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The Topological Symmetry Group of a Canonically Embedded Complete Graph in S^3

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Abstract. We show that the topological symmetry group of a canonically embedded complete graph of $n\geq 7$ vertices in the 3-sphere is isomorphic to a dihedral group of order $2n$.

1. Introduction.

Throughout this paper graphs are assumed to be finite and simple. The *topological symmetry group* of an embedded graph in the three-sphere S^3 was introduced by Jonathan Simon on his lecture at Tokyo in June 1991. On the other hand Takashi Otsuki defined a *canonical embedding* of a complete graph K_n of n vertices into S^3 [2] [8]. The purpose of this paper is to show that the topological symmetry group of a canonical embedding of K_n is isomorphic to a dihedral group D_n of order $2n$ for $n\geq 7$.

Let $V(G)$ be the set of the vertices of G. Let $\mathrm{Aut}(G)$ be the automorphism group of G. Namely

$$\mathrm{Aut}(G)=\{h\colon V(G)\to V(G)\mid h \text{ is a bijection preserving the adjacency of the vertices}\}\,.$$

Let $f\colon G\to S^3$ be an embedding. Then the topological symmetry group of f, denoted by $\mathrm{TSG}(f)$, is a subgroup of $\mathrm{Aut}(G)$ defined by

$$\mathrm{TSG}(f)=\{h\in\mathrm{Aut}(G)\mid \text{there is a homeomorphism } \varphi\colon S^3\to S^3 \text{ with } \varphi(f(G))=f(G) \text{ such that } f\circ h=\varphi\circ f|_{V(G)}\}\,.$$

We remark that φ is not necessarily orientation preserving. Thus our definition of $\mathrm{TSG}(f)$ is somewhat different from that in [7] and [9].

Let $P_1, P_2, \cdots, P_m$ be smoothly embedded disks in S^3 such that $P_i\cap P_j=\partial P_i=\partial P_j$ for $1\leq i<j\leq m$. Let $B_m=\bigcup_{i=1}^m P_i$ and we call it an *m-bud*. We further assume that the disks are arranged in this order in S^3. Namely $P_i\cup P_{i+1}$ bounds a 3-ball Q_i in S^3 such that $Q_i\cap B_m=\partial Q_i=P_i\cup P_{i+1}$, here we consider the suffixes modulo m, i.e. $m+1=1$.

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We set $\partial P_1 = \partial P_2 = \cdots = \partial P_m = C$. An *n-cycle* is a graph with n vertices that is homeomorphic to a circle. An *n-cycle* of a graph G is a subgraph of G that is an n-cycle. Let K_n be the complete graph on $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. We consider the suffixes modulo n. Let C_n be an n-cycle of K_n consisting of the edges joining v_i and v_{i+1}. First we consider the case that $n = 2k$ for some integer k. Let f_n: $K_n \to B_k \subset S^3$ be an embedding illustrated in Fig. 1.1 where $\hat{v}_i = f_n(v_i)$. Next we consider the case $n = 2k+1$ for some integer k. Let f_n: $K_n \to B_{k+1} \subset S^3$ be an embedding illustrated in Fig. 1.2. Then we say that the embedding f_n: $K_n \to S^3$ is a *canonical bud presentation* of K_n with respect to the cycle C_n.

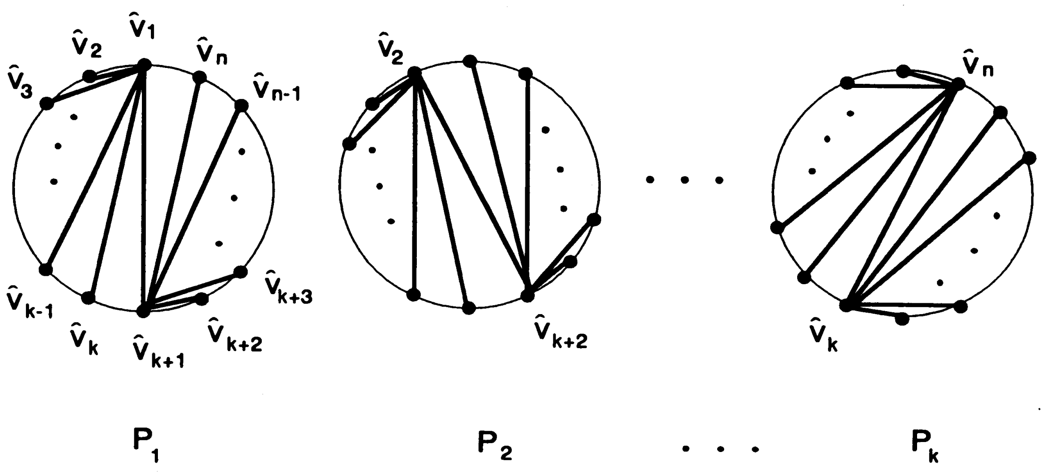

FIGURE 1.1

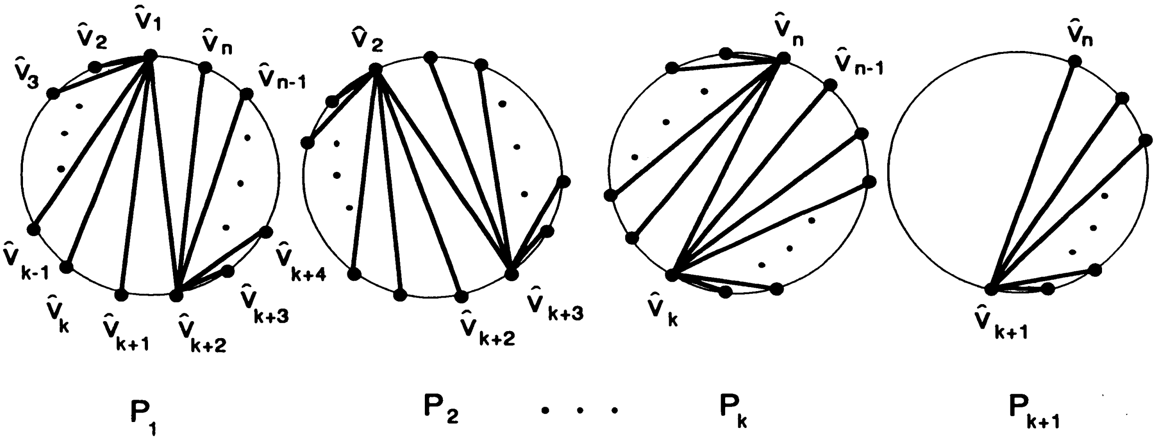

FIGURE 1.2

Let $D_n = \mathrm{Aut}(C_n) \subset \mathrm{Aut}(K_n)$ be the dihedral group of order $2n$. Then we have the following main theorem.

THEOREM 1.1. *Suppose that* $n \geq 7$, *then* $\mathrm{TSG}(f_n) = D_n$.

In section 2 we will give a proof of Theorem 1.1.

We note that a bud in S^3 is a one-point compactification of a *book* in R^3. Therefore a bud presentation is a *book presentation* in the sense of [2] [5] [6] [7] [8] etc. Our f_n is a *left canonical book presentation* in [8]. A *right canonical book presentation* is obtained from a left one by an orientation reversing homeomorphism of S^3. Therefore they have isomorphic topological symmetry groups.

In section 3 we consider bud presentations with $n \leq 7$.

We refer the reader to [3] and [4] for related results on topological symmetries of complete graphs in S^3.

2. Proof of Theorem 1.1.

We divide the proof of Theorem 1.1 into the following two lemmas.

LEMMA 2.1. $\mathrm{TSG}(f_n) \supset D_n$.

LEMMA 2.2. $\mathrm{TSG}(f_n) \subset D_n$.

PROOF OF LEMMA 2.1. Let $\rho: V(K_n) \to V(K_n)$ be a bijection defined by $\rho(v_i) = v_{i+1}$. Let $\tau: V(K_n) \to V(K_n)$ be a bijection defined by $\tau(v_i) = v_{n+2-i}$. Then $D_n = \mathrm{Aut}(C_n)$ is generated by ρ and τ. Therefore it is sufficient to show that $\rho, \tau \in \mathrm{TSG}(f_n)$. First we consider the case $n = 2k$. Then ρ is realized by a $2\pi/n$ rotation of S^3 along C followed by a $2\pi/k$ rotation of S^3 around C. By a π rotation of S^3 around the edge $\hat{v}_1 \hat{v}_{k+1}$ we have an embedding illustrated in Fig. 2.1.

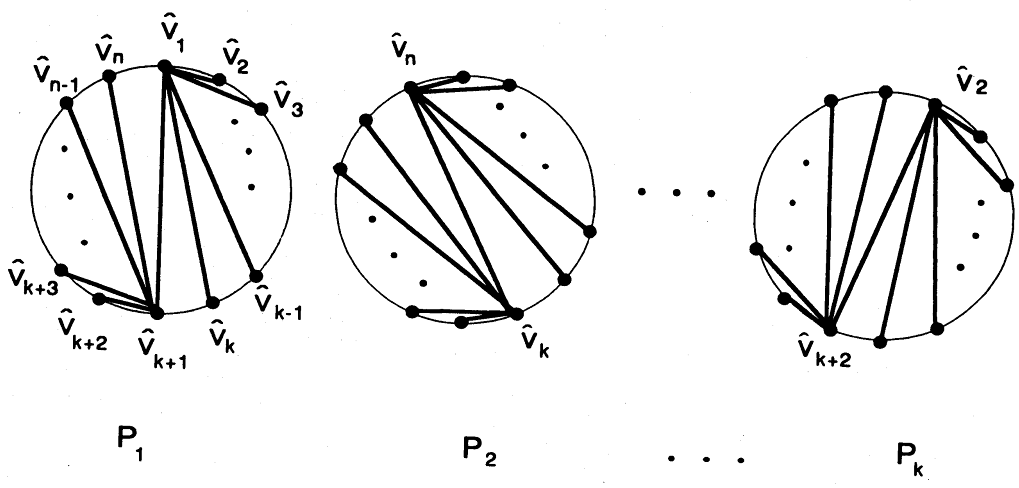

FIGURE 2.1

By the result of Otsuki [2] [8] we have that the image of this embedding is deformed into that of f_n by an ambient isotopy of S^3 fixing the vertices. Thus τ is realized by a homeomorphism of S^3. The case $n=2k+1$ is similar. But we need an additional deformation. That is a translation of half of the edges in P_i into P_{i+1}. More precisely, suppose that P_{i+1} contains just k edges as P_{k+1} in Fig. 1.2. Then some k edges in P_i are transformed into P_{i+1} by an ambient isotopy fixing the vertices. Now the proof is analogous. We omit the details. □

Let $\{v_{i_1}, v_{i_2}, \cdots, v_{i_l}\}$ be a subset of $V(K_n)=\{v_1, v_2, \cdots, v_n\}$ such that $i_1 < i_2 < \cdots < i_l$. Let C_l be a cycle consisting of the edges joining v_{i_j} and $v_{i_{j+1}}$. Let K_l be a subgraph of K_n induced by $\{v_{i_1}, v_{i_2}, \cdots, v_{i_l}\}$. Then K_l is a complete graph of l vertices. It is shown in [8] that $f_n|_{K_l}$ is ambient isotopic to a canonical bud presentation of K_l with respect to C_l.

Proof of Lemma 2.2. Case 1. $n=7$. It is not hard to check that $f_7(K_7)$ contains just one nontrivial knot as illustrated in Fig. 2.2, cf. [8].

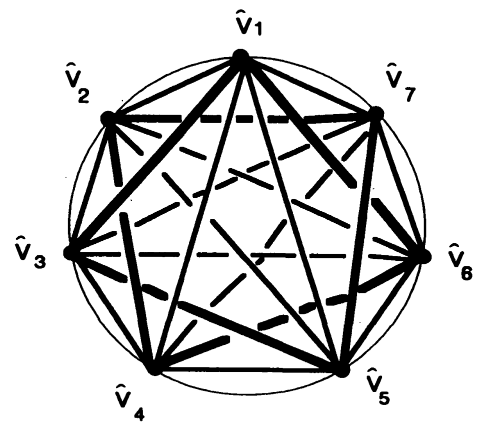

Figure 2.2

Therefore TSG(f_7) is a subgroup of Aut(C'_7) where C'_7 is the cycle consisting of the edges joining v_i and v_{i+2}. Since Aut$(C'_7)=$Aut(C_7) we have the result.

Case 2. $n=8$. Let H be a subgraph of K_8 consisting of the edges of C_8 and the edges joining v_i and v_{i+4}. Then by the results mentioned above we have that no edges in $f_8(H)$ are contained in a nontrivially knotted 7-cycle and other edges are contained in a knotted 7-cycle. Therefore we have TSG$(f_8) \subset$ Aut(H). It is easy to see Aut$(H)=$ Aut(C_8).

Case 3. $n \geq 9$. Similarly we have that an edge in $f_n(K_n)$ is on a knotted 7-cycle if and only if the edge is not on $f_n(C_n)$. Thus TSG$(f_n) \subset$ Aut(C_n). □

3. Minimal bud presentations.

An embedding $f: K_n \to B_m \subset S^3$ is called a *bud presentation* if $f^{-1}(C)=V(K_n)$. It is shown in [1] that $m \geq n/2$ is a necessary and sufficient condition for the existence of a bud presentation $f: K_n \to B_m$. A bud presentation is called *minimal* if $n = 2m$ or $n = 2m - 1$. We remark here that in case $n = 2m - 1$ our minimal bud presentation is slightly different from a minimal book presentation in [2] and [8]. Suppose that $n \leq 6$. Then $m \leq 3$. Since P_i and P_{i+1} are transformed into each other by an orientation reversing homeomorphism of S^3 fixing P_{i+2} we have that a minimal bud presentation is a canonical bud presentation when $n \leq 6$, cf [2].

The following results are shown by Yoshimatsu and Toba respectively.

THEOREM 3.1 [9]. *Let* $n \leq 5$. *Let* $f_n: K_n \to S^3$ *be a minimal (hence canonical) bud presentation. Then* $\mathrm{TSG}(f_n) = \mathrm{Aut}(K_n) \cong S_n$ *where* S_n *is the symmetric group on n points.*

SKETCH PROOF. The case $n \leq 4$ is easy. We can view $f_5(K_5)$ as a 1-skeleton of a 4-simplex where S^3 is viewed as the boundary of the 4-simplex. Thus we have $\mathrm{TSG}(f_5) = \mathrm{Aut}(K_5)$. □

THEOREM 3.2 [7]. *Let* $f_6: K_6 \to S^3$ *be a minimal (hence canonical) bud presentation. Then* $\mathrm{TSG}(f_6)$ *is isomorphic to* $S_2[S_3]$ *where* $S_2[S_3]$ *is the automorphism group of a disjoint union of two 3-cycles.*

SKETCH PROOF. The image $f_6(K_6)$ contains just one Hopf link of a disjoint union of two 3-cycles. We can see that other edges are placed in a symmetric mannar with respect to this Hopf link. □

EXAMPLE 3.3. Let $f: K_7 \to S^3$ be a minimal bud presentation illustrated in Fig. 3.1. Then f is not a canonical bud presentation with respect to any 7-cycle. In fact $f(K_7)$ contains a 6-cycle trefoil $\hat{v}_1 \hat{v}_4 \hat{v}_7 \hat{v}_5 \hat{v}_3 \hat{v}_6 \hat{v}_1$.

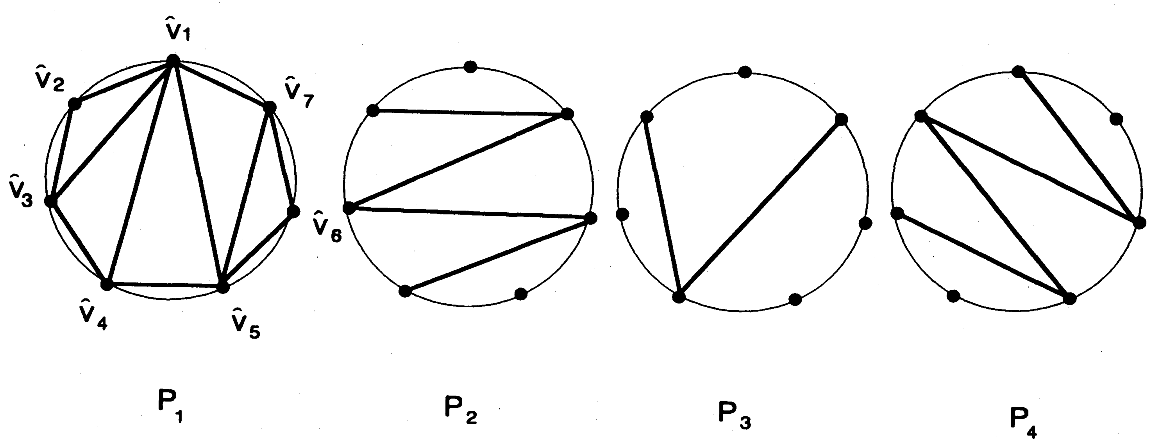

FIGURE 3.1

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