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Conjugate Connections and Moduli Spaces of Connections

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1. Introduction.

Let G be a Lie group with Lie algebra $\mathbf{g}$, and H a closed subgroup with Lie algebra $\mathbf{h}$. Let $\mathrm{Aut}(G, H)$ be the group of automorphisms of G leaving all elements of H fixed, and $\mathrm{Inn}(G, H)$ its normal subgroup consisting of inner automorphisms.

Given a principal G-bundle P, let $\mathscr{C}(P)$ be the space of connections in P and $\mathscr{G}(P)$ the group of gauge transformations of P. The main purpose of this note is to prove the following statement, (see Theorem 2 for more details).

If the structure group G of P can be reduced to H, then the group $\mathrm{Aut}(G, H)/\mathrm{Inn}(G, H)$ *acts on the moduli space $\mathscr{C}(P)/\mathscr{G}(P)$, and the action is free on the generic part of $\mathscr{C}(P)/\mathscr{G}(P)$.*

2. Definitions and theorems.

In our previous paper [2] we introduced the concept of conjugate connection in principal bundles. We first recall its definition.

Let P be a principal G-bundle over a manifold M with projection π. Let Q be a principal H-subbundle of P; in general, such a subbundle Q may not exist. We cover M by open sets U_i with local sections s_i: $U_i \to Q$. Then the transition functions a_{ij} are defined by

$$s_j(x) = s_i(x)a_{ij}(x) \qquad x \in U_i \cap U_j .$$

It is important to take local sections of the subbundle Q so that the a_{ij}'s are H-valued.

A connection in P is given by a family of $\mathbf{g}$-valued 1-form $\{\omega_i\}$, where each ω_i is defined on U_i and the forms ω_i and ω_j are related by (see [1; Proposition 1.4 on p. 66])

$$\omega_j = a_{ij}^{-1}\omega_i a_{ij} + a_{ij}^{-1} da_{ij} \qquad \text{on} \quad U_i \cap U_j . \tag{1}$$

Given $\sigma \in \mathrm{Aut}(G, H)$, we set

$$\omega_i^\sigma = \sigma(\omega_i) .$$

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Apply σ to (1). Since a_{ij} is H-valued and hence invariant by σ, $\{\omega_i^\sigma\}$ defines a connection in P. We call it the *σ-conjugate of the connection $\{\omega_i\}$ relative to Q*. This defines an action of $\mathrm{Aut}(G, H)$ on the space $\mathscr{C}(P)$ of connections in P.

Given a gauge transformation $\varphi \in \mathscr{G}(P)$ and an automorphism $\sigma \in \mathrm{Aut}(G, H)$, we constructed a gauge transformation φ^σ so that $\mathrm{Aut}(G, H)$ acts on $\mathscr{G}(P)$ as an automorphism group. We recall the definition. A gauge transformation φ of P is a transformation of P commuting with the right action of G and inducing the identity transformation on the base space M. With respect to a local section s_i of Q we express φ by a map $\varphi_i: U_i \to G$ as follows:

(2) $$\varphi(s_i(x)) = s_i(x)\varphi_i(x) \qquad x \in U_i .$$

Then

(3) $$\varphi_j(x) = a_{ij}(x)^{-1}\varphi_i(x)a_{ij}(x) \qquad x \in U_i \cap U_j .$$

Conversely, a family $\{\varphi_i\}$ which is related by (3) defines a gauge transformation φ of P.

Applying $\sigma \in \mathrm{Aut}(G, H)$ to (3), we obtain

$$\sigma(\varphi_j(x)) = a_{ij}(x)^{-1}\sigma(\varphi_i(x))a_{ij}(x) \qquad x \in U_i \cap U_j .$$

Therefore, the family $\{\sigma(\varphi_i)\}$ defines a gauge transformation of P, which we denote by φ^σ.

Given a connection form $\{\omega_i\}$ on P, a gauge transformation $\varphi = \{\varphi_i\}$ induces a new connection $\{\theta_i\}$ by

(4) $$\theta_i = \varphi_i^{-1}\omega_i\varphi_i + \varphi_i^{-1}d\varphi_i .$$

In [2] we proved

THEOREM 1. *If two connections $\{\omega_i\}$ and $\{\theta_i\}$ in P are gauge equivalent under φ, then their σ-conjugates $\{\omega_i^\sigma\}$ and $\{\theta_i^\sigma\}$ are gauge equivalent under φ^σ. Thus, the group* $\mathrm{Aut}(G, H)$ *acts on the moduli space of connections* $\mathscr{C}(P)/\mathscr{G}(P)$.

It was pointed out by M. Itoh that $\mathrm{Inn}(G, H)$ acts trivially on $\mathscr{C}(P)/\mathscr{G}(P)$. Thus $\mathrm{Aut}(G, H)/\mathrm{Inn}(G, H)$ acts on $\mathscr{C}(P)/\mathscr{G}(P)$. The purpose of this note is to show that this action is generically free in the following sense.

Fix a point $u_0 \in Q$, and let Ψ_{u_0} be the holonomy group of the connection ω with respect to the reference point u_0. We call a connection in P *generic* if its holonomy group coincides with G.

THEOREM 2. *Let $\sigma \in \mathrm{Aut}(G, H)$ and $\{\omega_i\} \in \mathscr{C}(P)$. Assume that $\{\omega_i^\sigma\}$ is gauge equivalent to $\{\omega_i\}$ under a gauge transformation φ. If we define an element $a \in G$ by $\varphi(u_0) = u_0 a$, then,*

$$\sigma(g) = a^{-1}ga \qquad \text{for} \quad g \in \Psi_{u_0} .$$

In particular, if the holonomy group is G, then σ is the inner automorphism defined by

a^{-1} *above.*

As a consequence, $\mathrm{Aut}(G, H)/\mathrm{Inn}(G, H)$ *acts freely on the generic part of* $\mathscr{C}(P)/\mathscr{G}(P)$.

3. Proof of Theorem 2.

We shall first show that if $\sigma \in \mathrm{Aut}(G, H)$ is inner so that

$$\sigma(g) = a^{-1}ga, \qquad g \in G,$$

then there is a gauge transformation φ_a such that a connection $\{\omega_i\}$ and its σ-conjugate $\{\omega_i^\sigma\}$ are gauge equivalent under φ_a.

Since σ leaves every element of H fixed, a commutes with every element of H. Define a constant map $\varphi_i: U_i \to G$ by

$$\varphi_i(x) = a, \qquad x \in U_i.$$

Then $\{\varphi_i\}$ satisfies (3) and defines a gauge transformation of P, which we call φ_a. Then from (4) it is evident that φ_a sends $\{\omega_i\}$ to $\{\omega_i^\sigma\}$. This proves that $\mathrm{Aut}(G, H)/\mathrm{Inn}(G, H)$ acts on $\mathscr{C}(P)/\mathscr{G}(P)$.

Before we start the proof of Theorem 2, we first need to express a connection $\{\omega_i\}$ by a globally defined $\mathbf{g}$-valued 1-form ω on P. Identifying $U_i \times G$ with $\pi^{-1}(U_i)$ by

$$(x, g) \longmapsto s_i(x)g, \qquad x \in U_i, \; g \in G,$$

we use (x, g) as a local coordinate for P. Then

$$\omega = g^{-1}\omega_i g + g^{-1}dg, \tag{5}$$

so that $\omega_i = s_i^*(\omega)$.

Every element $u \in P$ is of the form $u = s_i(x)g$, $g \in G$. Given $\sigma \in \mathrm{Aut}(G, H)$, define a transformation h_σ of P by

$$h_\sigma(s_i(x)g) = s_i(x)\sigma(g), \qquad g \in G. \tag{6}$$

As a transformation of $U_i \times G$, h_σ is given by

$$h_\sigma: (x, g) \longmapsto (x, \sigma(g)). \tag{7}$$

We note that $\sigma(\omega)$ is not, in general, a connection in P, let alone the σ-conjugate of ω. The σ-conjugate ω^σ of ω is given by

$$\omega^\sigma = (h_\sigma^{-1})^*(\sigma(\omega)) = \sigma(((h_\sigma^{-1})^*\omega)). \tag{8}$$

In fact, by (5) and (7)

$$\omega^\sigma = g^{-1}\sigma(\omega_i)g + g^{-1}dg = (h_\sigma^{-1})^*(\sigma(g^{-1}\omega_i g + g^{-1}dg)) = (h_\sigma^{-1})^*(\sigma(\omega)).$$

We quickly recall the definition of the holonomy group. Fix points x_0 in M and $u_0 \in Q \subset P$ such that $\pi(u_0) = x_0$. Given a curve $x(t)$, $0 \le t \le 1$, in M with $x(0) = x_0$, the parallel displacement of u_0 along $x(t)$ is a curve $u(t)$ in P such that

(a) $\pi(u(t))=x(t)$ for $0\leq t\leq 1$,

(b) The velocity vector $u'(t)$ of $u(t)$ is horizontal, i.e., $\omega(u'(t))=0$,

(c) $u(0)=u_0$.

A curve $u(t)$ satisfying (a) and (b) is called a horizontal lift of $x(t)$.

Assume that $c=x(t)$, $0\leq t\leq 1$, is a closed curve, i.e., $x(0)=x(1)=x_0$. Let $\tilde{c}=u(t)$ be the parallel displacement of u_0 along $x(t)$. Then $u(0)=u_0$ and $u(1)$ are in the same fiber of P. So, there is a unique element τ_c of G such that $u(1)=u_0\tau_c$. Consider all piecewise smooth closed curves c starting from and ending at x_0. Then the set of all τ_c forms a subgroup of G. This group, denoted by Ψ_{u_0}, is the holonomy group with reference to u_0.

LEMMA 1. *Let $\sigma\in \mathrm{Aut}(G,H)$. If a curve $u(t)$ in P is horizontal with respect to a connection ω, i.e., $\omega(u'(t))=0$, then $h_\sigma(u(t))$ is horizontal with respect to ω^σ.*

PROOF. By (8),

$$\begin{aligned}\omega^\sigma(h_\sigma(u'(t)))&=((h_\sigma^{-1})^*(\sigma(\omega))(h_\sigma(u'(t)))=(\sigma(\omega))(h_\sigma^{-1}(h_\sigma(u'(t)))\\&=(\sigma(\omega))(u'(t))=\sigma(\omega(u'(t)))=0\,.\end{aligned}$$

The following two lemmas are even more trivial.

LEMMA 2. *Let φ be the gauge transformation of P. If a curve $u(t)$ is horizontal with respect to ω, then the curve $\varphi^{-1}(u(t))$ is horizontal with respect to $\varphi^*\omega$, i.e., $(\varphi^*\omega)(\varphi^{-1}(u(t)))=0$.*

LEMMA 3. *If two curves $v(t)$ and $w(t)$ of P are horizontal lifts of a curve $x(t)$ in M with respect to a connection θ, then there is a constant element $a\in G$ such that*

$$w(t)=v(t)a \qquad \textit{for all} \quad t\,.$$

Using these lemmas we shall now complete the proof of Theorem 2.

Assuming that ω^σ is gauge equivalent to ω under φ, we set $\theta=\omega^\sigma=\varphi^*\omega$. Let $c=x(t)$, $0\leq t\leq 1$, be a closed curve in M. Let $u(t)$ be the horizontal lift of $x(t)$ with respect to ω such that $u(0)=u_0\in Q$.

Then, by Lemma 2, $(\varphi^{-1}(u(t)))$ is horizontal with respect to θ, and by Lemma 1, $h_\sigma(u(t))$ is horizontal with respect to θ. Since $\varphi^{-1}(u(t))$ and $h_\sigma(u(t))$ are lifts of $x(t)$,

$$h_\sigma(u(t))=\varphi^{-1}(u(t))a(t) \qquad \text{with} \quad a(t)\in G\,.$$

By Lemma 3, $a(t)$ is a constant element, say a, of G. Hence,

$$h_\sigma(u(t))=\varphi^{-1}(u(t))a\,.$$

Setting $t=0$, 1, we have

$$\begin{aligned}h_\sigma(u(0))&=\varphi^{-1}(u(0))a\,,\\h_\sigma(u(1))&=\varphi^{-1}(u(1))a\,.\end{aligned}$$

Let $\tau_c \in G$ be the holonomy element defined by c, i.e., $u(1)=u(0)\tau_c$. Since $u(0)=u_0 \in Q$ is fixed by h_σ, we have

$$h_\sigma(u(1))=h_\sigma(u(0)\tau_c)=u(0)\sigma(\tau_c)\,.$$

On the other hand, since the gauge transformation φ^{-1} commutes with the right action of G, we have

$$\begin{aligned}u(0)\sigma(\tau_c)&=h_\sigma(u(1))=\varphi^{-1}(u(1))a=\varphi^{-1}(u(0)\tau_c)a\\&=(\varphi^{-1}(u(0)))\tau_c a=h_\sigma(u(0))a^{-1}\tau_c a=u(0)a^{-1}\tau_c a\,.\end{aligned}$$

Hence, $\sigma(\tau_c)=a^{-1}\tau_c a$. Since every element g of Ψ_{u_0} is of the form τ_c for some c, we have $\sigma(g)=a^{-1}ga$ for all $g \in \Psi_{u_0}$, completing the proof of Theorem 2.

4. Example.

Consider the symmetric pair $(SO(r),\ SO(p)\times SO(q))$, $p+q=r$, which defines the Grassmann manifold of oriented p-planes in an r-dimensional real vector space. Since the case p or q is 2 is a little exceptional in that the Grassmann manifold is a hyperquadric and a Hermitian symmetric space, we assume that neither p nor q is 2. Then $\mathrm{Aut}(SO(r), SO(p)\times SO(q))$ consists of two elements, namely, the identity and the symmetry σ given by

$$\sigma: X \longmapsto I_{p,q}^{-1} X I_{p,q}\,,$$

where

$$I_{p,q}=\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

The automorphism σ of $SO(r)$ is inner if at least one of p, q is even. (If p is even and q is odd, use $-I_{p,q} \in SO(r)$ to define σ.)

We shall show that it is not inner if both p and q are odd. Let X be an element of the Lie algebra $\mathbf{so}(r)$. Since r is even, $\det(X)=p(X)^2$, where $p(X)$ is the Pfaffian of X. We have

$$p(\sigma(X))=p({}^tI_{p,q}XI_{p,q})=\det(I_{p,q})p(X)=-p(X)\,.$$

If σ is an inner automorphism given by some element $A \in SO(r)$, then

$$p(\sigma(X))=p({}^tAXA)=\det(A)p(X)=p(X)\,,$$

which is a contradiction.

Thus, if P is an $SO(r)$-bundle that is reducible to an $SO(p)\times SO(q)$-subbundle Q, then the group $\mathbf{Z}_2$ acts on the moduli space $\mathscr{C}(P)/\mathscr{G}(P)$ in such a way that its action on the generic part is free.

As we pointed out in [2], $\mathrm{Aut}(G, H)$ acts also on the moduli space of Yang-Mills connections, and the statement above holds also as an action on the moduli space of Yang-Mills connections.

References

[1] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry, vol. 1*, Wiley (1963).
[2] S. KOBAYASHI and E. SHINOZAKI, Conjugate connections in principal bundles, *Geometry and Topology of Submanifolds, VII, (in honor of K. Nomizu)*, World Scientific Publ., to appear.

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