s_2))=\mathrm{Re}(\langle P_1, P_2\rangle+8(\tau s_1)s_2)$, $(P_1, s_1), (P_2, s_2)\in T_{[1^-]}(z(EIX))$. Then, by [1], the volume of $Z(EIX)$ with respect to the metric g is given by

$$\mu(Z(EIX), g)=\frac{2^{12}\,3^2\,5^2\,7\,31\,37\,41\,43\,47\,53}{57!}\pi^{57}\,.$$

On the other hand, the restriction of the Killing form B_8 of $\mathfrak{e}_8$ on $T_{[1^-]}(Z(EIX))$ is given by

$$\begin{aligned}B_8((P_1, s_1), (P_2, s_2))&=-30\,\mathrm{Re}\langle P_1, P_2\rangle-120\,\mathrm{Re}((\tau s_1)s_2)\\&=-30g(((P_1, 0), (P_2, 0))-15g((0, s_1), (0, s_2))\,.\end{aligned}$$

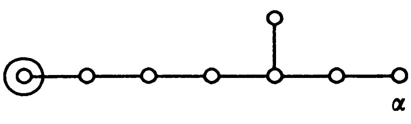
Hence the volume of $Z(EIX)$ with respect to the natural metric induced by the Killing form B_8 is

$$\mu_{E_8}(Z(EIX))=\sqrt{30}^{\,112}\sqrt{15}^{\,2}\mu(Z(EIX), g)\,.$$

Let $\phi_* : \mathfrak{u}(1)\to\mathfrak{e}_8$ be a Lie algebra homomorphism induced by the inclusion $\phi : U(1)\to E_8$ (see [1]). Since $B_8(\phi_*(i), \phi_*(i))=-120$, by Lemma 1 we have that

$$\mu_{E_8}(U(1))=\sqrt{120}\,2\pi\,.$$

Since $(\alpha, \alpha)_{\mathfrak{e}_7}=1/18$ and $(\alpha, \alpha)_{\mathfrak{e}_8}=1/30$ in the Dynkin diagram of $\mathfrak{e}_8$ below,



by Lemma 1 and Lemma 4 we have that

$$\mu_{E_8}(E_7)=\sqrt{30/18}^{\,133}\mu(E_7)\,.$$

Thus it follows from Lemma 3 and 3.3 that

$$\begin{aligned}\mu(E_8)&=\tfrac{1}{2}\mu_{E_8}(U(1))\mu_{E_8}(E_7))\mu_{E_8}(Z(EIX))\\&=\frac{2^{279}\,3^{77}\,5^{103}}{7^{14}\,11^{8}\,13^{6}\,17^{4}\,19^{3}\,23^{2}\,29}\pi^{128}\,.\end{aligned}$$

4. Volumes of compact classical symmetric spaces.

4.1. $AI=SU(n)/SO(n)$ $(n\geq 3)$. Since $B_{\mathfrak{so}(n)}(X, Y)=(n-2)\operatorname{tr}(XY)$, X, $Y\in\mathfrak{so}(n)$ and $B_{\mathfrak{su}(n)}(XY)=2n\operatorname{tr}(XY)$, X, $Y\in\mathfrak{su}(n)$, by Lemma 1 we have that

$$\mu_{SU(n)}(SO(n))=\sqrt{(2n)/(n-2)}^{\,n(n-1)/2}\mu(SO(n))\,.$$

Thus it follows from Lemma 3, 2.1 and 2.2 that

$$\begin{aligned}\mu(AI)&=\mu(SU(n))/\mu_{SU(n)}(SO(n))\\&=\frac{2^{(n-1)(n+3)/2}\,n^{n(n+1)/4}}{1!\,2!\cdots(n-1)!\,S_1S_2\cdots S_{n-1}}\pi^{(n-1)(n+2)/2}\,.\end{aligned}$$

4.2. $AII=SU(2n)/Sp(n)$. The inclusion $\varphi: Sp(n)\to SU(2n)$ is given by $\varphi(A+B\mathbf{j})=\begin{pmatrix}A & B\\ -\bar{B} & \bar{A}\end{pmatrix}$, $A, B\in M(n, \mathbf{C})$. For $D=\operatorname{diag}(\mathbf{i}, 0, \cdots, 0)\in\mathfrak{sp}(n)$, $B_{\mathfrak{sp}(n)}(D, D)=-4(n+1)$ and $B_{\mathfrak{su}(n)}(\varphi_*D, \varphi_*D)=-8n$. Hence by Lemma 1 we have that

$$\mu_{SU(2n)}(Sp(n))=\sqrt{(8n)/4(n+1)}^{\,n(2n+1)}\mu(Sp(n))\,.$$

Thus it follows from Lemma 3, 2.1 and 2.3 that

$$\mu(AII)=\mu(SU(n))/\mu_{SU(2n)}(Sp(n))=\frac{2^{(n-1)(4n+3)/2}\,n^{n(2n-1)/2}}{2!\,4!\cdots(2n-2)!}\pi^{n^2-1}\,.$$

4.3. $AIII=SU(m+n)/S(U(m)\times U(n))$. Let $\varphi: U(1)\times SU(m)\times SU(n)\to S(U(m)\times U(n))$ be a map defined by $\varphi(z, A, B)=\begin{pmatrix}z^{-n/d}A & 0\\ 0 & z^{m/d}B\end{pmatrix}$. Here $z\in U(1)$, $A\in SU(m)$, $B\in SU(n)$ and d is the greatest common factor of m and n. Then φ induces an isomorphism

$$(U(1)\times SU(m)\times SU(n))/\mathbf{Z}_{mn/d}\cong S(U(m)\times U(n))\,.$$

Hence by Lemma 1 and Lemma 2 we have that

$$\begin{aligned}&\mu_{SU(m+n)}(S(U(m)\times U(n)))\\&=\frac{d}{mn}\sqrt{\frac{2(m+n)^2mn}{d^2}}2\pi\sqrt{\frac{m+n}{m}}^{\,m^2-1}\mu(SU(m))\sqrt{\frac{m+n}{n}}^{\,n^2-1}\mu(SU(n))\,.\end{aligned}$$

Thus it follows from Lemma 3 and 2.1 that

$$\mu(AIII)=\mu(SU(m+n))/\mu_{SU(m+n)}(S(U(m)\times U(n)))$$
$$=2^{2mn}(m+n)^{mn}\frac{1!\,2!\cdots(m-1)!\,1!\,2!\cdots(n-1)!}{1!\,2!\cdots(m+n-1)!}\pi^{mn}\,.$$

4.4. $BDI=SO(m+n)/(SO(m)\times SO(n))$ $(m, n\geq 3)$. By Lemma 1 and Lemma 2 we have that

$$\mu_{SO(m+n)}(SO(m)\times SO(n))$$
$$=\sqrt{\frac{m+n-2}{m-2}}^{\,m(m-1)/2}\mu(SO(m))\sqrt{\frac{m+n-2}{n-2}}^{\,n(n-1)/2}\mu(SO(n))\,.$$

Thus it follows from Lemma 3 and 2.2 that

$$\mu(BDI)=\mu(SO(m+n))/\mu_{SO(m+n)}(SO(m)\times SO(n))$$
$$=(2(m+n-2))^{mn/2}\frac{S_1S_2\cdots S_{m+n-1}}{S_1S_2\cdots S_{m-1}S_1S_2\cdots S_{n-1}}\,.$$

4.5. $DIII=SO(2n)/U(n)$. Let $\varphi: U(1)\times SU(n)\to U(n)$ be a map defined by $\varphi(z, A)=zA$, $z\in U(1)$, $A\in SU(n)$. Then φ induces an isomorphism $(U(1)\times SU(n))/\mathbf{Z}_n\cong U(n)$. Hence, in the similar way as in 4.3 we have that

$$\mu_{SO(2n)}(U(n))=\frac{1}{n}\sqrt{(2n-2)2n}\,2\pi\sqrt{\frac{2(2n-2)}{2n}}^{\,n^2-1}\mu(SU(n))\,.$$

Thus it follows from Lemma 3 and 2.1 that

$$\mu(DIII)=\mu(SO(2n))/\mu_{SO(2n)}(U(n))$$
$$=2^{3n(n-1)/2}(n-1)^{n(n-1)/2}\frac{1!\,2!\cdots(n-2)!}{1!\,3!\cdots(2n-3)!}\pi^{n(n-1)/2}\,.$$

4.6. $CI=Sp(n)/U(n)$. Since $(U(1)\times SU(n))/\mathbf{Z}_n\cong U(n)$, in the similar way as in 4.5 we have that

$$\mu_{Sp(n)}U(n))=\frac{1}{n}\sqrt{4(n+1)n}\,2\pi\sqrt{\frac{4(n+1)}{2n}}^{\,n^2-1}\mu(SU(n))\,.$$

Thus it follows from Lemma 3, 2.1 and 2.3 that

$$\mu(CI)=\mu(Sp(n))/\mu_{Sp(n)}(U(n))$$
$$=2^{n(3n+1)/2}(n+1)^{n(n+1)/2}\frac{1!\,2!\cdots(n-1)!}{1!\,3!\cdots(2n-1)!}\pi^{n(n+1)/2}\,.$$

4.7. $CII=Sp(m+n)/(Sp(m)\times Sp(n))$. As in 4.5, we have that

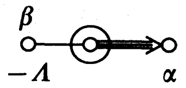
$$\mu_{Sp(m+n)}(Sp(m)\times Sp(n))$$
$$=\sqrt{\frac{4(m+n+1)}{4(m+1)}}^{\,m(2m+1)}\mu(Sp(m))\sqrt{\frac{4(m+n+1)}{4(n+1)}}^{\,n(2n+1)}\mu(Sp(n))\,.$$

Thus it follows from Lemma 3 and 2.3 that

$$\mu(CII)=\mu(Sp(m+n))/\mu_{Sp(m+n)}(Sp(m)\times Sp(n))$$
$$=2^{6mn}(m+n+1)^{2mn}\frac{1!\,3!\cdots(2m-1)!\;1!\,3!\cdots(2n-1)!}{1!\,3!\cdots(2m+2n-1)!}\pi^{2mn}\,.$$

5. Volumes of compact exceptional symmetric spaces.

5.1. $G=G_2/(Sp(1)\times Sp(1))/\mathbf{Z}_2$. Since $(\alpha,\alpha)_{\mathfrak{sp}(1)}=1/2$, $(\alpha,\alpha)_{\mathfrak{g}_2}=1/12$, and $(\beta,\beta)_{\mathfrak{sp}(1)}=1/2$, $(-\Lambda,-\Lambda)_{\mathfrak{g}_2}=1/4$ in the extended Dynkin diagram of $\mathfrak{g}_2$ below,



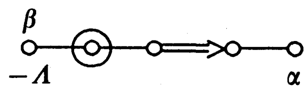
by Lemma 4 we have that

$$\mu_{G_2}((Sp(1)\times Sp(1))/\mathbf{Z}_2)=\tfrac{1}{2}\sqrt{4/2}^{\,3}\mu(Sp(1))\sqrt{12/2}^{\,3}\mu(Sp(1))\,.$$

Thus it follows from Lemma 3, 2.3 and 3.1 that

$$\mu(G)=\mu(G_2)/\mu_{G_2}((Sp(1)\times Sp(1))/\mathbf{Z}_2)=\frac{12^{13}\,3}{5}\pi^4\,.$$

5.2. $FI=F_4/(Sp(1)\times Sp(3))/\mathbf{Z}_2$. Since $(\alpha,\alpha)_{\mathfrak{sp}(3)}=1/8$, $(\alpha,\alpha)_{\mathfrak{f}_4}=1/18$ and $(\beta,\beta)_{\mathfrak{sp}(1)}=1/2$, $(-\Lambda,-\Lambda)_{\mathfrak{f}_4}=1/9$ in the extended Dynkin diagram of $\mathfrak{f}_4$ below,



by Lemma 4 we have that

$$\mu_{F_4}((Sp(1)\times Sp(3))/\mathbf{Z}_2)=\tfrac{1}{2}\sqrt{9/2}^{\,3}\mu(Sp(1))\sqrt{18/8}^{\,21}\mu(Sp(3))\,.$$

Thus it follows from Lemma 3, 2.3 and 5.2 that

$$\mu(FI)=\mu(F_4)/\mu_{F_4}((Sp(1)\times Sp(3))/\mathbf{Z}_2)=\frac{2^{23}\,3^{23}}{5^3\,7^2\,11}\pi^{14}\,.$$

5.3. $EI=E_6/Sp(4)/\mathbf{Z}_2$. Put $\mathfrak{J}(3,\mathbf{H}^C)=\{M\in M(3,\mathbf{H}^C)\,|\,M^*=M\}$. We identify $\mathfrak{J}(3,\mathbf{H}^C)\oplus(\mathbf{H}^C)^3$ with $\mathfrak{J}^C$ under the correspondence

$$M+\mathbf{a}\to M+\begin{pmatrix} 0 & a_3e_4 & -a_2e_4 \\ -a_3e_4 & 0 & a_1e_4 \\ a_2e_4 & -a_1e_4 & 0 \end{pmatrix},$$

where $\mathbf{a}={}^t(a_1, a_2, a_3)$, $a_i\in\mathbf{H}^C$. Put $\mathfrak{J}(4, \mathbf{H}^C)_0=\{P\in M(4, \mathbf{H}^C)\,|\,P^*=P, \mathrm{tr}(P)=0\}$. Let g: $\mathfrak{J}^C\to\mathfrak{J}(4, \mathbf{H}^C)_0$ be a map defined by

$$g(M+\mathbf{a})=\begin{pmatrix} \frac{1}{2}\mathrm{tr}(M) & i\mathbf{a} \\ i\mathbf{a}^* & M-\frac{1}{2}\mathrm{tr}(M)E \end{pmatrix}.$$

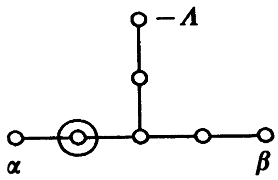
Then g is a C-linear isomorphism. Let $\varphi: Sp(4)\to E_6$ be a map defined by $\varphi(A)X=g^{-1}(A(gX)A^*)$, $X\in\mathfrak{J}^C$. Then φ induces the inclusion $Sp(4)/\mathbf{Z}_4 \hookrightarrow E_6$ and a Lie algebra homomorphism $\varphi_*: \mathfrak{sp}(4)\to\mathfrak{e}_6$. Put $D=\mathrm{diag}(\mathrm{i}, 0, 0, 0)\in\mathfrak{sp}(4)$ and $\phi=\varphi_*(D)$. Since $B_{\mathfrak{sp}(4)}(D, D)=-20$ and $B_6(\phi, \phi)=-48$, we have that

$$\mu_{E_6}(SP(4)/\mathbf{Z}_2)=\frac{1}{2}\sqrt{48/20}^{\,36}\mu(Sp(4)).$$

Thus it follows from Lemma 2, 2.3 and 3.3 that

$$\mu(EI)=\mu(E_6)/\mu_{E_6}(Sp(4)/\mathbf{Z}_2)=\frac{2^{55}\,3^{15}\sqrt{3}}{5^3\,7^2\,11}\pi^{22}.$$

5.4. $EII=E_6/(Sp(1)\times SU(6))/\mathbf{Z}_2$. Since $(\alpha, \alpha)_{\mathfrak{sp}(1)}=1/2$, $(\alpha, \alpha)_{\mathfrak{e}_6}=1/12$ and $(\beta, \beta)_{\mathfrak{su}(6)}=1/6$, $(-\Lambda, -\Lambda)_{\mathfrak{e}_6}=1/12$ in the extended Dynkin diagram of $\mathfrak{e}_6$ below,



by Lemma 4 we have that

$$\mu_{E_6}((Sp(1)\times SU(6))/\mathbf{Z}_2)=\frac{1}{2}\sqrt{12/2}^{\,3}\mu(Sp(1))\sqrt{12/6}^{\,35}\mu(SU(6)).$$

Thus it follows from Lemma 3, 2.1, 2.2 and 3.3 that

$$\mu(EII)=\mu(E_6)/\mu_{E_6}((Sp(1)\times SU(6))/\mathbf{Z}_2)=\frac{2^{63}\,3^{13}}{5^4\,7^3\,11}\pi^{20}.$$

5.5. $EIV=E_6/F_4$. The group E_6 has the subgroup F_4 as $F_4=\{\alpha\in E_6\,|\,\alpha E_1=E_1\}$. An element $\phi\in\mathfrak{e}_6$ has the expression as

$$\phi=\delta+i\tilde{T}, \qquad \delta\in\mathfrak{f}_4,\ T\in\mathfrak{J} \text{ with } \mathrm{tr}(T)=0.$$

The Killing form B_6 of $\mathfrak{e}_6$ is related to that of $\mathfrak{f}_4$ by the following equation:

$$B_6(\phi_1, \phi_2)=\tfrac{4}{3}B_4(\delta_1, \delta_2)-12(T_1, T_2)\,,$$

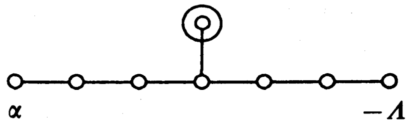
where $\phi_k=\delta_k+i\tilde{T}_k$, $\delta_k\in\mathfrak{f}_4$, $T_k\in\mathfrak{J}$ with $\mathrm{tr}(T_k)=0$. Hence by Lemma 1 we have that

$$\mu_{E_6}(F_4)=\sqrt{4/3}^{\,52}\,\mu(F_4)\,.$$

Thus it follows from Lemma 3, 3.2 and 3.3 that

$$\mu(EIV)=\mu(E_6)/\mu_{E_6}(F_4)=\frac{2^{30}\,3^{10}\sqrt{3}}{5\cdot 7}\pi^{14}\,.$$

5.6. $EV=E_7/SU(8)/\mathbf{Z}_2$. Since $(\alpha, \alpha)_{\mathfrak{su}(8)}=1/8$ and $(\alpha, \alpha)_{\mathfrak{e}_7}=1/18$ in the extended Dynkin diagram of $\mathfrak{e}_7$ below,



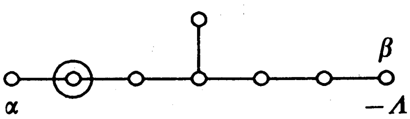
by Lemma 4 we have that

$$\mu_{E_7}(SU(8)/\mathbf{Z}_2)=\tfrac{1}{2}\sqrt{18/8}^{\,63}\,\mu(SU(8))\,.$$

Thus it follows from Lemma 3, 2.1 and 3.4 that

$$\mu(EV)=\mu(E_7)/\mu_{E_7}(SU(8)/\mathbf{Z}_2)=\frac{2^{74}\,3^{55}}{5^7\,7^5\,11^3\,13^2\,17}\pi^{35}\,.$$

5.7. $EVI=E_7/(SU(2)\times Spin(12))/\mathbf{Z}_2$. Since $(\alpha, \alpha)_{\mathfrak{su}(2)}=1/2$, $(\alpha, \alpha)_{\mathfrak{e}_7}=1/18$ and $(\beta, \beta)_{\mathfrak{spin}(12)}=1/10$, $(-\Lambda, -\Lambda)_{\mathfrak{e}_7}=1/18$ in the extended Dynkin diagram of $\mathfrak{e}_7$ below,



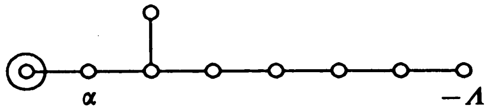
by Lemma 2 and Lemma 4 we have that

$$\mu_{E_7}((SU(2)\times Spin(12))/\mathbf{Z}_2)=\tfrac{1}{2}\sqrt{18/2}^{\,3}\,\mu(SU(2))\sqrt{18/10}^{\,66}\,\mu(Spin(12))\,.$$

Thus it follows from Lemma 3, 2.1 and 2.2 that

$$\mu(EVI)=\mu(E_7)/\mu_{E_7}((SU(2)\times Spin(12))/\mathbf{Z}_2)=\frac{2^{67}\,3^{51}}{5^6\,7^4\,11^3\,13^2\,17}\pi^{32}\,.$$

5.8. $EVIII=E_8/Ss(16)$. Since $(\alpha, \alpha)_{\mathfrak{so}(16)}=1/14$, $(\alpha, \alpha)_{\mathfrak{e}_8}=1/30$ in the extended Dynkin diagram of $\mathfrak{e}_8$ below,



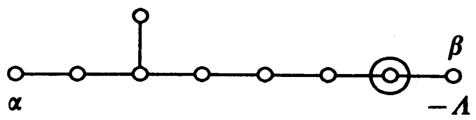
by Lemma 4 we have that

$$\mu_{E_8}(Ss(16))=\sqrt{30/14}^{\,120}\mu(Ss(16))=\sqrt{30/14}^{\,120}\mu(SO(16))\,.$$

Thus it follows from Lemma 3, 2.3 and 5.5 that

$$\mu(EVIII)=\mu(E_8)/\mu_{E_8}(Ss(16))=\frac{2^{132}\,3^{36}\,5^{51}}{7^9\,11^6\,13^6\,17^4\,19^3\,23^2\,29}\pi^{64}\,.$$

5.9. $EIX=E_8/(SU(2)\times E_7)/\mathbf{Z}_2$. Since $(\alpha,\alpha)_{\mathfrak{e}_7}=1/18$, $(\alpha,\alpha)_{\mathfrak{e}_8}=1/30$ and $(\beta,\beta)_{\mathfrak{su}(2)}=1/2$, $(-\Lambda,-\Lambda)_{\mathfrak{e}_8}=1/30$ in the extended Dynkin diagram of $\mathfrak{e}_8$ below,



by Lemma 2 and Lemma 4 we have that

$$\mu_{E_8}((SU(2)\times E_7)/\mathbf{Z}_2)=\frac{1}{2}\sqrt{30/2}^{\,3}\mu(SU(2))\sqrt{30/18}^{\,133}\mu(E_7)\,.$$

Thus it follows from Lemma 3, 3.1, 3.4 and 3.5 that

$$\mu(EIX)=\mu(E_8)/\mu_{E_8}((SU(2)\times E_7)/\mathbf{Z}_2)=\frac{2^{118}\,3^{31}\,5^{45}}{7^8\,11^5\,13^4\,17^3\,19^3\,23^3\,29}\pi^{56}\,.$$

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