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The Sym-Bobenko Formula and Constant Mean Curvature Surfaces in Minkowski 3-Space

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1. Introduction.

Recently, Dorfmeister, Pedit and Wu discovered a Weierstrass-type representation for harmonic maps from a Riemann surface into symmetric spaces [DPW]. In their formula, the Weierstrass data are defined as meromorphic potentials, i.e. meromorphic 1-forms on a Riemann surface with values in an infinite-dimensional loop algebra. They regarded a harmonic map as a map taking values in a twisted loop group and showed that every harmonic map from a Riemann surface into a symmetric space is obtained by integrating the potential. In a related paper, Dorfmeister and Haak have constructed constant mean curvature surfaces by applying the Sym-Bobenko formula [DH] to the loop-group-valued maps given by integrating the potentials.

On the other hand, Kenmotsu discovered a representation formula for immersions with prescribed mean curvature from a simply connected Riemann surface into Euclidean 3-space. In particular, he obtained a formula for an immersion with constant mean curvature whose Gauss map is a given harmonic map [K]. And Akutagawa and Nishikawa constructed the Minkowski 3-space version of the above formula [AN].

Motivated by these results, the present paper has two aims. The first is to establish a natural correspondence between the following two spaces: the space of conformal spacelike immersions with constant mean curvature from a simply connected Riemann surface Σ into Minkowski 3-space, and that of nowhere anti-holomorphic harmonic maps from Σ into the Poincaré half plane, regarded as the riemannian symmetric space $SL(2, \mathbf{R})/SO(2)$. The second is to prove the Lorentzian version of the Sym-Bobenko formula and apply it to construct spacelike immersions with constant mean curvature.

In section 2 we shall first prepare notations used in the later sections and recall the identification of the riemannian symmetric space $SL(2, \mathbf{R})/SO(2)$ with the unit disk equipped with the Poincaré metric and also with the Poincaré half plane. In section 3 we shall define a $sl(2, \mathbf{R})$-valued 1-form A^f on a Riemann surface Σ associated to a

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smooth map $f\colon \Sigma \to SL(2, \mathbf{R})/SO(2)$ and show that the harmonicity of f is equivalent to the d-closedness of A^f. By using this fact, we shall establish the correspondence between the two spaces above-mentioned. In section 4 we shall prove the Sym-Bobenko-type formula and give examples of spacelike immersions with constant mean curvature.

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2. Preliminaries.

We begin with fixing our terminology and notation. Let $L^3=(\mathbf{R}^3, \bar{g})$ denote Minkowski 3-space. Here $\bar{g}$ is the flat Lorentzian metric of signature $(+, +, -)$. In terms of the canonical coordinates (x^1, x^2, x^3) of $\mathbf{R}^3$, the metric $\bar{g}$, denoted also by $\langle\ ,\ \rangle$, is expressed as $\bar{g}=(dx^1)^2+(dx^2)^2-(dx^3)^2$. Let Σ be a Riemann surface and $\Phi\colon \Sigma \to L^3$ a smooth map from Σ into L^3. Let $\tilde{\Sigma}$ be the open subset of Σ defined by

$$\tilde{\Sigma}=\{p\in\Sigma \mid \Phi^*\bar{g} \text{ is positive definite at } p\}\,.$$

We call $\Phi\colon \Sigma \to L^3$ a spacelike immersion if $\Sigma=\tilde{\Sigma}$. Throughout this paper, we assume that Φ is weakly conformal, namely,

$$\Phi^*\bar{g}=\lambda^2(d\xi^1\otimes d\xi^1+d\xi^2\otimes d\xi^2)\,, \qquad \lambda\geqq 0\,,$$

where $\xi=\xi^1+\sqrt{-1}\xi^2$ is a local complex coordinate on Σ. Let I be the first fundamental form of the immersion $\Phi|_{\tilde{\Sigma}}$, that is, the riemannian metric on $\tilde{\Sigma}$ obtained by restricting $\Phi^*\bar{g}$ to $\tilde{\Sigma}$.

We define a local Lorentzian frame field (e_1, e_2, e_3) adapted to $\Phi|_{\tilde{\Sigma}}$ as follows. Let $\Phi(\xi)=(\Phi^1(\xi^1, \xi^2), \Phi^2(\xi^1, \xi^2), \Phi^3(\xi^1, \xi^2))$ be a local expression of the smooth map Φ with respect to a local complex coordinate $\xi=\xi^1+\sqrt{-1}\xi^2$ on $\tilde{\Sigma}$. For $i=1, 2$, let

$$e_i=\frac{1}{\lambda}\frac{\partial\Phi}{\partial\xi^i}=\frac{1}{\lambda}\left(\frac{\partial\Phi^1}{\partial\xi^i}, \frac{\partial\Phi^2}{\partial\xi^i}, \frac{\partial\Phi^3}{\partial\xi^i}\right). \tag{2.1}$$

We define $e_3=e_1\times e_2$. Here the exterior product $v\times w$ of two vectors $v={}^t(x_1, x_2, x_3)$, $w={}^t(y_1, y_2, y_3)$ in L^3 is defined by

$$v\times w={}^t(x_3y_2-x_2y_3,\ x_1y_3-x_3y_1,\ x_1y_2-x_2y_1) \qquad \text{(cf. [AN])}\,.$$

Let II denote the second fundamental form of $\Phi|_{\tilde{\Sigma}}$. We denote the covariant differentiation in L^3 by D. If we set

$$Q=-\langle D_\partial\partial, e_3\rangle\,, \qquad H=-\frac{2}{\lambda^2}\langle D_\partial\bar{\partial}, e_3\rangle\,,$$

II is expressed as

$$II=Q d\xi\otimes d\xi+(1/2)H\lambda^2 d\xi\otimes d\bar{\xi}+(1/2)H\lambda^2 d\bar{\xi}\otimes d\xi+\bar{Q} d\bar{\xi}\otimes d\bar{\xi}\,,$$

where $\partial = \partial/\partial\xi$ and $\bar{\partial} = \partial/\partial\bar{\xi}$. Notice that H is nothing but the mean curvature of the immersion.

Next we define the Gauss map $G(\Phi)\colon \tilde{\Sigma} \to \mathscr{H} \subset L^3$ of $\Sigma|_{\tilde{\Sigma}}$ by $p \mapsto e_3(p)$, where $\mathscr{H}$ is the unit pseudosphere defined by $\mathscr{H} = \{{}^t(x, y, z) \in L^3 \mid x^2 + y^2 - z^2 = -1\}$.

Next we recall the relation among various models of the hyperbolic plane defined by

$$\mathbf{H} = \left(\{p + \sqrt{-1}q \in \mathbf{C} \mid q > 0\}, \frac{dp \otimes dp + dq \otimes dq}{q^2} \right).$$

Let $\mathscr{H}^+$ be the upper unit pseudosphere in L^3 defined by $\mathscr{H}^+ = \{{}^t(x, y, z) \in \mathscr{H} \mid z > 0\}$. Let $\psi\colon \mathscr{H}^+ \to \mathbf{D}$ be the stereographic projection from $\mathscr{H}^+$ into the unit disk $\mathbf{D} = \{\alpha \in \mathbf{C} \mid |\alpha| < 1\}$ given by

$${}^t(x, y, z) \mapsto \frac{x}{1+z} + \sqrt{-1}\,\frac{y}{1+z}.$$

Let $\gamma\colon \mathbf{D} \to \mathbf{H}$ be the Cayley transform, that is, the map given by

$$\alpha \mapsto -\sqrt{-1}\,\frac{\alpha + \sqrt{-1}}{\alpha - \sqrt{-1}},$$

and $\varphi\colon \mathscr{H}^+ \to \mathbf{H}$ the map defined by $\varphi = \gamma \circ \psi$. Let J be the map from L^3 to $sl(2, \mathbf{R})$ defined by

$${}^t(x, y, z) \mapsto (x/2)\eta_1 + (y/2)\eta_2 + (z/2)\eta_3,$$

where

$$\eta_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By a simple calculation, we see that J satisfies

$$J(r_1 \times r_2) = [J(r_1), J(r_2)] \qquad \text{for any two vectors } r_1, r_2 \text{ in } L^3,$$

where [,] denotes the Lie bracket of $sl(2, \mathbf{R})$ and $\times$ is the exterior product defined as above.

Now we have the natural bijection $\rho\colon SL(2, \mathbf{R})/SO(2) \to \mathbf{H}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} SO(2) \mapsto \frac{a\sqrt{-1} + b}{c\sqrt{-1} + d}.$$

The canonical metric on $SL(2, \mathbf{R})/SO(2)$ as a riemannian symmetric space coincides with the pull back

$$\rho^*\left(\frac{dp \otimes dp + dq \otimes dq}{q^2} \right)$$

of the Poincaré metric by ρ.

PROPOSITION 2.2. *For any element g in $SL(2, \mathbf{R})$, we have the following identity*

$$\varphi\circ J^{-1}\left(Ad(g)\frac{1}{2}\eta_3\right)=\rho\circ\pi(g)\,,$$

where π is the natural projection from $SL(2, \mathbf{R})$ to $SL(2, \mathbf{R})/SO(2)$.

PROOF. Setting $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$Ad(g)\frac{1}{2}\eta_3=(ac+bd)\frac{1}{2}\eta_1+\frac{1}{2}(a^2+b^2-c^2-d^2)\frac{1}{2}\eta_2+\frac{1}{2}(a^2+b^2+c^2+d^2)\frac{1}{2}\eta_3\,.$$

So $J^{-1}(Ad(g)\frac{1}{2}\eta_3)$ lies in $\mathscr{H}^+$. Mapping this point of $\mathscr{H}^+$ by ψ, we get

$$\psi\circ J^{-1}\left(Ad(g)\frac{1}{2}\eta_3\right)=\frac{2(ac+bd)}{2+a^2+b^2+c^2+d^2}+\sqrt{-1}\,\frac{a^2+b^2-c^2-d^2}{2+a^2+b^2+c^2+d^2}\,.$$

By a straightforward computation, $\gamma^{-1}\circ\rho\circ\pi(g)$ is equal to the right-hand side of this formula. This completes the proof of Proposition 2.2. □

3. $sl(2, \mathbf{C})$-valued 1-forms on a Riemann surface.

Let Σ be a Riemann surface and $f\colon \Sigma\to SL(2, \mathbf{R})/SO(2)$ a smooth map. We define an $sl(2, \mathbf{C})$-valued 1-form ω^f on Σ as follows. Take any point p of Σ, and let $(U(p), \xi)$ be a local coordinate system around p so that there exists a local lift $F\colon U(p)\to SL(2, \mathbf{R})$. Let A, B, C be complex-valued smooth functions on $U(p)$ such that

$$A\eta_1+B\eta_2+C\eta_3=F^{-1}\frac{\partial}{\partial\xi}F\,. \tag{3.1}$$

Let ω^f be the $sl(2, \mathbf{C})$-valued 1-form on Σ defined by

$$(\omega^f)_p=m(p)Ad(F(p))\sigma_-\otimes(d\xi)_p\,, \tag{3.2}$$

where the complex-valued smooth function m and the element σ_- of $sl(2, \mathbf{C})$ are defined respectively by

$$m=\sqrt{-1}A-B\,,\qquad \sigma_-=\frac{1}{2}(\eta_1-\sqrt{-1}\eta_2)\,. \tag{3.3}$$

Let ω denote the map $f\in C^\infty(\Sigma, SL(2, \mathbf{R})/SO(2))\mapsto\omega^f\in\Gamma(sl(2, \mathbf{C})\otimes T^{*1,0}_{\mathbf{C}}\Sigma)$.

LEMMA 3.4. *The map ω is well-defined.*

PROOF. It is easy to check that the definition of ω is indepednent of the choice of coordinate system. We verify that the definition of ω is independent of the choice of the lift F. To do this, let $\tilde{F}$ be another lift of f, i.e. $\tilde{F}=Fk$, wehre

$$k=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

is a map from $U(p)$ to $SO(2)$. Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{m}$ be the corresponding functions on $U(p)$. Setting $\mu=e^{\sqrt{-1}\theta}$, we get

$$\tilde{m}=\mu^2 m\,, \qquad Ad(\tilde{F})\sigma_-=\mu^{-2}Ad(F)\sigma_-\,.$$

Thus we see that

$$\tilde{m}Ad(\tilde{F})\sigma_-\otimes d\xi=mAd(F)\sigma_-\otimes d\xi\,.$$

This completes the proof of Lemma 3.4. □

Let $A^f=\omega^f+\overline{\omega^f}$, twice the real part of ω^f, and A the map $f\in C^\infty(\Sigma, SL(2,\mathbf{R})/SO(2))\mapsto A^f\in\Gamma(sl(2,\mathbf{R})\otimes T^*_{\mathbf{R}}\Sigma)$.

LEMMA 3.5. *Let f be a smooth map from Σ to $SL(2,\mathbf{R})/SO(2)$. Then f is harmonic if and only if $dA^f=0$.*

PROOF. To start the proof, we quote the following

THEOREM 3.6 [GO]. *Let G/K be a symmetric space and $\pi\colon G\to G/K$ the natural projection. We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$ respectively, and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition. Let $F\colon\Sigma\to G$ be a smooth map from a Riemann surface Σ into G, and let $\alpha=\alpha_0+\alpha_1=F^{-1}dF$ where α_0 and α_1 are $\mathfrak{k}$- and $\mathfrak{p}$-valued. Then $\pi\circ F$ is harmonic if and only if*

$$\bar{\partial}\alpha_1'+[\alpha_0\wedge\alpha_1']=0\,, \tag{3.7}$$

where α_1' is the $(1,0)$-component of α_1.

We apply this theorem to $G=SL(2,\mathbf{R})$ and $K=SO(2)$. Notice that $\mathfrak{k}=\mathbf{R}\eta_3$ and $\mathfrak{p}=\mathbf{R}\eta_1\oplus\mathbf{R}\eta_2$. Then by substituting (3.1), equation (3.7) becomes the system

$$\begin{cases}\dfrac{\partial}{\partial\bar{\xi}}B=2\bar{C}A\,,\\[2mm] \dfrac{\partial}{\partial\bar{\xi}}A=-2\bar{C}B\,.\end{cases} \tag{3.8}$$

By a direct calculation,

$$\begin{aligned}dA^f=&-(\sqrt{-1}A_{\bar{\xi}}-B_{\bar{\xi}}-2\sqrt{-1}m\bar{C})Ad(F)\sigma_-\otimes d\xi\wedge d\bar{\xi}\\ &+(-\sqrt{-1}\bar{A}_{\xi}-\bar{B}_{\xi}+2\sqrt{-1}\bar{m}C)Ad(F)\sigma_+\otimes d\xi\wedge d\bar{\xi}\,,\end{aligned} \tag{3.9}$$

where $\sigma_+=\frac{1}{2}(\eta_1+\sqrt{-1}\eta_2)$. If f is harmonic, then equation (3.8) holds, and substituting this into (3.9), we get $dA^f=0$, proving the "only if" part of the proposition. To prove the "if" part, we consider the Maurer-Cartan equation

$$d\theta+\frac{1}{2}[\theta\wedge\theta]=0\,,$$

where θ is the left Maurer-Cartan form of $SL(2, \mathbf{R})$. Since $\alpha=F^{-1}dF=F^*(\theta)$, we get

$$d\alpha+\frac{1}{2}[\alpha\wedge\alpha]=0\,. \tag{3.10}$$

Taking the $\mathfrak{p}$-part of this, we see that

$$d\alpha_1+[\alpha_0\wedge\alpha_1]=0\,. \tag{3.11}$$

If we use A, B, and C, this unravels to become

$$\begin{cases} A_{\bar{\xi}}-\bar{A}_{\xi}+2B\bar{C}-2\bar{B}C=0\,, \\ B_{\bar{\xi}}-\bar{B}_{\xi}-2A\bar{C}+2\bar{A}C=0\,. \end{cases} \tag{3.12}$$

On the other hand, since $dA^f=0$, it follows from (3.9) that

$$\begin{cases} \sqrt{-1}A_{\bar{\xi}}-B_{\bar{\xi}}-2\sqrt{-1}m\bar{C}=0\,, \\ -\sqrt{-1}\bar{A}_{\xi}-\bar{B}_{\xi}+2\sqrt{-1}\bar{m}C=0\,. \end{cases} \tag{3.13}$$

Solving equations (3.12) and (3.13), we get (3.8). We have completed the proof of Lemma 3.5. □

Let H be a fixed positive constant. For a smooth map $f: \Sigma\to SL(2, \mathbf{R})/SO(2)$, we define L^3-valued 1-form L^f on Σ by

$$L^f(X)=J^{-1}\left(\frac{1}{H}A^f(X)\right), \tag{3.14}$$

where X is an arbitrary tangent vector to Σ.

LEMMA 3.15. *Let Σ be a connected, simply connected Riemann surface and $f: \Sigma\to SL(2, \mathbf{R})/SO(2)$ a harmonic map. Then there exists a smooth map $\Phi^f: \Sigma\to L^3$, unique up to an additive constant, such that*

$$(\Phi^f)^*(\Omega)=L^f\,,$$

where $\Omega={}^t(dx, dy, dz)$.

PROOF. Since f is harmonic, we have $dA^f=0$ and so $dL^f=0$ by Lemma 3.5. Thus we can integrate L^f to get a smooth map $\Phi^f: \Sigma\to L^3$, determined up to an additive constant, such that $(\Phi^f)^*(\Omega)=L^f$. □

From this point on, we shall always assume that our Riemann surface Σ is connected and simply connected.

LEMMA 3.16. *Let $f: \Sigma \to SL(2, \mathbf{R})/SO(2)$ be a harmonic map and Φ^f as above. Then the pull back $(\Phi^f)^* \bar{g}$ is given by*

$$\Phi^{f*}\bar{g} = \frac{2m\bar{m}}{H^2}(d\xi \otimes d\bar{\xi} + d\bar{\xi} \otimes d\xi),$$

where m is defined as in (3.3).

PROOF. Set $\partial = \partial/\partial\xi$, $\bar{\partial} = \partial/\partial\bar{\xi}$. The $d\xi \otimes d\bar{\xi}$ component of I is given by

$$\langle \Phi^f_*\partial, \Phi^f_*\bar{\partial} \rangle = \left\langle \frac{1}{H} A^f(\partial), \frac{1}{H} A^f(\bar{\partial}) \right\rangle_{sl} = \frac{2m\bar{m}}{H^2},$$

where $\langle\ ,\ \rangle_{sl}$ is the Ad-invariant inner product of $sl(2, \mathbf{R})$ given by

$$\langle \eta_1, \eta_1 \rangle_{sl} = \langle \eta_2, \eta_2 \rangle_{sl} = -\langle \eta_3, \eta_3 \rangle_{sl} = 4, \qquad \langle \eta_i, \eta_j \rangle_{sl} = 0 \text{ if } i \neq j.$$

Similary we can compute other components, getting the desired formula. □

PROPOSITION 3.17. *Let $f: \Sigma \to SL(2, \mathbf{R})/SO(2)$ be a smooth map. Take any point p of Σ, and let $(U(p), \xi)$ be a coordinate neighborhood of p such that f has a local lift*

$$F: U(p) \to SL(2, \mathbf{R});\ \xi \mapsto \begin{pmatrix} a(\xi), b(\xi) \\ c(\xi), d(\xi) \end{pmatrix}.$$

Let $\Psi: \Sigma \to \mathbf{H}$ be the map defined by

$$\Psi = \rho \circ f.$$

Then $\Psi_*(\partial/\partial\xi)$ is given by

$$\Psi_*\left(\frac{\partial}{\partial\xi}\right) = \{m(\beta^1 - \sqrt{-1}\beta^2)\}\frac{\partial}{\partial w} + \{(\sqrt{-1}A + B)(-\beta^1 - \sqrt{-1}\beta^2)\}\frac{\partial}{\partial\bar{w}}$$

on $U(p)$, where m, A, and B are defined as in (3.1) *and* (3.3), *w is the complex coordinate of $\mathbf{H}$, and*

$$\beta^1 = \frac{2(d^2 - c^2)}{(d^2 + c^2)^2}, \qquad \beta^2 = \frac{4cd}{(d^2 + c^2)^2}.$$

In particular, the equation $\partial\Psi/\partial\xi(p) = 0$ holds if and only if $m(p) = 0$.

PROOF. A straightforward computation. □

COROLLARY 3.18. *Let f and Φ^f be as in Lemma* 3.15. *Let $\mathcal{N}$ be the subset of Σ defined by $\mathcal{N} = \{p \in \Sigma \mid \partial\Psi/\partial\xi(p) \neq 0\}$, where $\Psi = \rho \circ f$. Then $\tilde{\Sigma} = \mathcal{N}$.*

PROOF. This follows from the definition of $\tilde{\Sigma}$, Lemma 3.16, and Proposition 3.17. □

LEMMA 3.19. *Let f and Φ^f be as in Lemma* 3.15. *Then the image of the Gauss* *map* $G(\Phi^f): \tilde{\Sigma} \to \mathscr{H}$ *is contained in* $\mathscr{H}^+$ *and*

$$\varphi \circ G(\Phi^f) = \rho \circ f \quad on\ \tilde{\Sigma}.$$

PROOF. Take any point $p \in \tilde{\Sigma}$, and let $(U(p), \xi) \subset \tilde{\Sigma}$ be a coordinate neighborhood of p such that f has a local lift $F: U(p) \to SL(2, \mathbf{R})$. We can choose

$$e_1 = \Phi^f_*\left(\frac{H}{2\sqrt{m\bar{m}}} \frac{\partial}{\partial \xi^1}\right), \qquad e_2 = \Phi^f_*\left(\frac{H}{2\sqrt{m\bar{m}}} \frac{\partial}{\partial \xi^2}\right)$$

as an orthonormal frame on $U(p)$. By definition, the value of the Gauss map $G(\Phi^f)$ at p is given by

$$\begin{aligned} G(\Phi^f)(p) &= e_3(p) = e_1(p) \times e_2(p) \\ &= J^{-1}\left(\left[\frac{1}{H} A^f\left(\frac{H}{2\sqrt{m\bar{m}}} \frac{\partial}{\partial \xi^1}\right), \frac{1}{H} A^f\left(\frac{H}{2\sqrt{m\bar{m}}} \frac{\partial}{\partial \xi^2}\right)\right]\right) \\ &= J^{-1}\left(Ad(F(p)) \frac{1}{2} \eta_3\right). \end{aligned}$$

Using Proposition 2.2, $G(\Phi^f)(p)$ lies in $\mathscr{H}^+$ and

$$\begin{aligned} \varphi \circ G(\Phi^f)(p) &= \varphi \circ \left(J^{-1}(Ad(F(p)) \frac{1}{2} \eta_3)\right) \\ &= \rho \circ \pi(F(p)) = \rho \circ f(p). \end{aligned}$$

□

LEMMA 3.20. *Let* Φ^f *be as in Lemma* 3.15, *and II the second fundamental form of the immersion* $\Phi^f|_{\tilde{\Sigma}}: \tilde{\Sigma} \to L^3$. *Then II is given by*

$$II = \frac{2(A^2+B^2)}{H} d\xi \otimes d\xi + \frac{2m\bar{m}}{H} d\xi \otimes d\bar{\xi} + \frac{2m\bar{m}}{H} d\bar{\xi} \otimes d\xi + \frac{\overline{2(A^2+B^2)}}{H} d\bar{\xi} \otimes d\bar{\xi},$$

where A, B, and m are defined as in (3.1) *and* (3.3).

PROOF. *Set* $\Phi = \Phi^f$. The $d\xi \otimes d\xi$ component of II is given by

$$-\langle D_\partial(\Phi_* \partial), e_3 \rangle = -\left\langle \frac{1}{H} \partial(A^f(\partial)), Ad(F) \frac{1}{2} \eta_3 \right\rangle_{sl} = \frac{2(A^2+B^2)}{H}.$$

The other components can be computed in a similar way, and we get the desired formula. □

COROLLARY 3.21. *Let* Φ^f *be as in Lemma* 3.15. *Then the mean curvature of* $\Phi^f|_{\tilde{\Sigma}}: \tilde{\Sigma} \to L^3$ *is equal to the constant H.*

PROOF. Let $I = \Phi^{f*}\tilde{g}|_{\tilde{\Sigma}}$. The mean curvature of $\Phi^f|_{\tilde{\Sigma}}$ is given by $(1/2)\mathrm{trace}_I(II)$. Using Lemma 3.16 and Lemma 3.20, we get $(1/2)\mathrm{trace}_I(II) = H$. □

Let Harm denote the space of harmonic maps from Σ into $SL(2, \mathbf{R})/SO(2)$, and let $\tilde{C}^{\infty}(\Sigma, L^3)$ be the space of equivalence classes of maps from Σ into L^3, where two elements $\Phi_1, \Phi_2 : \Sigma \to L^3$ are equivalent if $\Phi_2 = \Phi_1 + c$ for some constant vector c in L^3. By Lemma 3.15 we have the map

$$R: \mathrm{Harm} \to \tilde{C}^{\infty}(\Sigma, L^3); \qquad f \mapsto [\Phi^f],$$

where Φ^f satisfies $\Phi^{f*}(\Omega) = L^f$. Let Harm* be the set of elements of Harm which are nowhere anti-holomorphic, and denote by $\mathrm{Imm}_H(\Sigma, L^3)$ the set of elements of $\tilde{C}^{\infty}(\Sigma, L^3)$ whose representatives are conformal spacelike immersion with constant mean curvature $H > 0$ having Gauss images in $\mathscr{H}^+$.

THEOREM 3.22. *The image of* Harm* *by* R *is contained in* $\mathrm{Imm}_H(\Sigma, L^3)$. *Moreover* $R^*: \mathrm{Harm}^* \to \mathrm{Imm}_H(\Sigma, L^3)$ *is bijective, where* R^* *is the restriction of* R *to* Harm*.

PROOF. The first statement follows immediately from Lemma 3.16, Corollary 3.18, Lemma 3.19 and Corollary 3.21. Let us prove the bijectivity of R^*. Since the Gauss map of an immersion with constant mean curvature is harmonic [M], we can define the map $\tilde{G}: \mathrm{Imm}_H(\Sigma, L^3) \to \mathrm{Harm}$ by $[\Phi] \mapsto \rho^{-1} \circ \varphi \circ G(\Phi)$. First we shall show that the image of $\tilde{G}$ is contained in Harm*. Assume that $\tilde{G}([\Phi])$ is not a nowhere anti-holomorphic map, i.e. $\Psi = \psi \circ G(\Phi)$ has some point $p \in \Sigma$ such that $\partial\Psi/\partial\xi(p) = 0$. Since the induced metric $\Phi^*\bar{g}$ is given by

$$\Phi^*\bar{g} = \left[\frac{1}{H} \frac{2}{(1 - |\Psi|^2)} \left| \frac{\partial\Psi}{\partial\xi} \right| \right]^2 ((d\xi^1)^2 + (d\xi^2)^2)$$

(See [AN].), Φ is degenerate at the point p. (Note that the orientation of $\mathbf{D}$ in this paper is opposite to the one in [AN]. So the above expression of the metric is slightly different from that in [AN].) This contradicts the assumption that Φ is an immersion. So $\tilde{G}$ maps $\mathrm{Imm}_H(\Sigma, L^3)$ into Harm*. We have $\tilde{G} \circ R^* = \mathrm{id}$, as is easily derived from Lemma 3.19. So to prove the bijectivity of R^*, it remains to show that $\tilde{G}$ is injective. Let $[\Phi_1]$, $[\Phi_2]$ be two elements of $\mathrm{Imm}_H(\Sigma, L^3)$ such that $\tilde{G}([\Phi_1]) = \tilde{G}([\Phi_2])$. Since the Gauss maps and the mean curvatures of Φ_1 and Φ_2 agree, their first and the second fundamental forms must also agree. By the fundamental theorem of differential geometry, there exists a rotational isometry σ of L^3 and a constant vector c of L^3 such that $\Phi_2 = \sigma \circ (\Phi_1 + c)$ and $\sigma(0) = 0$. Combining this and the assumption that $G(\Phi_1) = G(\Phi_2)$, we have $G(\Phi_1) = \sigma(G(\Phi_1))$. Suppose that σ is not the identity. Then since σ is an orientation preserving rotational isometry of L^3, which fixes $0 \in L^3$ and satisfies $\sigma(\mathscr{H}^+) = \mathscr{H}^+$, σ has at most one fixed point in $\mathscr{H}^+$. So $\tilde{G}([\Phi_1])$ is a constant map and, in particular, is an anti-holomorphic map. This contradiction shows that σ must be the identity, and so $\Phi_1 = \Phi_2 + c$, i.e. $[\Phi_1] = [\Phi_2]$. Thus $\tilde{G}$ is injective. Theorem 3.22 has been proved. □

4. The Sym-Bobenko formula for Minkowski 3-space.

We shall derive the Sym-Bobenko formula for Minkowski 3-space.

THEOREM 4.1. *Let Σ be a simply connected Riemann surface and $f: \Sigma \to SL(2, \mathbf{R})/SO(2)$ a hormonic map. Let $F: \Sigma \to SL(2, \mathbf{R})$ be a lift of f. Let $F(\cdot): (-\varepsilon, \varepsilon) \times \Sigma \to SL(2, \mathbf{R})$, $\varepsilon > 0$, be a smooth map such that*

(1) $F(0) = F$,

(2)

$$\frac{d}{dt}\bigg|_{t=0}\left(F(t)^{-1}\frac{\partial}{\partial\xi}F(t)\right) = -\sqrt{-1}(A\eta_1 + B\eta_2),$$

where $A\eta_1 + B\eta_2 + C\eta_3 = F^{-1}\frac{\partial}{\partial\xi}F$. *Then* $R(f)$ *is given by*

$$R(f) = \left[J^{-1}\left\{-\frac{1}{2H}\left(\left(\frac{d}{dt}\bigg|_{t=0}F(t)\right)F^{-1} - Ad(F)\frac{1}{2}\eta_3\right)\right\}\right].$$

PROOF. Set $[\Phi] = R(f)$. It suffices to show that

$$\frac{\partial\Phi}{\partial\xi} = \frac{\partial}{\partial\xi}\left\{J^{-1}\left\{-\frac{1}{2H}\left(\left(\frac{d}{dt}\bigg|_{t=0}F(t)\right)F^{-1} - Ad(F)\frac{1}{2}\eta_3\right)\right\}\right\}. \tag{4.2}$$

Equation (4.2) is equivalent to

$$J\left(\frac{\partial\Phi}{\partial\xi}\right) = \frac{\partial}{\partial\xi}\left\{-\frac{1}{2H}\left(\left(\frac{d}{dt}\bigg|_{t=0}F(t)\right)F^{-1} - Ad(F)\frac{1}{2}\eta_3\right)\right\}. \tag{4.3}$$

By the definition of $R(f)$ and the property of L^f, the left-hand side of (4.3) is equal to

$$\frac{1}{H}A^f(\partial) = \frac{1}{H}mAd(F)\sigma_- . \tag{4.4}$$

On the other hand, $-2H$ times the right-hand side of (4.3) is equal to

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\left(\left(\frac{d}{dt}\bigg|_{t=0}F(t)\right)F^{-1}\right) - \frac{\partial}{\partial\xi}\left(Ad(F)\frac{1}{2}\eta_3\right) \\
&= \left(\partial\left(\frac{d}{dt}\bigg|_{t=0}F(t)\right)\right)F^{-1} + \frac{d}{dt}\bigg|_{t=0}F(t)\partial(F^{-1}) - Ad(F)\left[F^{-1}\partial F, \frac{1}{2}\eta_3\right] \\
&= Ad(F)\left\{\frac{d}{dt}\bigg|_{t=0}(F(t)^{-1}\partial F(t))\right\} - Ad(F)(A\eta_2 - B\eta_1) \\
&= Ad(F)\{-\sqrt{-1}(A\eta_1 + B\eta_2) - (A\eta_2 - B\eta_1)\} = -2mAd(F)\sigma_- .
\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we get (4.3). □

EXAMPLE 4.6. Let $\tilde{f}: \mathbf{C} \to \mathbf{H}$ be the harmonic map defined by

$$\xi = x + \sqrt{-1}y \mapsto \sqrt{-1}\exp(4y),$$

and let $f: \mathbf{C} \to SL(2, \boldsymbol{R})/SO(2)$ be the harmonic map defined by $f = \rho^{-1} \circ \tilde{f}$. Let $F(\cdot): (-\pi, \pi) \times \mathbf{C} \to SL(2, \mathbf{R})$ be the map defined by

$$(t, x + \sqrt{-1}y) \mapsto \begin{pmatrix} \exp(-\sqrt{-1}\lambda^{-1}\xi + \sqrt{-1}\lambda\bar{\xi}) & 0 \\ 0 & \exp(\sqrt{-1}\lambda^{-1}\xi - \sqrt{-1}\lambda\bar{\xi}) \end{pmatrix},$$

where $\lambda = \exp(\sqrt{-1}t)$. Set $H = 1$. Then $F(\cdot)$ satisfies the conditions of Theorem 4.1, and $R(f)$ is given by

$$R(f) = \left[x + \sqrt{-1}y \mapsto {}^t\!\left(2x, \frac{\sinh 4y}{2}, \frac{\cosh 4y}{2} \right) \right].$$

This is a hyperbolic cylinder of mean curvature 1 in L^3.

EXAMPLE 4.7. Let $\tilde{f}: \mathbf{H} \to \mathbf{H}$ be the identity map, and let $f = \rho^{-1} \circ \tilde{f}: \mathbf{H} \to SL(2, \mathbf{R})/SO(2)$. Let $F(\cdot): (-\pi, \pi) \times \mathbf{H} \to SL(2, \mathbf{R})$ be the map defined by

$$(t, x + \sqrt{-1}y) \mapsto \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos(1/2)t & \sin(1/2)t \\ -\sin(1/2)t & \cos(1/2)t \end{pmatrix}.$$

Set $H = 1$. Then $F(\cdot)$ satisfies the conditions of Theorem 4.1, and $R(f)$ is given by

$$R(f) = \left[x + \sqrt{-1}y \mapsto {}^t\!\left(\frac{x}{y}, -\frac{1}{2y} + \frac{y}{2} + \frac{x^2}{2y}, \frac{1}{2y} + \frac{y}{2} + \frac{x^2}{2y} \right) \right].$$

This is a hyperboloid of mean curvature 1 in L^3.

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