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Asymptotics of Scattering Phases for the Schrödinger Operator with Magnetic Fields

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1. Introduction.

In this paper we shall consider the asymptotics of scattering phases of Schrödinger equation with magnetic fields. The equation is described as follows:

$$\sum_{j=1}^{d} \{i\partial_j + b_j(x)\}^2 u + q(x)u = \lambda u .$$

In particular we do *not* assume that the scalar potential $q(x)$ and the vector potential $b_j(x)$ are spherically symmetric. D. R. Yafaev defined in [8] the scattering phases of Schrödinger equation

$$-\Delta u + q(x)u = \lambda u ,$$

which has the scalar potential $q(x)$ *without* spherical symmetry, and he studied the asymptotics. One of his results is that the asymptotics of scattering phases depend on the asymptotics of *the even part* $q_e(x)$ *of the scalar potential* (i.e., $q_e(x) = (q(x) + q(-x))/2$). So we shall extend the result to the case for Schödinger equation with magnetic fields.

Yafaev gave a definition of scattering phases related with the eigenvalue of the modified scattering matrix $\Sigma(\lambda)$. $\Sigma(\lambda)$ is defined by $S(\lambda)J$ where $S(\lambda)$ is a scattering matrix and J is a reflection operator. In fact, the scattering phases defined by D. R. Yafaev make sense in physical point of view (cf. [9]). In a similar way we can also define scattering phases of Schrödinger equation with magnetic fields. In our case we can find that the asymptotics of scattering phases depend on the asymptotics of *the even part* $q_e(x)$ *of scalar potential* and *the odd part* $b_{j,o}$ *of the vector potential* (i.e., $b_{j,o}(x) = (b_j(x) - b_j(-x))/2$).

This paper is organized as follows. In section 2 we give the correct definition of the scattering phases related to the eigenvalues of the operator $\Sigma(\lambda)$. And we give main theorems without proofs. In section 3 some properties of compact operators are given. In particular the properties of singular values play an important role in this paper. In

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section 4 we give a stationary representation of scattering matrix $S(\lambda)$, and some results obtained by M. Sh. Birman and D. R. Yafaev ([2], [3], [8]). In section 5 we define the operator B by using the operator $\Sigma(\lambda)$ and consider the asymptotics of its eigenvalues. By considering the construction of B, the asymptotics of the scattering phases of $\Sigma(\lambda)$ reduce to the asymptotics of the eigenvalues of the operator B. In section 7 we give the proofs of main theorems.

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2. Main theorems.

Let Ξ^d be the dual space of $\mathbf{R}^d$ and $\mathbf{S}^{d-1}$ the unit sphere in Ξ^d. Let Λ_ω be a $(d-1)$-dimensional subspace in Ξ^d which is orthogonal to $\omega \in \mathbf{S}^{d-1}$. Assume that in case $d>2$ the sphere $\mathbf{S}_\omega^{d-2} \equiv \mathbf{S}^{d-1} \cap \Lambda_\omega$ is endowed with $(d-2)$-surface measure and in case $d=2$ the set $\mathbf{S}_\omega^{d-2}$ consists of two points which have measure 1. Let $\mathscr{H}=L^2(\mathbf{R}^d)$, $\mathbf{H}=L^2(\mathbf{S}^{d-1})$. We denote the function $(\partial_j f)(x)$ by $(\partial_j f)$ and the operator $\partial_j(f\times\cdot)$ by $\partial_j f$. We denote $x=r\omega$, $r=|x|$, $\omega\in\mathbf{S}^{d-1}$.

ASSUMPTION (V). *We assume that the functions $q(x)$, $b_j(x)$ and $(\partial_j b_j)(x)$ satisfy*

$$(2.1)\qquad \begin{cases} |q(x)|\le C(1+|x|)^{-\beta}, & q=\bar{q}, \\ |b_j(x)|\le C(1+|x|)^{-\beta}, & b_j=\bar{b}_j, \\ |(\partial_j b_j)(x)|\le C(1+|x|)^{-\beta}, & \beta>1,\ x\in\mathbf{R}^d,\ d\ge 2. \end{cases}$$

In this paper we shall consider operators in $\mathscr{H}$ as follows:

$$H_0=-\Delta, \qquad V=\sum_{j=1}^{d}\{i\partial_j b_j(x)+ib_j(x)\partial_j+b_j^2(x)\}+q(x),$$

$$H=H_0+V=\sum_{j=1}^{d}\{i\partial_j+b_j(x)\}^2+q(x).$$

Now we denote the resolvents of the operators H_0 and H by $R_0(z)=(H_0-z)^{-1}$, $\Im z\neq 0$ and $R(z)=(H-z)^{-1}$, $\Im z\neq 0$, respectively.

Before we describe main theorems, we shall give some definitions and lemmas. (Proofs of the theorems and the lemmas will be given in section 7 and section 8, respectively.)

Let $S(\lambda)$ be a scattering matrix (the precise definition of the scattering matrix will be given by (4.2)) related with the operators H_0 and H. Let J be an operator defined by $(Jf)(\omega)=f(-\omega)$, $f\in\mathbf{H}$.

LEMMA 2.1. *The spectrum of J consists of the eigenvalues ± 1, and the eigenspace corresponding to the eigenvalue ± 1 is $\mathbf{H}_\pm$, respectively (i.e., $\mathbf{H}_+$ is the subspace of the even functions, and $\mathbf{H}_-$ is that of the odd functions). Moreover*

(2.2) $$P_{\pm}=2^{-1}(I\pm J)$$

is the orthogonal projection in $\mathbf{H}$ *onto the subspace* $\mathbf{H}_{\pm}$, *respectively.*

DEFINITION 2.2. We define a unitary operator $\Sigma(\lambda)$ as follows:

(2.3) $$\Sigma(\lambda)=S(\lambda)J\,.$$

LEMMA 2.3. *The spectrum of* $\Sigma(\lambda)$ *consists of eigenvalues with finite multiplicity. Moreover the eigenvalues accumulate only at the points* $+1$, -1.

We shall describe scattering phases defined by D. R. Yafaev ([8]).

DEFINITION 2.4 (cf. [8]). We shall denote eigenvalues of $\Sigma(\lambda)$ accumulating at $+1$ by

$$\exp(\mp 2i\delta_n^{\pm})\,,\qquad 0<\delta_n^{\pm}\le\pi/4\,,\quad \delta_{n+1}^{\pm}\le\delta_n^{\pm}$$

and eigenvalues of $\Sigma(\lambda)$ accumulating at -1 by

$$-\exp(\mp 2i\eta_n^{\pm})\,,\quad 0<\eta_n^{\pm}<\pi/4\,,\quad \eta_{n+1}^{\pm}\le\eta_n^{\pm}\,.$$

Then we call $\delta_n^{\pm}$ and $\eta_n^{\pm}$ *scattering phases.*

Under the above preliminaries, we shall give two main theorems.

THEOREM A. *Suppose* $\beta>(\alpha+1)/2$ *and Assumption* (V). *We shall denote the even part of* $q(x)$ *and the odd part of* $b_j(x)$ *by*

$$q_e(x)=\frac{q(x)+q(-x)}{2}\,,\qquad b_{j,o}(x)=\frac{b_j(x)-b_j(-x)}{2}\,,$$

respectively. We shall assume that $q_e(x)$ *and* $b_{j,o}(x)$ *satisfy*

(2.4) $$\begin{cases} q_e(x)=|x|^{-\alpha}g_e(\omega)+o(|x|^{-\alpha})\,,\\ b_{j,o}(x)=|x|^{-\alpha}g_{j,o}(\omega)+o(|x|^{-\alpha})\,,\\ (\partial_j b_j)_e(x)=(\partial_j b_{j,o})(x)=o(|x|^{-\alpha})\,,\end{cases}$$

as $|x|\to\infty$, *where* g_e, $g_{j,o}\in C^\infty(\mathbf{S}^{d-1})$, $\alpha>1$. *Putting*

$$\Omega(\lambda;\omega,\psi)=\int_0^\pi\{-2\lambda^{1/2}\langle B_o(\omega\cos\theta+\psi\sin\theta),\omega\rangle+g_e(\omega\cos\theta+\psi\sin\theta)\}\sin^{\alpha-2}\theta d\theta\,,$$

where $\langle\ ,\ \rangle$ *is the usual inner product,*

$$\psi\in\mathbf{S}_\omega^{d-2}\,,\qquad B_o(\omega)=(g_{1,o}(\omega),g_{2,o}(\omega)\cdots,g_{d,o}(\omega))\,,$$

we define

$$\mathbf{a}_{\pm}(\mathbf{S}_0)=2^{-1}(d-1)^{-\rho}(2\pi)^{-\alpha}\lambda^{-1+\alpha/2}\left(\int_{\mathbf{S}_0}d\omega\int_{\mathbf{S}_\omega^{d-2}}d\psi(\Omega_{\pm}(\lambda,\omega,\psi))^{1/\rho}\right)^{\rho}\,,$$

where $\mathbf{S}_0$ *is any fixed hemisphere in* $\mathbf{S}^{d-1}$, $\rho=(\alpha-1)(d-1)^{-1}$ *and* $\Omega_+=\max\{\Omega, 0\}$, $\Omega_-=\Omega_+-\Omega$. *Then we have*

(i) $\mathbf{a}_\pm(\mathbf{S}_0)$ *is independent on* $\mathbf{S}_0$ *and*

$$\pi^{-1}a_\pm \equiv \mathbf{a}_\pm(\mathbf{S}_0)=2^{-\rho}\mathbf{a}_\pm(\mathbf{S}^{d-1}); \tag{2.5}$$

(ii)

$$\lim_{n\to\infty} n^\rho \delta_n^\pm = \lim_{n\to\infty} n^\rho \eta_n^\pm = a_\pm . \tag{2.6}$$

THEOREM B. *Suppose that Assumption* (V) *is satisfied and that*

$$q_e(x)=O(|x|^{-\alpha}), \quad b_{j,o}(x)=O(|x|^{-\alpha}), \quad (\partial_j b_j)_e(x)=(\partial_j b_{j,o})(x)=O(|x|^{-\alpha}),$$
$$\beta>(\alpha+1)/2,$$

Then we have

$$\delta_n^\pm=O(n^{-\rho}), \quad \eta_n^\pm=O(n^{-\rho}), \tag{2.7}$$

where $\rho=(\alpha-1)(d-1)^{-1}$.

REMARK. The result (ii) of Theorem A are gauge invariant for the gauge functions $f(x)$ such that $f(x)\to C$ as $|x|\to\infty$. In fact, for different vector potentials $\mathbf{b}$ and $\tilde{\mathbf{b}}=\mathbf{b}+\nabla f$, $\Omega(\lambda; \omega, \psi)$ is invariant (see [3] pp. 346–347).

3. Properties of compact operators.

We shall describe some properties of compact operators (cf. [1], [7]).

NOTATIONS. (i) For a compact self-adjoint operator K, we shall denote positive eigenvalues and negative eigenvalues by $\lambda_n^+(K)$ and $-\lambda_n^-(K)$, respectively, which are enumerated with their multiplicities and $\lambda_n^\pm \downarrow 0$ as $n\to\infty$.

(ii) For a compact operator K we call $s_n(K)\equiv(\lambda_n^+(K^*K))^{1/2}$ singular values.

LEMMA 3.1 (Inequalities of singular values [Ky Fan's inequalities]). *Let A be a bounded operator and K_j ($j=1, 2$) compact operators. Then the following inequalities hold.*

$$s_n(AK)\le \|A\|s_n(K), \quad s_n(KA)\le \|A\|s_n(K), \tag{3.1}$$

$$s_{n_1+n_2-1}(K_1+K_2)\le s_{n_1}(K_1)+s_{n_2}(K_2), \tag{3.2}$$

$$s_{n_1+n_2-1}(K_1K_2)\le s_{n_1}(K_1)s_{n_2}(K_2), \qquad n_j\in\mathbf{N}. \tag{3.3}$$

By Lemma 3.1 we have the following lemmas.

LEMMA 3.2. *If A_k are compact operators then we have*

$$s_N\left(\sum_{k=1}^m A_k\right)\le \sum_{k=1}^m s_n(A_k), \qquad N=mn-(m-1).$$

LEMMA 3.3. *Let A_k, C_k be compact operators and B_k bounded operators. Then we have*

$$s_N\left(\sum_{k=1}^{m} A_kB_kC_k\right) \le \sum_{k=1}^{m} \|B_k\| s_n(A_k)s_n(C_k)\,, \qquad N=2m(n-1)+1\,.$$

LEMMA 3.4. *Let A_k, C_k be compact operators and B_k bounded operators. If $s_n(A_k)=O(n^{-\rho/2})$ and $s_n(C_k)=O(n^{-\rho/2})$ then we have*

$$s_n\left(\sum_{k=1}^{m} A_kB_kC_k\right)=O(n^{-\rho})\,.$$

PROPOSITION 3.5. *Let K_l $(l=1, 2)$ be compact self-adjoint operators. If*

$$\lambda_n^{\pm}(K_1) \sim k_{\pm}n^{-\rho}\,, \quad s_n(K_2)=o(n^{-\rho})\,, \qquad n \to \infty\,,$$

then

$$\lambda_n^{\pm}(K_1+K_2) \sim k_{\pm}n^{-\rho}\,, \qquad n \to \infty\,.$$

We shall consider the perturbation of an isolated eigenvalue of infinite multiplicity. Let $\mathfrak{X}$ be a Hilbert space and A be a bounded self-adjoint operator in $\mathfrak{X}$. Assume that λ is an isolated eigenvalue of A with infinite multiplicity. If $\varepsilon>0$ is small enough then the both intervals $[\lambda-\varepsilon, \lambda)$ and $(\lambda, \lambda+\varepsilon]$ can be gap in the spectrum.

Let K be any compact self-adjoint operator and put $B=A+K$. By Weyl's theorem $\sigma_e(A)=\sigma_e(B)$ ($\sigma_e(A)$ means the essential spectrum of A). Therefore the spectrum of B in $[\lambda-\varepsilon, \lambda)$ and $(\lambda, \lambda+\varepsilon]$ consists of eigenvalues which have a finite multiplicity and accumulate only at λ.

Now we denote the eigenvalues of B in $(\lambda, \lambda+\varepsilon]$ (resp. $[\lambda-\varepsilon, \lambda)$) by $\mu_n^+(B)$ (resp. $\mu_n^-(B)$) enumerated with their multiplicities and $\lambda<\mu_{n+1}^+(B)\le\mu_n^+(B)$ (resp. $\mu_n^-(B)\le \mu_{n+1}^-(B)<\lambda$).

Taking account of the above facts we have the following two theorems.

THEOREM 3.6 ([8]). *Let A, K and λ be the same as above and $B=A+K$. Let P be an orthogonal projection onto the eigenspace of A corresponding to the isolated eigenvalue λ.*

If for some $\rho>0$

$$\lambda_n^{\pm}(PKP) \sim k_{\pm}n^{-\rho}\,, \qquad s_n(K)=o(n^{-\rho/2})\,, \qquad n\to\infty\,, \tag{3.4}$$

then

$$\mu_n^{\pm}(B)=\lambda\pm k_{\pm}n^{-\rho}+o(n^{-\rho})\,, \qquad n\to\infty\,, \tag{3.5}$$

holds.

THEOREM 3.7 ([8]). *If for some $\rho>0$*

$$s_n(PKP)=O(n^{-\rho})\,, \qquad s_n(K)=O(n^{-\rho/2})\,,$$

then

$$|\mu_n^{\pm}(B)-\lambda|=O(n^{-\rho}).$$

4. Auxiliary facts.

We denote the Fourier transform of a function $f\in\mathscr{H}$ by $\hat{f}$, i.e.,

$$\hat{f}(p)=(2\pi)^{-d/2}\int_{\mathbf{R}^d}\exp(-i\langle x,p\rangle)f(x)dx\,,\qquad p\in\Xi^d\,.$$

We define $\Gamma_0(\lambda)$ as an operator from $\mathscr{H}$ onto $L^2(\mathbf{R}^+;\mathbf{H})$ as follows:

$$
\begin{aligned}
(\Gamma_0(\lambda)f)(\omega)&=2^{-1/2}\lambda^{(d-2)/4}\hat{f}(\lambda^{1/2}\omega)\\
&=2^{-1/2}\lambda^{(d-2)/4}(2\pi)^{-d/2}\int_{\mathbf{R}^d}\exp(-i\langle x,\lambda^{1/2}\omega\rangle)f(x)dx\,,\qquad \lambda>0\,.
\end{aligned}
\tag{4.1}
$$

By the definition of $\Gamma_0(\lambda)$ we have $\Gamma_0^*(\lambda)$ as follows:

$$(\Gamma_0^*(\lambda)v)(x)=2^{-1/2}\lambda^{(d-2)/4}(2\pi)^{-d/2}\int_{\mathbf{S}^{d-1}}\exp(i\langle x,\lambda^{1/2}\omega\rangle)v(\omega)d\omega\,.$$

LEMMA 4.1. *Let X_γ be a multiplication operator by the function $(1+x^2)^{-\gamma/2}$. Then for $\gamma>1/2$*

$$Z_0(\lambda)=Z_0^{(\gamma)}(\lambda)=\Gamma_0(\lambda)X_\gamma$$

is a compact operator from $\mathscr{H}$ to $\mathbf{H}$ and continuous on $\lambda>0$.

PROOF. It holds from Sobolev's trace theorem. □

PROPOSITION 4.2. *Let Assumption* (V) *be satisfied and $\gamma>1/2$. Then*

$$G(z)=G^\gamma(z)=X_\gamma R(z)X_\gamma$$

is continuous in norm with respect to z in the complex plane except interval $[0,\infty)$ *and $G(\lambda\pm i\varepsilon)$ have finite boundary values as $\varepsilon\downarrow 0$.*

PROOF. It holds from Mourre estimate ([4]). □

We shall denote the stationary representation of scattering matrix.

DEFINITION 4.3 ([10]). For the operators H_0 and H, the scattering matrix $S(\lambda)$ is defined by

$$S(\lambda)=I-2\pi i\Gamma_0(\lambda)(V-VR(\lambda+i0)V)\Gamma_0^*(\lambda)\,.\tag{4.2}$$

This form is called *the stationary representation of the scattering matrix.*

It is well known that $S(\lambda)$ is a unitary operators in $\mathbf{H}$.

Note that the operator V can be rewritten as follows:

$$V=\sum_{j=1}^{d}\{i\partial_j b_j(x)+ib_j(x)\partial_j+b_j^2(x)\}+q(x)$$
$$=\sum_{j=1}^{d}\{2i\partial_j b_j(x)-i(\partial_j b_j)(x)+b_j^2(x)\}+q(x)\,.$$

DEFINITION (V_1, V_2). We define V_1, V_2 as follows:

$$V_1=\sum_{j=1}^{d}2i\partial_j b_j(x)\,,\qquad V_2=-\sum_{j=1}^{d}i(\partial_j b_j)(x)+\sum_{j=1}^{d}b_j^2(x)+q(x)\,,$$
$$V=V_1+V_2\,.$$

LEMMA 4.4. *For $\Gamma_0\equiv\Gamma_0(\lambda)$ and V_1, the following equality holds.*

$$(\Gamma_0V_1f)(\omega)=-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j f)(\omega)\,.$$

Here ω_j is the j-th component of $\omega\in\mathbf{S}^{d-1}$.

PROOF. We can see easily from the following calculations.

$$(\Gamma_0V_1f)(\omega)=\left\{\Gamma_0\left(\sum_{j=1}^{d}2i\partial_j b_j(x)\right)f\right\}(\omega)=2i\sum_{j=1}^{d}2^{-1/2}\lambda^{(d-2)/4}\widehat{\partial_j(b_jf)}(\lambda^{1/2}\omega)$$
$$=2i\sum_{j=1}^{d}2^{-1/2}\lambda^{(d-2)/4}(i\lambda^{1/2}\omega_j)(\widehat{b_jf})(\lambda^{1/2}\omega)=-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j f)(\omega)\,.\qquad\square$$

COROLLARY 4.5. *For Γ_0 and V_1, the following equality holds.*

$$\Gamma_0V_1\Gamma_0^*=-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j\Gamma_0^*)\,.$$

PROOF. It is clear by Lemma 4.4. $\square$

LEMMA 4.6. *For Γ_0 and V_1, the following equality holds.*

$$(V_1\Gamma_0^*u)(x)=2\sum_{j=1}^{d}\{(-\lambda^{1/2}b_j\Gamma_0^*\omega_j u)(x)+(i(\partial_j b_j)\Gamma_0^*u)(x)\}\,.$$

PROOF. We shall divide a proof into 3 steps.

Step 1. By integration by parts we have

$$(V_1\Gamma_0^*u)(x)=((\Gamma_0V_1^*)^*u)(x)=\left\{\left(\Gamma_0\left(2\sum_{j=1}^{d}ib_j(x)\partial_j\right)\right)^*u\right\}(x)$$
$$=2\sum_{j=1}^{d}\{(\Gamma_0(ib_j(x)\partial_j))^*u\}(x)\,.$$

Step 2. Here we shall consider $\{(\Gamma_0(ib_j\partial_j))^*u\}(x)$. For $f\in C_0^\infty$ we know $\partial_jb_j(x)f(x)=(\partial_jb_j)(x)f(x)+b_j(x)(\partial_jf)(x)$ and so we have $b_j(x)(\partial_jf)(x)=\partial_jb_j(x)f(x)-(\partial_jb_j)(x)f(x)$. Using this and Lemma 4.4 we have

$$\begin{aligned}(\Gamma_0(ib_j\partial_jf))(\omega)&=(\Gamma_0(i\partial_jb_jf))(\omega)-(\Gamma_0(i(\partial_jb_j)f))(\omega)\\&=-\lambda^{1/2}\omega_j(\Gamma_0b_jf)(\omega)-(\Gamma_0(i(\partial_jb_j)f))(\omega)\,.\end{aligned}$$

By the above equality for $u\in\mathbf{H}$ we have

$$\begin{aligned}(\Gamma_0ib_j\partial_jf,u)_{\mathbf{H}}&=(-\lambda^{1/2}\omega_j\Gamma_0b_jf,u)_{\mathbf{H}}-(\Gamma_0i(\partial_jb_j)f,u)_{\mathbf{H}}\\&=(f,(-\lambda^{1/2}\omega_j\Gamma_0b_j)^*u)_{\mathscr{H}}-(f,-i(\partial_jb_j)\Gamma_0^*u)_{\mathscr{H}}\\&=(f,\{(-\lambda^{1/2}\omega_j\Gamma_0b_j)^*+i(\partial_jb_j)\Gamma_0^*\}u)_{\mathscr{H}}\,.\end{aligned}$$

Consequently we obtain

$$(\Gamma_0(ib_j\partial_j))^*u=\{(-\lambda^{1/2}\omega_j\Gamma_0b_j)^*+i(\partial_jb_j)\Gamma_0^*\}u=\{(-\lambda^{1/2}b_j\Gamma_0^*\omega_j)+i(\partial_jb_j)\Gamma_0^*\}u\,.$$

Step 3. By Step 1 and Step 2, we can find

$$(V_1\Gamma_0^*u)(x)=2\sum_{j=1}^d\{(\Gamma_0(ib_j\partial_j))^*u\}(x)=\left[2\sum_{j=1}^d\{(-\lambda^{1/2}b_j\Gamma_0^*\omega_j)+i(\partial_jb_j)\Gamma_0^*\}u\right](x)$$

and hence lemma is proved. □

DEFINITION (q_2, W). We shall define a multiplication operator W by the function $q_2(x)(1+x^2)^{\beta/2}$ where $q_2(x)=-\sum_{j=1}^d i(\partial_jb_j)(x)+\sum_{j=1}^d b_j^2(x)+q(x)$.

Then the following lemma holds.

LEMMA 4.7. *Let $b_j(x)$, $(\partial_jb_j)(x)$ and $q(x)$ satisfy Assumption* (V) *and let W be a multiplication operator by $q_2(x)(1+x^2)^{\beta/2}$. Then W is a bounded operator.*

PROOF. This is clear from Assumption (V). □

Let $\gamma_1+\gamma_2=\beta$, $\gamma_j>1/2$. Then we see that

$$V_2=X_{\gamma_1}WX_{\gamma_2}=X_{\gamma_2}WX_{\gamma_1}\,.$$

LEMMA 4.8. *$\Gamma_0V_l\Gamma_0^*$, $\Gamma_0V_lR(\lambda+i0)V_m\Gamma_0^*$ $(l,m=1,2)$ are compact operators in* **H**.

PROOF. We shall treat these operators separately.

(1) $\Gamma_0V_1\Gamma_0^*$: By Corollary 4.5 we have $\Gamma_0V_1\Gamma_0^*=-2\sum_{j=1}^d\lambda^{1/2}\omega_j(\Gamma_0b_j\Gamma_0^*)$. Here ω_j in the right hand side is the j-th component of ω and $|\omega|=1$. The multiplication operator by ω_j is a bounded operator in **H**. Next we shall write $\Gamma_0b_j\Gamma_0^*=(\Gamma_0X_\gamma)(X_\gamma^{-1}b_jX_\gamma^{-1})(X_\gamma\Gamma_0^*)$. By Lemma 4.1 the operators Γ_0X_γ and $X_\gamma\Gamma_0^*$ are compact operators, and by Assumption (V), $|X_\gamma^{-1}b_jX_\gamma^{-1}|=|b_j(x)(1+x^2)^\gamma|\le C$, $\gamma=\beta/2>1/2$. So $\Gamma_0b_j\Gamma_0^*$ is a compact operator. Hence $\Gamma_0V_1\Gamma_0^*$ is a compact operator.

(2) $\Gamma_0V_2\Gamma_0^*$: We can write $\Gamma_0V_2\Gamma_0^*=\Gamma_0X_{\gamma_1}WX_{\gamma_2}\Gamma_0^*=Z_0^{\gamma_1}W(Z_0^{\gamma_2})^*$. By Lemma 4.1 Z_0^γ is compact and by Lemma 4.7 W is bounded. Hence $\Gamma_0V_2\Gamma_0^*$ is a compact operator.

(3) $\Gamma_0 V_1 R(\lambda+i0) V_2 \Gamma_0^*$: By Lemma 4.4, $(\Gamma_0 V_1 f)(\omega) = -2\sum_{j=1}^{d} \lambda^{1/2}\omega_j(\Gamma_0 b_j f)(\omega)$, and $V_2 = X_{\gamma_1} W X_{\gamma_2} = X_{\gamma_2} W X_{\gamma_1}$. So we can write

$$\begin{aligned}\Gamma_0 V_1 R V_2 \Gamma_0^* &= -2\sum_{j=1}^{d} \lambda^{1/2}\omega_j \Gamma_0 b_j R V_2 \Gamma_0^* \\ &= -2\sum_{j=1}^{d} \lambda^{1/2}\omega_j (\Gamma_0 X_{\gamma_2})(X_{\gamma_2}^{-1} b_j X_{\gamma_1}^{-1})(X_{\gamma_1} R X_{\gamma_1}) W (X_{\gamma_2}\Gamma_0^*)\,.\end{aligned}$$

$\Gamma_0 X_{\gamma_2}$ and $X_{\gamma_2}\Gamma_0^*$ are compact operators (by Lemma 4.1), $G^{\gamma_1}(z) = X_{\gamma_1} R X_{\gamma_1}$ is a bounded operator (by Proposition 4.2), W is a bounded operator (by Lemma 4.7) and $X_{\gamma_2}^{-1} b_j X_{\gamma_1}^{-1} = b_j(x)(1+x^2)^{(\gamma_1+\gamma_2)/2} = b_j(x)(1+x^2)^{\beta/2}$ is a bounded operator (by Assumption (V)). Therefore $\Gamma_0 V_1 R V_2 \Gamma_0^*$ is a compact operator.

(4) $\Gamma_0 V_2 R(\lambda+i0) V_1 \Gamma_0^*$: Using Lemma 4.6 we write as follows:

$$\begin{aligned}\Gamma_0 V_2 R V_1 \Gamma_0^* &= \Gamma_0 V_2 R\left\{-2\sum_{j=1}^{d} \lambda^{1/2} b_j \Gamma_0^* \omega_j + 2\sum_{j=1}^{d} i(\partial_j b_j)\Gamma_0^*\right\} \\ &= -2\lambda^{1/2}\sum_{j=1}^{d} (\Gamma_0 X_{\gamma_2}) W (X_{\gamma_1} R X_{\gamma_1})(X_{\gamma_1}^{-1} b_j X_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\omega_j \\ &\quad + 2\sum_{j=1}^{d} (\Gamma_0 X_{\gamma_2}) W (X_{\gamma_1} R X_{\gamma_1})(X_{\gamma_1}^{-1} i(\partial_j b_j) X_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\,.\end{aligned}$$

By Lemma 4.1, Proposition 4.2, Lemma 4.7 and Assumption (V), $\Gamma_0 V_2 R V_1 \Gamma_0^*$ is a compact operator.

(5) $\Gamma_0 V_1 R(\lambda+i0) V_1 \Gamma_0^*$: Using Lemma 4.4 and Lemma 4.6 we have

$$\begin{aligned}\Gamma_0 V_1 R V_1 \Gamma_0^* &= -2\sum_{j=1}^{d} \lambda^{1/2}\omega_j \Gamma_0 b_j R V_1 \Gamma_0^* \\ &= -2\sum_{j=1}^{d} \lambda^{1/2}\omega_j \Gamma_0 b_j R\left\{-2\sum_{k=1}^{d} \lambda^{1/2} b_k \Gamma_0^* \omega_k + 2\sum_{k=1}^{d} i(\partial_k b_k)\Gamma_0^*\right\} \\ &= 4\lambda \sum_{j,k=1}^{d} \omega_j \Gamma_0 b_j R b_k \Gamma_0^* \omega_k - 4\lambda^{1/2}\sum_{j,k=1}^{d} \omega_j \Gamma_0 b_j R i(\partial_k b_k)\Gamma_0^*\,.\end{aligned}$$

Here we can write each term in the first sum in the right hand side as follows:

$$\omega_j \Gamma_0 b_j R b_k \Gamma_0^* \omega_k = \omega_j (\Gamma_0 X_\gamma)(X_\gamma^{-1} b_j X_\gamma^{-1})(X_\gamma R X_\gamma)(X_\gamma^{-1} b_k X_\gamma^{-1})(X_\gamma \Gamma_0^*)\omega_k\,,$$

where $\gamma = \beta/2 > 1/2$. We know that $\Gamma_0 X_\gamma$ is a compact operator and the others are bounded. Hence $\omega_j \Gamma_0 b_j R b_k \Gamma_0^* \omega_k$ is a compact operator. Similarly we can easily know that the rest of the sum in the right hand side is a compact operator. Hence $\Gamma_0 V_1 R V_1 \Gamma_0^*$ is a compact operator.

(6) $\Gamma_0 V_2 R(\lambda+i0) V_2 \Gamma_0^*$: Similarly, by $\Gamma_0 V_2 R V_2 \Gamma_0^* = (\Gamma_0 X_{\gamma_1}) W (X_{\gamma_2} R X_{\gamma_2}) W (X_{\gamma_1}\Gamma_0^*)$, we

have that $\Gamma_0 V_2 R V_2 \Gamma_0^*$ is a compact operator. □

COROLLARY 4.9. *$S(\lambda)-I$ is a compact operator.*

PROOF. By (4.2) and $V=V_1+V_2$ we have

$$S(\lambda)=I-2\pi i\Gamma_0(V-VR(\lambda+i0)V)\Gamma_0^*$$
$$=I-2\pi i\sum_{k=1}^{2}\Gamma_0 V_k\Gamma_0^*+2\pi i\sum_{k,j=1}^{2}\Gamma_0 V_k R(\lambda+i0)V_j\Gamma_0^*\,.$$

By Lemma 4.8, $S(\lambda)-I$ is a compact operator. □

We denote a subset of $\mathbf{S}^{d-1}$ by $\mathscr{Y}_j$ $(j=1, 2)$. Let Y_j be a multiplication operator by the characteristic function of $\mathscr{Y}_j$. We set

$$T=T(\lambda)=\Gamma_0(\lambda)V\Gamma_0^*(\lambda)\,,$$

and consider the asymptotics of the spectrum of Y_1TY_2.

The following proposition was given by M. Sh. Birman and D. R. Yafaev.

PROPOSITION 4.10 ([2], [3]). *Let $q(x)$ and $b_j(x)$ have the following asymptotics*

$$\left\{\begin{array}{l} q(x)=|x|^{-\alpha}g(\omega)+o(|x|^{-\alpha})\,, \\ b_j(x)=|x|^{-\alpha}g_j(\omega)+o(|x|^{-\alpha})\,, \\ (\partial_j b_j)(x)=o(|x|^{-\alpha})\,, \end{array}\right. \tag{4.3}$$

as $|x|\to\infty$, g, $g_j\in C^\infty(\mathbf{S}^{d-1})$, $\alpha>1$. Putting

$$\Omega(\lambda;\omega,\psi)=\int_0^\pi\{-2\lambda^{1/2}\langle B(\omega\cos\theta+\psi\sin\theta),\omega\rangle+g(\omega\cos\theta+\psi\sin\theta)\}\sin^{\alpha-2}\theta d\theta\,, \tag{4.4}$$

where

$$\psi\in\mathbf{S}_\omega^{d-2}\,,\qquad B(\omega)=(g_1(\omega),g_2(\omega)\cdots,g_d(\omega))\,,$$

we define

$$\mathbf{a}_\pm(\mathscr{Y})=2^{-1}(d-1)^{-\rho}(2\pi)^{-\alpha}\lambda^{-1+\alpha/2}\left(\int_{\mathscr{Y}}d\omega\int_{\mathbf{S}_\omega^{d-2}}d\psi(\Omega_\pm(\lambda;\omega,\psi))^{1/\rho}\right)^\rho\,, \tag{4.5}$$

where $\rho=(\alpha-1)(d-1)^{-1}$ and $\Omega_+=\max\{\Omega,0\}$, $\Omega_-=\Omega_+-\Omega$. Then

$$\lambda_n^\pm(YTY)=\mathbf{a}_\pm(\mathscr{Y})n^{-\rho}+o(n^{-\rho})\,. \tag{4.6}$$

Moreover if $(d-1)$-surface measure of $\mathscr{Y}_1\cap\mathscr{Y}_2$ is equal to 0 then we have

$$s_n(Y_1TY_2)=o(n^{-\rho})\,. \tag{4.7}$$

By Proposition 4.10 we have the following corollary.

COROLLARY 4.11. *If $\gamma>1/2$ then for the compact operator $Z_0^\gamma(\lambda)=\Gamma_0(\lambda)X_\gamma$ in Lemma 4.1 we have*

$$s_n(Z_0^\gamma(\lambda)) = O(n^{-(\gamma-1/2)/(d-1)}).$$

PROOF. We shall calculate $s_n^2(Z_0^\gamma(\lambda))$ first.

$$s_n^2(Z_0^\gamma(\lambda)) = \lambda_n^\pm(Z_0^\gamma(\lambda)(Z_0^\gamma(\lambda))^*) = \lambda_n^\pm(\Gamma_0(\lambda)X_\gamma X_\gamma \Gamma_0^*(\lambda))$$
$$= \lambda_n^\pm(\Gamma_0(\lambda)X_{2\gamma}\Gamma_0^*(\lambda)).$$

Putting $V = X_{2\gamma}$ in the definition of T and $Y = I$ in (4.6) we have

$$s_n^2(Z_0^\gamma(\lambda)) = \lambda_n^\pm(T) = \mathbf{a}_\pm(\mathbf{S}^{d-1})n^{-\rho} + o(n^{-\rho}), \quad \rho = (2\gamma - 1)(d-1)^{-1}.$$

Hence we obtain

$$s_n(Z_0^\gamma(\lambda)) = O(n^{-\rho/2}) = O(n^{-(\gamma-1/2)/(d-1)}). \qquad \square$$

5. Asymptotics of the eigenvalues of the operator B.

DEFINITION 5.1 (cf. [8]). We define an operator B by

(5.1) $$B = 2^{1/2}\Im(\tau\Sigma(\lambda)) = 2^{-1/2}i(\bar{\tau}JS^* - \tau SJ), \qquad \tau = \exp(\pi i/4).$$

Putting $R = R(\lambda + i0)$,

(5.2) $$T_l = \Gamma_0 V_l \Gamma_0^*, \qquad T_{lm} = -\Gamma_0 V_l R V_m \Gamma_0^* \quad (l, m = 1, 2),$$

(5.3) $$K_l = \tau T_l J + \bar{\tau} J T_l^*, \qquad K_{lm} = \tau T_{lm} J + \bar{\tau} J T_{lm}^* \quad (l, m = 1, 2),$$

(5.4) $$K = -2^{1/2}\pi(K_1 + K_2 + K_{11} + K_{12} + K_{21} + K_{22}),$$

we know that

(5.5) $$B = J + K.$$

REMARK. An advantage of the representation (5.5) of B is that J, K are self-adjoint operators. By Lemma 4.8 the operators T_l, T_{lm} $(l, m = 1, 2)$ are compact and so K is compact self-adjoint. Hence by combining the fact mentioned above and Lemma 2.3 the eigenvalues of the operator B accumulate only at the points ± 1, i.e., denoting the eigenvalue of B by $\mu_n^\pm$, $\nu_n^\pm$ we see

$$\mu_n^\pm \to 1 \pm 0, \qquad \nu_n^\pm \to -1 \pm 0.$$

LEMMA 5.2. *If there exists $\rho > 0$ such that $\mu_n^\pm \sim 1 \pm k_\pm n^{-\rho}$ (resp. $\nu_n^\pm \sim -1 \pm l_\pm n^{-\rho}$) as $n \to \infty$ then $\delta_n^\mp \sim 2^{-1}k_\pm n^{-\rho}$ (resp. $\eta_n^\pm \sim 2^{-1}l_\pm n^{-\rho}$). Here δ_n (resp. η_n) is the same as in Definition 2.4.*

PROOF. We shall prove only that $\delta_n^\mp \sim 2^{-1}k_\pm n^{-\rho}$. Taking account of $\mu_n^\pm \to 1 \pm 0$ and $\mu_n^\pm = 2^{1/2}\Im(\tau \exp(\pm 2i\delta_n^\mp))$ we know that $\pm 2\delta_n^\mp \sim \mu_n^\pm - 1 \sim \pm k_\pm n^{-\rho}$. (In the case of $\nu_n^\pm$, remark that $\nu_n^\pm = -2^{1/2}\Im(\tau \exp(\mp 2i\eta_n^\pm))$.) $\square$

Our purpose is to study the asymptotics of scattering phases. Because of Lemma

5.2, it is reduced to the study of the asymptotics of the eigenvalues of B. So we consider the asymptotics of the eigenvalues of B. Using Theorem 3.6 we can find the asymptotics of the spectrum of the operator B. To apply Theorem 3.6 to the operator $B=J+K$, we consider the following two problems.
(1) Estimate of the singular value of the operator K;
(2) Investigation about the asymptotics of the spectrum of the operators $P_{\pm}KP_{\pm}$.

LEMMA 5.3. *Let* $\rho_0=(\beta-1)(d-1)^{-1}$ *and* $\rho_1<2(\beta-1)(d-1)^{-1}$. *If Assumption* (V) *is satisfied then for* T_l, T_{lm} *we have*

$$s_n(T_l)=O(n^{-\rho_0}), \qquad s_n(T_{lm})=O(n^{-\rho_1}) \quad (l, m=1, 2).$$

PROOF. We shall treat these singular values separately.
(1) $s_n(T_1)=O(n^{-\rho_0})$: By Corollary 4.5 we have

$$s_n(T_1)=s_n(\Gamma_0 V_1 \Gamma_0^*)=s_n\left(-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j\Gamma_0 b_j\Gamma_0^*\right).$$

Taking account of the facts that the operators $\Gamma_0 X_{\beta/2}$ and $X_{\beta/2}\Gamma_0^*$ are compact and that the others are bounded, and using Lemma 3.3, we have

$$\begin{aligned} s_N\left(-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j\Gamma_0 b_j\Gamma_0^*\right)&=s_N\left(-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j\Gamma_0 X_{\beta/2}X_{\beta/2}^{-1}b_jX_{\beta/2}^{-1}X_{\beta/2}\Gamma_0^*\right)\\ &\leq\sum_{j=1}^{d}|2\lambda^{1/2}|\,\|\omega_j\|\,\|X_{\beta/2}^{-1}b_jX_{\beta/2}^{-1}\|s_n^2(\Gamma_0X_{\beta/2}),\end{aligned}$$

where $N=2d(n-1)+1$. By Corollary 4.11 we know $s_n^2(\Gamma_0X_{\beta/2})=O(n^{-(\beta-1)/(d-1)})=O(n^{-\rho_0})$ and so we have $s_n(-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j\Gamma_0b_j\Gamma_0^*)=O(n^{-\rho_0})$. Hence we get $s_n(T_1)=O(n^{-\rho_0})$.
(2) $s_n(T_2)=O(n^{-\rho_0})$: By (3.1) and (3.3) we can see

$$s_{2n-1}(T_2)=s_{2n-1}(\Gamma_0V_2\Gamma_0^*)=s_{2n-1}(\Gamma_0X_{\beta/2}WX_{\beta/2}\Gamma_0^*)\leq\|W\|s_n^2(\Gamma_0X_{\beta/2}).$$

Using Corollary 4.11 we have $s_n(T_2)=O(n^{-\rho_0})$.
(3) $s_n(T_{12})=O(n^{-\rho_1})$: We can get the following equality in the same way as the proof of Lemma 4.8 (3).

$$\begin{aligned} s_n(T_{12})&=s_n(\Gamma_0V_1RV_2\Gamma_0^*)\\ &=s_n\left(-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0X_{\gamma_2})(X_{\gamma_2}^{-1}b_jX_{\gamma_1}^{-1})(X_{\gamma_1}RX_{\gamma_1})W(X_{\gamma_2}\Gamma_0^*)\right).\end{aligned}$$

Here putting $\gamma_1=\gamma$, $\gamma_2=\beta-\gamma$, we have $s_n(\Gamma_0X_{\beta-\gamma})=O(n^{-(2\beta-2\gamma-1)/2(d-1)})$ by Corollary 4.11. We put $\rho_1=(2\beta-2\gamma-1)/(d-1)$, then $\rho_1<2\rho_0$ and $s_n(\Gamma_0X_{\beta-\gamma})=O(n^{-\rho_1/2})$. Moreover taking account of the fact that the rest of operators in the right hand side are bounded (by Assumption (V) and Proposition 4.2), and using Lemma 3.4, we have $s_n(T_{12})=O(n^{-\rho_1})$.

(4) $s_n(T_{21})=O(n^{-\rho_1})$: By the equality in the proof of Lemma 4.8 (4) and the inequality (3.1) we know easily that

$$
\begin{aligned}
s_{2n-1}(T_{21}) &= s_{2n-1}(\Gamma_0 V_2 R V_1 \Gamma_0^*)\\
&= s_{2n-1}\Big(2\lambda^{1/2}\sum_{j=1}^{d}(\Gamma_0 X_{\gamma_2})W(X_{\gamma_1}RX_{\gamma_1})(X_{\gamma_1}^{-1}b_jX_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\omega_j\\
&\qquad -2\sum_{j=1}^{d}(\Gamma_0 X_{\gamma_2})W(X_{\gamma_1}RX_{\gamma_1})(X_{\gamma_1}^{-1}i(\partial_j b_j)X_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\Big)\\
&\le s_n\Big(2\lambda^{1/2}\sum_{j=1}^{d}(\Gamma_0 X_{\gamma_2})W(X_{\gamma_1}RX_{\gamma_1})(X_{\gamma_1}^{-1}b_jX_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\omega_j\Big)\\
&\qquad + s_n\Big(2\sum_{j=1}^{d}(\Gamma_0 X_{\gamma_2})W(X_{\gamma_1}RX_{\gamma_1})(X_{\gamma_1}^{-1}i(\partial_j b_j)X_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\Big).
\end{aligned}
$$

We can treat the right hand side in the same way as the proof of $s_n(T_{12})$. Hence we know $s_n(T_{21})=O(n^{-\rho_1})$.

(5) $s_n(T_{11})=O(n^{-\rho_1})$: By the equality in the proof of Lemma 4.8 (5) and the inequality (3.1) we know easily that

$$
\begin{aligned}
s_{2n-1}(T_{11}) &= s_{2n-1}(\Gamma_0 V_1 R V_1 \Gamma_0^*)\\
&= s_{2n-1}\Big(-4\lambda\sum_{j,k=1}^{d}\omega_j\Gamma_0 b_j R b_k\Gamma_0^*\omega_k + 4\lambda^{1/2}\sum_{j,k=1}^{d}\omega_j\Gamma_0 b_j R i(\partial_k b_k)\Gamma_0^*\Big)\\
&\le s_n\Big(-4\lambda\sum_{j,k=1}^{d}\omega_j\Gamma_0 b_j R b_k\Gamma_0^*\omega_k\Big) + s_n\Big(4\lambda^{1/2}\sum_{j,k=1}^{d}\omega_j\Gamma_0 b_j R i(\partial_k b_k)\Gamma_0^*\Big)\\
&= s_n\Big(-4\lambda\sum_{j,k=1}^{d}\omega_j(\Gamma_0 X_{\gamma_2})(X_{\gamma_2}^{-1}b_jX_{\gamma_1}^{-1})(X_{\gamma_1}RX_{\gamma_1})(X_{\gamma_1}^{-1}b_kX_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\omega_k\Big)\\
&\qquad + s_n\Big(4\lambda^{1/2}\sum_{j,k=1}^{d}\omega_j(\Gamma_0 X_{\gamma_2})(X_{\gamma_2}^{-1}b_jX_{\gamma_1}^{-1})(X_{\gamma_1}RX_{\gamma_1})(X_{\gamma_1}^{-1}i(\partial_k b_k)X_{\gamma_2}^{-1})(X_{\gamma_2}\Gamma_0^*)\Big).
\end{aligned}
$$

Using Lemma 3.2 we can treat the right hand side in the same way as the proof of $s_n(T_{12})$. Therefore we know $s_n(T_{11})=O(n^{-\rho_1})$.

(6) $s_n(T_{22})=O(n^{-\rho_1})$: We have

$$
\begin{aligned}
s_{2n-1}(T_{22}) &= s_{2n-1}(\Gamma_0 V_2 R V_2 \Gamma_0^*) = s_{2n-1}((\Gamma_0 X_{\gamma_2})W(X_{\gamma_1}RX_{\gamma_1})W(X_{\gamma_2}\Gamma_0^*))\\
&\le \|W\|^2\|X_{\gamma_1}RX_{\gamma_1}\|s_n^2(\Gamma_0 X_{\gamma_2}).
\end{aligned}
$$

So we get an estimate of $s_n(T_{22})$ in the same way as the proof of $s_n(T_{12})$. □

COROLLARY 5.4. *Suppose that Assumption* (V) *is satisfied. Then for the operator* K *in* (5.4) *we have*

$$s_n(K)=O(n^{-\rho_0}), \qquad \rho_0=(\beta-1)(d-1)^{-1}.$$

Moreover if $\beta>(\alpha+1)/2$ *then*

$$s_n(K)=o(n^{-\rho/2}), \qquad \rho=(\alpha-1)(d-1)^{-1}.$$

PROOF. Using the definition of K, Lemma 3.2 and Lemma 5.3 we find $s_n(K)=O(n^{-\rho_0})$. We shall give a proof of the second claim. If $\beta>(\alpha+1)/2$ then $2(\beta-1)>\alpha-1$. So we have $\rho=(\alpha-1)(d-1)^{-1}<2(\beta-1)(d-1)^{-1}=2\rho_0$. Hence $\rho/2<\rho_0$ and $s_n(K)=O(n^{-\rho_0})=o(n^{-\rho/2})$. □

We shall consider the asymptotics of the spectrum of the operators $P_\pm KP_\pm$.

PROPOSITION 5.5. *Suppose* $\beta>(\alpha+1)/2$. *Let* K_{lm} *be the operator in* (5.3) *and* P_σ (σ *is whether* "+" *or* "−") *the operator in* (2.2). *Then*

$$s_n\left(\sum_{l,m=1}^{2}(-2^{1/2}\pi P_\sigma K_{lm}P_\sigma)\right)=o(n^{-\rho})$$

holds.

PROOF. By (5.3) and (3.2) we have

$$s_{2n-1}(P_\sigma K_{lm}P_\sigma)=s_{2n-1}(\tau P_\sigma T_{lm}JP_\sigma+\bar{\tau}P_\sigma JT^*_{lm}P_\sigma)\leq 2|\tau|\,\|J\|s_n(T_{lm}).$$

By Lemma 5.3 we know

$$s_n(-2^{1/2}\pi P_\sigma K_{lm}P_\sigma)=O(n^{-\rho_1}), \qquad \rho_1<2(\beta-1)(d-1)^{-1}.$$

Therefore by Lemma 3.3 we see

$$s_n\left(\sum_{l,m=1}^{2}(-2^{1/2}\pi P_\sigma K_{lm}P_\sigma)\right)=O(n^{-\rho_1}).$$

Furthermore as $\beta>(\alpha+1)/2$, we know

$$\rho=(\alpha-1)(d-1)^{-1}<2(\beta-1)(d-1)^{-1}=2\rho_0.$$

Since ρ_1 is an arbitrary number less than $2\rho_0$, we can take ρ_1 such that $\rho<\rho_1<2\rho_0$. Hence

$$s_n\left(\sum_{l,m=1}^{2}(-2^{1/2}\pi P_\sigma K_{lm}P_\sigma)\right)=o(n^{-\rho})$$

holds. □

THEOREM 5.6. *Suppose* $\beta>(\alpha+1)/2$. *Let* K_l *be the operator in* (5.3) *and* P_σ (σ *is whether* "+" *or* "−") *the operator in* (2.2) *and* $\mathbf{a}_\pm(\mathbf{S}_0)$ *the function in* (4.5). *Fix an arbitrary hemisphere* $\mathbf{S}_0$ *of* $\mathbf{S}^{d-1}$ *and put* $a_\pm=\pi\mathbf{a}_\pm(\mathbf{S}_0)$. *Then*

$$\lambda_n^{\pm}\left(\sum_{l=1}^{2}(-2^{1/2}\pi P_\sigma K_l P_\sigma)\right)=2a_{\mp\sigma}n^{-\rho}+o(n^{-\rho})$$

holds.

We shall postpone proving Theorem 5.6 and give its proof in section 6.

THEOREM 5.7. *If $\beta>(\alpha+1)/2$ then for K defined in* (5.4) *we have*

$$\lambda_n^{\pm}(P_\sigma K P_\sigma)=2a_{\mp\sigma}n^{-\rho}+o(n^{-\rho}).$$

PROOF. By the definition of K we have

$$P_\sigma K P_\sigma=\sum_{l=1}^{2}(-2^{1/2}\pi P_\sigma K_l P_\sigma)+\sum_{l,m=1}^{2}(-2^{1/2}\pi P_\sigma K_{lm} P_\sigma).$$

Combining Theorem 5.6, Proposition 5.5 and Proposition 3.5 we can easily obtain this lemma. □

THEOREM 5.8. *If $\beta>(\alpha+1)/2$ then for $\mu_n^{\pm}$, $\nu_n^{\pm}$, which are the eigenvalues of the operator B in* (5.5), *the following equalities hold.*

$$\mu_n^{\pm}=1\pm 2a_{\mp}n^{-\rho}+o(n^{-\rho}),$$
$$\nu_n^{\pm}=-1\pm 2a_{\pm}n^{-\rho}+o(n^{-\rho}).$$

PROOF. By Theorem 5.7, Corollary 5.4 and Theorem 3.6 we can easily find this lemma. □

6. Proof of Theorem 5.6.

To prove Theorem 5.6, we shall define new operators and consider their properties. We define operators $L_\pm$, $\tilde{T}$ as follows (cf. [8]):

(6.1) $$L_\pm=\frac{P_\pm(K_1+K_2)P_\pm}{2^{1/2}},$$

(6.2) $$\tilde{T}=\frac{(T_1+T_2)+J(T_1+T_2)J}{2}.$$

Using $J^2=I$, (2.2) and (5.3) we find

(6.3) $$L_\pm=2^{-1}(\pm\tilde{T}+\tilde{T}J).$$

LEMMA 6.1. *For T_1, T_2 defined in* (5.2) *and for J we have*

$$\frac{T_1+JT_1J}{2}=\Gamma_0\left(\sum_{j=1}^{d}2i\partial_j b_{j,o}\right)\Gamma_0^*,$$

$$\frac{T_2+JT_2J}{2}=\Gamma_0\left(-\sum_{j=1}^{d}i(\partial_j b_j)_e(x)+\sum_{j=1}^{d}(b_j^2)_e(x)+q_e(x)\right)\Gamma_0^*.$$

Proof. First we prove the following two assertions:

$$(\Gamma_0 b_j(x)\Gamma_0^* Jf)(-\omega)=(\Gamma_0 b_j(-x)\Gamma_0^* f)(\omega)\,,$$
$$(J\Gamma_0 V_2(x)\Gamma_0^* Jf)(\omega)=(\Gamma_0 V_2(-x)\Gamma_0^* f)(\omega)\,.$$

We put $k=2^{-1/2}\lambda^{(d-2)/4}(2\pi)^{-d/2}$. By definition we can find that

$$\begin{aligned}
&(\Gamma_0 b_j(x)\Gamma_0^* Jf)(-\omega)\\
&=2^{-1/2}\lambda^{(d-2)/4}(2\pi)^{-d/2}\int_{\mathbf{R}^d}\exp(-i\langle x,\,-\lambda^{1/2}\omega\rangle)b_j(x)(\Gamma_0^* Jf)(x)dx\\
&=k^2\int_{\mathbf{R}^d}\exp(-i\langle x,\,-\lambda^{1/2}\omega\rangle)b_j(x)\int_{\mathbf{S}^{d-1}}(Jf)(\omega')\exp(i\langle x,\,\lambda^{1/2}\omega'\rangle)d\omega' dx\\
&=k^2\int_{\mathbf{R}^d}\exp(-i\langle x,\,-\lambda^{1/2}\omega\rangle)b_j(x)\int_{\mathbf{S}^{d-1}}f(-\omega')\exp(i\langle x,\,\lambda^{1/2}\omega'\rangle)d\omega' dx\\
&=k^2\int_{\mathbf{R}^d}\exp(-i\langle x,\,-\lambda^{1/2}\omega\rangle)b_j(x)\int_{\mathbf{S}^{d-1}}f(\omega')\exp(i\langle x,\,-\lambda^{1/2}\omega'\rangle)d\omega' dx\\
&=k\int_{\mathbf{R}^d}\exp(-i\langle x,\,-\lambda^{1/2}\omega\rangle)b_j(x)(\Gamma_0^* f)(-x)dx\\
&=k\int_{\mathbf{R}^d}\exp(-i\langle -x,\,-\lambda^{1/2}\omega\rangle)b_j(-x)(\Gamma_0^* f)(x)dx\\
&=(\Gamma_0 b_j(-x)\Gamma_0^* f)(\omega)\,.
\end{aligned}$$

So we have $(\Gamma_0 b_j(x)\Gamma_0^* Jf)(-\omega)=(\Gamma_0 b_j(-x)\Gamma_0^* f)(\omega)$. In a similar way, we also find $(J\Gamma_0 V_2(x)\Gamma_0^* Jf)(\omega)=(\Gamma_0 V_2(-x)\Gamma_0^* f)(\omega)$.

Now using the first assertion above and Lemma 4.4 we shall prove the first part of the lemma. We shall calculate the following.

$$\begin{aligned}
&\left(\frac{T_1+JT_1J}{2}f\right)(\omega)=\frac{1}{2}(\Gamma_0 V_1\Gamma_0^* f)(\omega)+\frac{1}{2}(J\Gamma_0 V_1\Gamma_0^* Jf)(\omega)\\
&=\frac{1}{2}\left\{-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j(x)\Gamma_0^* f)(\omega)\right\}+\frac{1}{2}J\left\{-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j(x)\,\Gamma_0^* Jf)(\omega)\right\}\\
&=-\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j(x)\Gamma_0^* f)(\omega)-\sum_{j=1}^{d}\lambda^{1/2}(-\omega_j)(\Gamma_0 b_j(x)\Gamma_0^* Jf)(-\omega)\\
&=-\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j(x)\Gamma_0^* f)(\omega)+\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_j(-x)\Gamma_0^* f)(\omega)\\
&=-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j\left(\Gamma_0\left(\frac{b_j(x)-b_j(-x)}{2}\right)\Gamma_0^* f\right)(\omega)\\
&=-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_{j,o}(x)\Gamma_0^* f)(\omega)=\Gamma_0\left(\sum_{j=1}^{d}2i\partial_j b_{j,o}(x)\Gamma_0^* f\right)(\omega)\,.
\end{aligned}$$

Hence we obtain the first part of the lemma.

Similarly, using the second assertion above and the definition of V_2, we can also find the second part of the lemma. □

COROLLARY 6.2. *The operator $\tilde{T}$ defined in* (6.2) *can be rewritten as follows*:

$$\tilde{T}=\Gamma_0\left\{\sum_{j=1}^{d}2i\partial_j b_{j,o}-\sum_{j=1}^{d}i(\partial_j b_j)_e(x)+\sum_{j=1}^{d}(b_j^2)_e(x)+q_e(x)\right\}\Gamma_0^* .$$

PROOF. This is clear from Lemma 6.1 and (6.2). □

DEFINITION 6.3. We define an operator $\tilde{V}$ by

$$\begin{aligned}\tilde{V}&=\sum_{j=1}^{d}2i\partial_j b_{j,o}(x)-\sum_{j=1}^{d}i(\partial_j b_j)_e(x)+\sum_{j=1}^{d}(b_j^2)_e(x)+q_e(x)\\ &=\sum_{j=1}^{d}i\partial_j b_{j,o}+\sum_{j=1}^{d}ib_{j,o}\partial_j+\sum_{j=1}^{d}(b_j^2)_e(x)+q_e(x) .\end{aligned}$$

Note that

$$\tilde{T}=\Gamma_0\tilde{V}\Gamma_0^* , \tag{6.4}$$

and $\tilde{T}$ is self-adjoint since $\tilde{V}$ is self-adjoint. Moreover by $\tilde{T}J=J\tilde{T}$ and (6.3) we can find

$$L_\pm J=JL_\pm=\pm L_\pm . \tag{6.5}$$

Let Y_0 be a multiplication operator by the characteristic function for a fixed hemisphere $\mathbf{S}_0\subset\mathbf{S}^{d-1}$. In general, for an arbitrary operator satisfying (6.5), we have the following lemma.

LEMMA 6.4. *Let $N_\pm$ be compact operators satisfying* (6.5), *i.e.*, $N_\pm J=JN_\pm=\pm N_\pm$, *and put $M_\pm=2Y_0N_\pm Y_0$. Then $N_\pm$ and $M_\pm$ have common non-zero eigenvalues with the same multiplicities.*

PROOF. Let $\mathbf{S}_0'=\mathbf{S}^{d-1}\backslash\mathbf{S}_0$ and $Y_0'=I-Y_0$. Since $L^2(\mathbf{S}^{d-1})=L^2(\mathbf{S}_0)\oplus L^2(\mathbf{S}_0')$, we have

$$N_\pm=\begin{pmatrix}Y_0N_\pm Y_0 & Y_0N_\pm Y_0'\\ Y_0'N_\pm Y_0 & Y_0'N_\pm Y_0'\end{pmatrix}.$$

By a simple calculation, we find $Y_0'=JY_0J$. Using this relation and (6.5), we can rewrite matrix $N_\pm$ as follows:

$$N_\pm=2^{-1}\begin{pmatrix}M_\pm & \pm M_\pm J\\ \pm JM_\pm & JM_\pm J\end{pmatrix}.$$

The matrix $N_\pm$ is unitarily equivalent to

$$D_\pm = 2^{-1}\begin{pmatrix} M_\pm & \pm M_\pm \\ \pm M_\pm & M_\pm \end{pmatrix}.$$

Therefore non-zero spectrum of $N_\pm$ and $D_\pm$ are the same. Moreover since non-zero spectrum of $D_\pm$ and $M_\pm$ are the same, $N_\pm$ and $M_\pm$ have common non-zero eigenvalues. □

LEMMA 6.5. *$M_\pm \equiv 2Y_0L_\pm Y_0$ can be written as follows:*

$$M_\pm = \pm Y_0\tilde{T}Y_0 + Y_0\tilde{T}Y_0'J, \qquad Y_0' = I - Y_0. \tag{6.6}$$

PROOF. Taking account of $Y_0' = JY_0J$, $J^2 = I$ and (6.3) we see

$$\begin{aligned} M_\pm &= 2Y_0L_\pm Y_0 = 2Y_0\{2^{-1}(\pm\tilde{T} + \tilde{T}J)\}Y_0 \\ &= Y_0(\pm\tilde{T} + \tilde{T}J)Y_0 = \pm Y_0\tilde{T}Y_0 + Y_0\tilde{T}JY_0 \\ &= \pm Y_0\tilde{T}Y_0 + Y_0\tilde{T}Y_0'J. \end{aligned}$$

□

LEMMA 6.6. *Under the same condition as Theorem* A *we have*
(i) *$\mathbf{a}_\pm(\mathrm{S}_0)$ is independent on S_0 and*

$$\pi^{-1}a_\pm \equiv \mathbf{a}_\pm(\mathrm{S}_0) = 2^{-\rho}\mathbf{a}_\pm(\mathrm{S}^{d-1});$$

(ii)

$$\lambda_n^\pm(M_\sigma) = \pi^{-1}a_{\pm\sigma}n^{-\rho} + o(n^{-\rho}), \qquad n \to \infty.$$

PROOF. We shall prove (i) first. We put

$$\Omega(\lambda; \omega, \psi) = \Omega_o(\lambda; \omega, \psi) + \Omega_e(\lambda; \omega, \psi),$$

where

$$\Omega_o \equiv \int_0^\pi \{-2\lambda^{1/2}\langle B_o(\omega\cos\theta + \psi\sin\theta), \omega\rangle \sin^{\alpha-2}\theta\} d\theta,$$

$$\Omega_e \equiv \int_0^\pi g_e(\omega\cos\theta + \psi\sin\theta)\sin^{\alpha-2}\theta d\theta.$$

Then by $B_o(\omega) = -B_o(-\omega)$ and $g_e(\omega) = g_e(-\omega)$, we have

$$\Omega_o(\lambda; \omega, \psi) = \Omega_o(\lambda; -\omega, -\psi), \quad \Omega_e(\lambda; \omega, \psi) = \Omega_e(\lambda; -\omega, -\psi).$$

By the properties above we obtain

$$\begin{aligned}
&\int_{\mathbf{S}^{d-1}}\int_{\mathbf{S}_\omega^{d-2}}(\Omega_\pm(\lambda;\omega,\psi))^{1/\rho}d\psi d\omega \\
&=\int_{\mathbf{S}_0}\int_{\mathbf{S}_\omega^{d-2}}(\Omega_\pm(\lambda;\omega,\psi))^{1/\rho}d\psi d\omega+\int_{\mathbf{S}_0'}\int_{\mathbf{S}_\omega^{d-2}}(\Omega_\pm(\lambda;\omega,\psi))^{1/\rho}d\psi d\omega \\
&=\int_{\mathbf{S}_0}\int_{\mathbf{S}_\omega^{d-2}}(\Omega_\pm(\lambda;\omega,\psi))^{1/\rho}d\psi d\omega+\int_{\mathbf{S}_0'}\int_{\mathbf{S}_\omega^{d-2}}(\Omega_\pm(\lambda;-\omega,-\psi))^{1/\rho}d\psi d\omega \\
&=2\int_{\mathbf{S}_0}\int_{\mathbf{S}_\omega^{d-2}}(\Omega_\pm(\lambda;\omega,\psi))^{1/\rho}d\psi d\omega\,,
\end{aligned}$$

where $\mathbf{S}_0'=\mathbf{S}^{d-1}\setminus\mathbf{S}_0$. Therefore by (4.5) we see $\mathbf{a}_\pm(\mathbf{S}_0)=2^{-\rho}\mathbf{a}_\pm(\mathbf{S}^{d-1})$. Hence $\mathbf{a}_\pm(\mathbf{S}_0)$ is independent of $\mathbf{S}_0$ and $\mathbf{a}_\pm(\mathbf{S}_0)=2^{-\rho}\mathbf{a}_\pm(\mathbf{S}^{d-1})$.

(ii) We put

$$\pi^{-1}a_\pm\equiv\mathbf{a}_\pm(\mathbf{S}_0)=2^{-\rho}\mathbf{a}_\pm(\mathbf{S}^{d-1})\,.$$

We shall apply Proposition 4.10 to $\tilde{T}=\Gamma_0\tilde{V}\Gamma_0^*$. By (4.6) in Proposition 4.10 we have

$$\lambda_n^\pm(Y_0\tilde{T}Y_0)=\mathbf{a}_\pm(\mathbf{S}_0)n^{-\rho}+o(n^{-\rho})=\pi^{-1}a_\pm n^{-\rho}+o(n^{-\rho})\,. \tag{6.7}$$

Since $|\mathbf{S}_0\cap\mathbf{S}_0'|=0$, we have, by (4.7), $s_n(Y_0\tilde{T}Y_0')=o(n^{-\rho})$, and so

$$s_n(Y_0\tilde{T}Y_0'J)=o(n^{-\rho})\,. \tag{6.8}$$

Recall that J, $\tilde{T}$, Y_0, Y_0' are self-adjoint and that $JY_0'=Y_0J$ and $\tilde{T}J=J\tilde{T}$. Taking account of these facts, we have

$$(Y_0\tilde{T}Y_0'J)^*=JY_0'\tilde{T}Y_0=Y_0(J\tilde{T}J)Y_0'J=Y_0\tilde{T}Y_0'J\,.$$

Hence $Y_0\tilde{T}Y_0'J$ is a self-adjoint operator. Taking account of the fact that for any compact operator A, $\lambda_n^\pm(-A)=\lambda_n^\mp(A)$ and applying Proposition 3.5 to the operator $\sigma Y_0\tilde{T}Y_0+Y_0\tilde{T}Y_0'J$, we have

$$\lambda_n^\pm(M_\sigma)=\lambda_n^\pm(\sigma Y_0\tilde{T}Y_0+Y_0\tilde{T}Y_0'J)=\pi^{-1}a_{\pm\sigma}n^{-\rho}+o(n^{-\rho})\,. \qquad \square$$

PROOF OF THEOREM 5.6. By (6.1) we have

$$\begin{aligned}
\lambda_n^\pm\left(\sum_{l=1}^{2}(-2^{1/2}\pi P_\sigma K_l P_\sigma)\right)&=2^{1/2}\pi\lambda_n^\pm(-P_\sigma(K_1+K_2)P_\sigma)\\
&=2^{1/2}\pi\lambda_n^\pm(-2^{1/2}L_\sigma)=\lambda_n^\pm(-2\pi L_\sigma)\\
&=2\pi\lambda_n^\mp(L_\sigma)\,.
\end{aligned}$$

Using Lemma 6.4 and Lemma 6.6 we see

$$\lambda_n^\pm(L_\sigma)=\pi^{-1}a_{\pm\sigma}n^{-\rho}+o(n^{-\rho})\,.$$

Therefore we can find

$$\lambda_n^{\pm}\left(\sum_{l=1}^{2}(-2^{1/2}\pi P_\sigma K_l P_\sigma)\right)=2\pi\lambda_n^{\mp}(L_\sigma)=2a_{\mp\sigma}n^{-\rho}+o(n^{-\rho})\,. \qquad \square$$

7. Proof of main theorems.

PROOF OF THEOREM A. (i) The proof is the same as in Lemma 6.6 (i).
(ii) By $\mu_n^{\pm}=2^{1/2}\Im(\tau\exp(\pm 2i\delta_n^{\mp}))$ and $\nu_n^{\pm}=-2^{1/2}\Im(\tau\exp(\mp 2i\eta_n^{\pm}))$, and adapting Theorem 5.8 to the result of Lemma 5.2, we find

$$\delta_n^{\mp}\sim 2^{-1}\times 2a_{\mp}\times n^{-\rho}=a_{\mp}n^{-\rho}\,,\qquad \eta_n^{\pm}\sim 2^{-1}\times 2a_{\pm}\times n^{-\rho}=a_{\pm}n^{-\rho}\,.$$

Hence we have $\lim n^\rho\delta_n^{\pm}=\lim n^\rho\eta_n^{\pm}=a_{\pm}$. $\square$

To prove Theorem B, we shall give two lemmas.

LEMMA 7.1. *We shall assume that $b_j(x)$ and $q(x)$ satisfy Assumption* (V) *and that*

$$q_e(x)=O(|x|^{-\alpha})\,,\quad b_{j,o}(x)=O(|x|^{-\alpha})\,,\quad (\partial_j b_j)_e(x)=(\partial_j b_{j,o})(x)=O(|x|^{-\alpha})\,.$$

If $\beta>(\alpha+1)/2$ then for the singular values $s_n^{(1)}$, $s_n^{(2)}$, $s_n^{(3)}$, $s_n^{(4)}$ of the operators

$$\Gamma_0\left(\sum_{j=1}^{d}2i\partial_j b_{j,o}\right)\Gamma_0^*\,,\quad \Gamma_0\left(\sum_{j=1}^{d}i(\partial_j b_j)_e(x)\right)\Gamma_0^*\,,\quad \Gamma_0\left(\sum_{j=1}^{d}(b_j^2)_e\right)\Gamma_0^*\,,\quad \Gamma_0 q_e\Gamma_0^*\,,$$

respectively, we have

$$s_n^{(k)}=O(n^{-\rho})\qquad (k=1,2,3,4)\,,$$

where $\rho=(\alpha-1)(d-1)^{-1}$.

PROOF. We shall take their proof in order. Recall that $X_\gamma=(1+x^2)^{-\gamma/2}$.
(1) $s_n^{(1)}$: By Lemma 4.4 we have

$$\begin{aligned}s_n\left(\Gamma_0\left(\sum_{j=1}^{d}2i\partial_j b_{j,o}\right)\Gamma_0^*\right)&=s_n\left(-2\sum_{j=1}^{d}\lambda^{1/2}\omega_j(\Gamma_0 b_{j,o}\Gamma_0^*)\right)\\&=s_n\left(-2\sum_{j=1}^{d}(\lambda^{1/2}\omega_j)(\Gamma_0X_{\alpha/2})(X_{\alpha/2}^{-1}b_{j,o}X_{\alpha/2}^{-1})(X_{\alpha/2}\Gamma_0^*)\right).\end{aligned}$$

By Corollary 4.11, $s_n(\Gamma_0X_{\alpha/2})$, $s_n(X_{\alpha/2}\Gamma_0^*)=O(n^{-\rho})$, by assumption the rest of the operators are bounded, and using Lemma 3.4 and Corollary 4.11 we have

$$s_n^{(1)}=O(n^{-(\alpha-1)/(d-1)})=O(n^{-\rho})\,.$$

(2) $s_n^{(2)}$: By assumption $X_{\alpha/2}^{-1}i(\partial_j b_j)_e(x)X_{\alpha/2}^{-1}=(1+x^2)^{\alpha/2}i(\partial_j b_j)_e(x)$ is bounded. Therefore in the same way as the proof of (1) we have

$$s_n^{(2)}=O(n^{-(\alpha-1)/(d-1)})=O(n^{-\rho})\,.$$

(3) $s_n^{(3)}$: Taking account of $(1+x^2)^{\alpha/2}\le C(1+|x|)^{1+\alpha}$ and $\beta>(\alpha+1)/2$ we have

$(1+x^2)^{\alpha/2}\le C(1+|x|)^{2\beta}$. So we find that

$$|X_{\alpha/2}^{-1}(b_j^2)_e(x)X_{\alpha/2}^{-1}|=|(1+x^2)^{\alpha/2}(b_j^2)_e(x)|=\left|(1+x^2)^{\alpha/2}\frac{b_j^2(x)+b_j^2(-x)}{2}\right|$$
$$\le C\{|(1+|x|)^{\beta}b_j(x)|^2+|(1+|x|)^{\beta}b_j(-x)|^2\}\,.$$

Hence by Assumption (V), $X_{\alpha/2}^{-1}(b_j^2)_e(x)X_{\alpha/2}^{-1}$ is bounded. Therefore in the same way as the proof of (1) we have

$$s_n^{(3)}=O(n^{-(\alpha-1)/(d-1)})=O(n^{-\rho})\,.$$

(4) $s_n^{(4)}$: By assumption $X_{\alpha/2}^{-1}q_e(x)X_{\alpha/2}^{-1}=(1+x^2)^{\alpha/2}q_e(x)$ is bounded. Therefore in the same way as the proof of (1) we have

$$s_n^{(4)}=O(n^{-(\alpha-1)/(d-1)})=O(n^{-\rho})\,.$$ □

LEMMA 7.2. *We shall assume that* $b_j(x)$ *and* $q(x)$ *satisfy Assumption* (V) *and that*

$$q_e(x)=O(|x|^{-\alpha})\,,\quad b_{j,o}(x)=O(|x|^{-\alpha})\,,\quad (\partial_j b_j)_e(x)=(\partial_j b_{j,o})(x)=O(|x|^{-\alpha})\,.$$

If $\beta>(\alpha+1)/2$ *then*

$$s_n(L_\sigma)=O(n^{-\rho})\,.$$

PROOF. By (6.4) and by the definition of $\tilde V$ (see Def. 6.3), we find

$$s_n(\tilde T)=s_n(\Gamma_0\tilde V\Gamma_0^*)$$
$$=s_n\left(\Gamma_0\left(\sum_{j=1}^d 2i\partial_j b_{j,o}(x)-\sum_{j=1}^d i(\partial_j b_j)_e(x)+\sum_{j=1}^d (b_j^2)_e(x)+q_e(x)\right)\Gamma_0^*\right).$$

By Lemma 3.3 and Lemma 7.1 we have $s_n(\tilde T)=O(n^{-\rho})$. Moreover by (6.3) we see

$$s_{2n-1}(L_\pm)=s_{2n-1}\left(\frac{\pm\tilde T+\tilde TJ}{2}\right)\le\frac{1}{2}s_n(\tilde T)+\frac{\|J\|}{2}s_n(\tilde T)=s_n(\tilde T)\,.$$

Therefore $s_n(L_\sigma)=O(n^{-\rho})$. □

PROOF OF THEOREM B. By (5.4) and (3.2) we find

$$s_{2n-1}(P_\sigma KP_\sigma)\le s_n\left(P_\sigma\left(-2^{1/2}\pi\sum_{l=1}^2 K_l\right)P_\sigma\right)+s_n\left(\sum_{l,m=1}^2(-2^{1/2}\pi P_\sigma K_{lm}P_\sigma)\right).$$

Taking account of (6.1) and Lemma 7.2, for the first term in the right hand side we have

$$s_n\left(P_\sigma\left(-2^{1/2}\pi\sum_{l=1}^2 K_l\right)P_\sigma\right)=2\pi s_n(2^{-1/2}P_\sigma(K_1+K_2)P_\sigma)$$
$$=2\pi s_n(L_\sigma)=O(n^{-\rho})\,. \tag{7.1}$$

For the second term in the right hand side, by Proposition 5.5, we have

(7.2)
$$s_n\left(\sum_{l,m=1}^{2}(-2^{1/2}\pi P_\sigma K_{lm} P_\sigma)\right)=o(n^{-\rho}).$$

Combining (7.1) and (7.2) we have

(7.3)
$$s_n(P_\sigma K P_\sigma)=O(n^{-\rho}).$$

Since $\beta>(\alpha+1)/2$, by Corollary 5.4 we have

(7.4)
$$s_n(K)=o(n^{-\rho/2}).$$

Applying Theorem 3.7 to the operator $B=J+K$ in (5.5) we can find, by (7.3) and (7.4),

$$|\mu_n^{\pm}-1|=O(n^{-\rho}), \quad |\nu_n^{\pm}+1|=O(n^{-\rho}).$$

Hence by Lemma 5.2 we have $\delta_n^{\pm}=O(n^{-\rho})$, $\eta_n^{\pm}=O(n^{-\rho})$. □

8. Appendix.

PROOF OF LEMMA 2.1. Suppose $(Jf)(\omega)=\lambda f(\omega)$. Then $f=J^2f=\lambda Jf=\lambda^2 f$ because of $J^2=I$, so $\lambda=\pm 1$. Hence the set of eigenvalues of J consists of only ± 1, and so the eigenspace for $+1$ (resp. -1) is the subspace of the space of the even (resp. odd) functions in **H**. Moreover by the following equalities, the latter statement of Lemma 2.1 is proved.

$$P_+f(\omega)=(f(\omega)+f(-\omega))/2=f_e(\omega), \quad P_-f(\omega)=(f(\omega)-f(-\omega))/2=f_o(\omega). \quad \square$$

PROOF OF LEMMA 2.3. We know $\Sigma(\lambda)-J=S(\lambda)J-J=(S(\lambda)-I)J$. Since $S(\lambda)-I$ is compact, the operator $\Sigma(\lambda)-J$ is compact by Corollary 4.9. Taking account of the fact that the set of eigenvalues of J consists of only ± 1 and using Weyl's theorem (see [1]) we have Lemma 2.3. □

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