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On the Greatest Regular Closed Subalgebras and the Apostol Algebras of L^p-Multipliers Whose Fourier Transforms Are Continuous and Vanish at Infinity

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Abstract. For certain algebras of continuous functions, the relationship between the greatest regular subalgebras, the algebras which consist of functions of which corresponding multiplication operators are decomposable, and the sets of functions with natural spectra are studied. In particular, spectral properties of certain Fourier multipliers are considered.

1. Introduction.

Let B be a commutative Banach algebra. Inoue-Takahasi [6] and Neumann [12] independently proved that there is the greatest regular closed subalgebra $\mathrm{Reg}B$ of B. The existence for the case that B is semi-simple and unital had been discovered by Albrecht [1]. Given a Banach space X, a bounded linear operator T on X is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane C, there exists T-invariant closed linear subspaces X_U and X_V of X such that $\sigma(T|X_U)\subset U$, $\sigma(T|X_V)\subset V$ and $X_U+X_V=X$, where $\sigma(\cdot)$ denotes the spectrum of an operator. A subset $\mathrm{Dec}B$ of B consists of $b\in T$ for which the corresponding multiplication operator T_b on B defined by $T_b(a)=ab$ for $a\in B$ is a decomposable operator on B. Neumann [11] proved that if B is semi-simple, then $b\in\mathrm{Dec}B$ if and only if the Gelfand transform $\check{b}$ of B is hull-kernel continuous on the maximal ideal space Φ_B of B. It follows that $\mathrm{Dec}B$ is a closed subalgebra of B and $\mathrm{Reg}B\subset\mathrm{Dec}B$. The algebra $\mathrm{Dec}B$ is called the Apostol algebra of B and dates back to classical work of Apostol [2]. The author does not know for which B the identity $\mathrm{Reg}B=\mathrm{Dec}B$ holds. Laursen and Neumann [10] studied the case where $B=M_0(A)$, where $M_0(A)$ is a closed subalgebra of the *multiplier algebra* $M(A)$ of a semi-simple commutative Banach algebra A which consists of $T\in M(A)$ such that

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the restriction of the Gelfand transform $\check{T}$ in $M(A)$ to Φ_A vanishes at infinity in the Gelfand topology of the maximal ideal space Φ_A of A. We denote

$$M_{00}(A)=\{T\in M(A) : \check{T}=0 \text{ on } \Phi_{M(A)}\backslash\Phi_A\} .$$

THEOREM (Laursen and Neumann). *Let A be a semi-simple regular commutative Banach algebra. Then*

$$\operatorname{Reg} M_0(A)=\operatorname{Dec} M_0(A)=M_{00}(A) .$$

Moreover if Φ_A is scattered, then

$$M_{00}(A)=\{T\in M_0(A) : \sigma(T) \text{ is countable}\}=\{T\in M_0(A) : \sigma(T)=\overline{\check{T}(\Phi_A)}\} .$$

Let G be a locally compact abelian group with dual group $\hat{G}$. For $1\leq p<\infty$, $L^p(G)$ will denote the usual L^p space with respect to the Harr measure on G. Let $M_p(G)$ designate the algebra of bounded operators on $L^p(G)$ which commute with all translations on $L^p(G)$. The algebra $M_p(G)$ is called the L^p-multiplier and is a commutative Banach algebra which is isometric and isomorphic to the measure algebra $M(G)$ if $p=1$ [16]. If $T\in M_p(G)$, then there exists a unique function $\hat{T}\in L^\infty(\hat{G})$ such that $\widehat{Tf}=\hat{T}\hat{f}$, for all integrable simple functions f on G, where $\hat{f}$ denotes the Fourier transform of f. We also say that $\hat{T}$ is the Fourier transform of $T\in M_p(G)$. We denote by $C_0M_p(G)$ those multipliers $T\in M_p(G)$ for which $\hat{T}$ is continuous and vanishes at infinity on $\hat{G}$. If $p=1$ or G is compact, then $L^p(G)$ is a semi-simple commutative algebra and $M_p(G)$ coincides with the multiplier algebra $M(L^p(G))$ of $L^p(G)$. Since the maximal ideal space of $L^p(G)$ is $\hat{G}$, we see that

$$M_0(L^p(G))=C_0M_p(G) .$$

By a theorem of Laursen and Neumann we see that

$$\operatorname{Reg} M_0(L^p(G))=\operatorname{Dec} M_0(L^p(G))=M_{00}(L^p(G))$$

if $p=1$ or G is compact. Moreover they coincide with

$$\{T\in M_0(L^p(G)) : \sigma(T) \text{ is countable}\}$$

and

$$\{T\in M_0(L^p(G)) : \sigma(T)=\overline{\hat{T}(\hat{G})}\}$$

if G is compact. If $1<p<\infty$, $p\neq 2$ and if G is not compact, then $L^p(G)$ is not a commutative Banach algebra under the convolution. Then a theorem of Laursen and Neumann cannot be directly applied for $L^p(G)$ and $C_0M_p(G)$.

In this paper, we show that $\operatorname{Reg} M=\operatorname{Dec} M$ for certain Banach algebras M of continuous functions on locally compact Hausdorff spaces Y. As a consequence of the result we show that $\operatorname{Reg} C_0M_p(G)$ coincides with $\operatorname{Dec} C_0M_p(G)$ for any locally compact abelian group G and $1\leq p<\infty$. We also consider the case where Y is scattered.

2. Spectra of continuous functions.

For a commutative Banach algebra B and $b \in B$, $\check{b}$ will denote the Gelfand transform of b, $\check{B} = \{\check{b} : b \in B\}$ and Φ_B the maximal ideal space for B. For an element b in B, we denote the spectrum of b in B by sp(b, B). Let Y be a locally compact Hausdorff space. The algebra of all complex-valued continuous functions which vanish at infinity on Y will be denoted by $C_0(Y)$. Let M be a commutative Banach algebra included in $C_0(Y)$ which *strongly* separates the points in Y, that is, M separates the points in Y and $\ker M = \{y \in Y : m(y) = 0, \forall m \in M\} = \varnothing$. The original norm for M is denoted by $\|\cdot\|_M$. On the other hand, for a subset K of Φ_M, $\|\cdot\|_{\infty(K)}$ stands for the uniform norm on K. For a point $y \in Y$, we denote

$$L_y = \{p \in \Phi_M : \check{m}(p) = m(y) \text{ for } \forall m \in \operatorname{Reg} M\},$$

$$L_\infty = \{p \in \Phi_M : \check{m}(p) = 0 \text{ for } \forall m \in \operatorname{Reg} M\}.$$

Let Ns(M) designate the set of functions with natural spectra, that is,

$$\operatorname{Ns}(M) = \{m \in M : \operatorname{sp}(m, M) = \overline{m(Y)}\}.$$

We also denote

$$M' = \{m \in M : \check{m}|L_y \text{ is constant for } \forall y \in Y, \check{m}|L_\infty = 0\}.$$

Then M' is a closed subalgebra of M since $\|\check{m}\|_{\infty(\Phi_M)} \leq \|m\|_M$, and contains Reg$M$. By a theorem of Neumann [11] we see that

$$\operatorname{Reg} M \subset \operatorname{Dec} M \subset \operatorname{Ns}(M).$$

THEOREM 2.1. *Let Y be a locally compact Hausdorff space and M a commutative Banach algebra included in $C_0(Y)$ which strongly separates the points in Y. When Y is compact we assume $1 \in M$. Suppose that* Reg$M$ *is dense in $C_0(Y)$. If a subset S of* Ns(M) *contains* Reg$M$ *and is closed under addition, then* Reg$M = S$. *In particular,* Reg$M =$ Dec$M = M'$.

PROOF. We first consider the case where Y is compact. In this case $L_\infty = \varnothing$ since $1 \in M$. Let x and y be a pair of distinct points in Y. Then $L_x \cap L_y = \varnothing$ since

$$L_a = \bigcap_{m \in \operatorname{Reg} M} \check{m}^{-1}(m(a))$$

for every $a \in Y$ and Reg$M$ separates the points in Y. Let Φ_M be the maximal ideal space of M. Then

$$\Phi_M = \bigcup_{x \in Y} L_x.$$

Suppose not. Choose $p \in \Phi_M \setminus \bigcup_{x \in Y} L_x$. Put $J = \{m \in \operatorname{Reg} M : \check{m}(p) = 0\}$. Since $1 \in \operatorname{Reg} M$, J is a proper ideal of Reg$M$. There exists a maximal ideal $\tilde{J}$ of Reg$M$ with $J \subset \tilde{J}$. Since the maximal ideal space $\Phi_{\mathrm{Reg}M}$ of $\mathrm{Reg}M$ coincides with Y, there exists $a\in Y$ such that

$$\tilde{J}=\{m\in \mathrm{Reg}M : m(a)=0\}\ .$$

There exists $m_p\in \mathrm{Reg}M$ such that $\check{m}_p(p)\neq m_p(a)$ for $p\in\Phi_M\backslash L_a$. Since $1\in \mathrm{Reg}M$ we may assume $\check{m}_p(p)=0$, so $m_p\in J\subset\tilde{J}$. We have $m_p(a)=0$, which is a contradiction. We have just proved that $\Phi_M=\bigcup_{x\in Y}L_x$.

Next we prove $S\subset M'$. Suppose that $m_S\in S\backslash M'$. Then the set $\check{m}_S(L_a)$ contains at least two points for some point $a\in Y$, so there exists $p\in L_a$ such that $\varepsilon=|\check{m}_S(p)-m_S(a)|/3$ is positive. Since $\mathrm{Reg}M$ is dense in $C_0(Y)$ we can choose $m_\varepsilon\in \mathrm{Reg}M$ which satisfies the inequality $\|m_S-m_\varepsilon\|_{\infty(Y)}<\varepsilon$. Since $\mathrm{Reg}M\subset S$ and $S+S\subset S\subset \mathrm{Ns}(M)$, we have $m_S-m_\varepsilon\in \mathrm{Ns}(M)$. Thus $(m_S-m_\varepsilon)\check{}(\Phi_M)\subset\{z\in C:|z|<\varepsilon\}$. Since $m_\varepsilon\in \mathrm{Reg}M$ we have $\check{m}_\varepsilon(p)=m_\varepsilon(a)$, so the inequalities

$$|(m_S-m_\varepsilon)\check{}(p)|\geq|\check{m}_S(p)-m_S(a)|-|m_S(a)-m_\varepsilon(a)|>2\varepsilon$$

hold, which is a contradiction.

Note that M' is a closed subalgebra of M. We also see that $M'\subset \mathrm{Ns}(M)$ since $\Phi_M=\bigcup_{x\in Y}L_x$ and $\mathrm{sp}(m, M)=\check{m}(\Phi_M)$ for $m\in M$. By a simple calculation we have $\mathrm{sp}(m, M')=\mathrm{sp}(m, M)$ for every $m\in M'$, so $\mathrm{sp}(m, M')=m(Y)$ for every $m\in M'$. We show that the maximal ideal space $\Phi_{M'}$ of M' coincides with Y. Let $m_1, \cdots, m_n\in M'$ be a corona data, that is, $\sum|m_i|>0$ on Y. Then there are functions $h_1, \cdots, h_n\in C_0(Y)$ such that $\sum_{i=1}^n m_ih_i=1$ on Y. Since $\mathrm{Reg}M$ is dense in $C_0(Y)$ and $\mathrm{Reg}M\subset M'$ we can choose functions $u_1, \cdots, u_n\in M'$ which satisfies the inequality

$$\left|\sum_{i=1}^n m_iu_i-1\right|<1/2$$

on Y. Put $F=\sum_{i=1}^n m_iu_i$. Then $F\in M'$, so $\mathrm{sp}(F, M')=F(Y)$. Thus $F^{-1}\in M'$. Put $g_1=u_1F^{-1}, \cdots, g_n=u_nF^{-1}$. Then $g_1, \cdots, g_n\in M'$ is a corona solution, that is, $\sum_{i=1}^n m_ig_i=1$. It follows that $\Phi_{M'}=Y$. (Suppose that $p\in\Phi_{M'}\backslash Y$. For every $y\in Y$ there is a function $f_y\in M'$ with $\check{f}_y(p)=0$ and $f_y(y)=1$. Since $\Phi_{M'}$ is compact there is a finite number of $y_1, \cdots, y_n\in Y$ with

$$\sum_{i=1}^n |f_{y_i}|>1/2$$

on Y. Thus there are functions $g_1, \cdots, g_n$ such that

$$\sum_{i=1}^n g_if_{y_i}=1$$

on Y, which is a contradiction since $\check{f}_{y_1}(p)=\cdots=\check{f}_{y_n}(p)=0$.)

Let K be a closed subset of Y and $x\in Y\backslash K$. Then there is $m\in \mathrm{Reg}M$ such that $m(K)=0$ and $m(x)=1$. Since $\mathrm{Reg}M\subset M'$ and $\Phi_{M'}=Y$ we see that M' is regular, henceforce $M'=\mathrm{Reg}M$. It follows that $\mathrm{Reg}M=S=M'$. Since M is semi-simple and Y

is hull-kernel dense in Φ_M, we have $\mathrm{Reg}\, M \subset \mathrm{Dec}\, M = \{m \in M : \check{m}$ is hull-kernel continuous on $\Phi_M\} \subset \mathrm{Ns}(M)$ by a theorem of Neumann [12]. Since $\mathrm{Dec}\, M$ is closed under addition we have $\mathrm{Reg}\, M = \mathrm{Dec}\, M$.

We next consider the case where Y is not compact. Let $M[e]$ be the unitalization of M. For a subset P of M, we denote by $P[e]$ the directed sum of P and the space of constant functions on Y. We denote by $\overline{\Phi_M} = \Phi_M \cup \{\infty\}$ the one point compactification of Φ_M. Then we may suppose that the one point compactification $\bar{Y}$ of Y is a closed subset of $\overline{\Phi_M}$. It is well-known that the maximal ideal space $\Phi_{M[e]}$ of $M[e]$ coincides with $\overline{\Phi_M}$ (cf. [9]). It is easy to see that $\mathrm{Reg}\, M[e] = (\mathrm{Reg}\, M)[e]$ and $\mathrm{Ns}(M[e]) = \mathrm{Ns}(M)[e]$. Henceforce $\mathrm{Reg}\, M[e] \subset S[e] \subset \mathrm{Ns}(M[e])$. It follows by the first part of the proof that $\mathrm{Reg}\, M[e] = S[e]$, in particular, $\mathrm{Reg}\, M[e] = \mathrm{Dec}\, M[e] = M[e]'$. By a simple calculation we have $\mathrm{Reg}\, M = S$ and $\mathrm{Reg}\, M = \mathrm{Dec}\, M = M'$. Q.E.D.

COROLLARY 2.2. *Suppose that M is the same as in Theorem 2.1. Then* $\mathrm{Reg}\, M = \mathrm{Dec}\, M = M' = \mathrm{Ns}(M)$ *if and only if* $\mathrm{Ns}(M)$ *is closed under addition.*

We can easily prove Corollary 2.2 by using Theorem 2.1 and a proof is omitted.

Eschmeier, Laursen and Neumann [3] proved that $\mathrm{NS}(M(G)) \cap M_0(G) = M_{00}(G)$ if and only if $\mathrm{NS}(M(G)) \cap M_0(G)$ is closed under addition, where $\overline{M_{00}}(G) = \{\mu \in M(G) : \check{\mu} = 0$ on $\Phi_{M(G)} \backslash \hat{G}\}$ and $\mathrm{NS}(M(G)) = \{\mu \in M(G) : \mathrm{sp}(\mu, M(G)) = \overline{\hat{\mu}(\hat{G})}\}$. Since $\mathrm{Reg}\, M_0(G) = \mathrm{Dec}\, M_0(G) = M_{00}(G)$, we see that Corollary 2.2 is a generalization of a result of Eschmeier, Laursen and Neumann [3].

In general $\mathrm{Ns}(M)$ need not be closed under addition. Zafran [18] showed that for an I-group G there exist measures μ and ν in $M(G)$ such that $\mathrm{sp}(\mu, M(G)) = \overline{\hat{\mu}(\hat{G})}$, $\mathrm{sp}(\nu, M(G)) = \overline{\hat{\nu}(\hat{G})}$ and $\mathrm{sp}(\mu+\nu, M(G))$ properly contains $\overline{(\mu+\nu)\hat{}(\hat{G})}$. The author [4] showed that for an infinite compact abelian group G, $\mathrm{NS}(M_p(G)) + \mathrm{NS}(M_p(G)) = M_p(G)$ for $1 < p < \infty$, $p \neq 2$, which means that $\mathrm{NS}(M_p(G))$ is not closed under addition since $\mathrm{NS}(M_p(G)) \neq M_p(G)$ by a theorem of Igari [5], where $\mathrm{NS}(M_p(G)) = \{T \in M_p(G) : \mathrm{sp}(T, M_p(G)) = \overline{\hat{T}(\hat{G})}\}$.

On the othr hand, the situation is relatively simple in the case where Y is scattered. Recall that a locally compact Hausdorff space Y is scattered if every non-empty compact subset of Y contains an isolated point. Every discrete space is scattered and every scattered space is totally disconnected. A locally compact Hausdorff space Y is scattered if and only if $f(Y)$ is countable for every $f \in C_0(Y)$ (cf. [13], [14]).

THEOREM 2.3. *Let Y be a locally compact Hausdorff scattered space and M a commutative Banach algebra included in $C_0(Y)$ which strongly separates the points in Y. We assume $1 \in M$ if Y is compact. Then we have* $\mathrm{Reg}\, M = \mathrm{Dec}\, M = \{m \in M : \mathrm{sp}(m, M)$ *is countable*$\} = \mathrm{Ns}(M)$.

To prove the theorem we need some lemmas. Put $M'' = \{m \in M : \mathrm{sp}(m, M)$ is countable$\}$.

LEMMA 2.4. $\mathrm{Ns}(M) = M''$.

PROOF. Suppose that $m \in \mathrm{Ns}(M)$. Then $\mathrm{sp}(m, M)=\overline{m(Y)} \subset m(Y) \cup \{0\}$, so $\mathrm{sp}(m, M)$ is countable since Y is scattered. We have $m \in M''$.

Suppose that there exists $m \in M'' \backslash \mathrm{Ns}(M)$. Then there exists $\lambda \in \mathrm{sp}(m, M) \backslash \overline{m(Y)}$. Put $S_r=\{z \in C: |z-\lambda|=r\}$ and $U_r=\{z \in C: |z-\lambda|<r\}$ for a positive real number r. Let r_0 be the distance from λ to $\overline{m(Y)}$. Since $\mathrm{sp}(m, M)$ is countable, there is a positive real number r with $r<r_0$ such that $S_r \cap \mathrm{sp}(m, M)=\varnothing$. Put $U=U_r$. Then $\check{m}^{-1}(U)=\check{m}^{-1}(U \cup S_r)$ is compact and $\check{m}^{-1}(U) \cap Y=\varnothing$, where $\check{m}$ is the Gelfand transformation of m in M. Since $\check{m}^{-1}(U)$ is also an open subset of Φ_M, by the Šilov idempotent theorem there exists $l \in M$ such that

$$\check{l}=\begin{cases} 1 & \text{on } \check{m}^{-1}(U) \\ 0 & \text{on } \Phi_M \backslash \check{m}^{-1}(U), \end{cases}$$

which is a contradiction since $Y \subset \Phi_M \backslash \check{m}^{-1}(U)$ and so $\check{l}=0$ on Y. Q.E.D.

LEMMA 2.5. *M'' is a closed subalgebra of M.*

PROOF. It is easy to see that M'' is a subalgebra of M. We show that M'' is closed in M. Suppose that $m_n \in M''$ and $m_n \to m$ in M. Then

$$\|\check{m}_n-\check{m}\|_{\infty(\Phi_M)} \leq \|m_n-m\|_M \to 0,$$

as $n \to \infty$. Suppose that $m \in M \backslash M''$. Then there exists $p \in \Phi_M$ such that $\check{m}(p) \in \check{m}(\Phi_M) \backslash \overline{m(Y)}$ since $M''=\mathrm{Ns}(M)$ by Lemma 2.4. By a simple calculation we can find $p_n \in Y$ such that $\check{m}_n(p)=m_n(p_n)$ since $m_n \in M''$ and $M''=\mathrm{Ns}(M)$ by Lemma 2.4. Let p_0 be a cluster point of $\{p_n\}$ in Y. (Suppose that Y is not compact. Then $|m(p_n)-\check{m}_n(p)|=|m(p_n)-m_n(p_n)| \leq \|m-m_n\|_M \to 0$ as $n \to \infty$. Since $\check{m}_n(p) \to \check{m}(p)$ and $\check{m}(p) \neq 0$ we may suppose that there exists $\varepsilon>0$ such that $|m(p_n)|>\varepsilon$ for every positive integer n. It follows that $\{p_n\}$ is included in a compact subset of Y. Thus there is a cluster point $p_0 \in Y$.) We may suppose that $m(p_n) \to m(p_0)$ as $n \to \infty$. Then we have

$$|\check{m}(p)-m(p_0)| \leq 2\|\check{m}-\check{m}_n\|_{\infty(\Phi_M)}+|m(p_n)-m(p_0)| \to 0$$

as $n \to \infty$, hence $\check{m}(p)=m(p_0)$, which is a contradiction. Q.E.D.

By a simple calculation we see

LEMMA 2.6. *For every $m \in M''$ we have* $\mathrm{sp}(m, M'')=\mathrm{sp}(m, M)$.

LEMMA 2.7. *M'' is a regular commutative Banach algebra.*

PROOF. By Lemma 2.5 M'' is a commutative Banach algebra, so we show regularity of M''. Let K be a closed subset of the maximal ideal space $\Phi_{M''}$ of M'' and $x \in \Phi_{M''} \backslash K$. Then there exists $m \in M''$ such that $\tilde{m}(x)=1$, where $\tilde{m}$ is the Gelfand transform of m in M''. By Lemma 2.4 and Lemma 2.6 there exists an open neighborhood U of 1 with $U \subset \{z \in C: |z-1|<1/2\}$ such that $F=\tilde{m}^{-1}(U)$ is a compact subset of $\Phi_{M''}$. Then by the Šilov idempotent theorem we can find $l_0 \in M''$ such that

$$\tilde{l}_0 = \begin{cases} 1 & \text{on} \quad F \\ 0 & \text{on} \quad \Phi_{M''} \backslash F \, . \end{cases}$$

For every $y \in K \cap F$ there exists $h_y \in M''$ such that $\tilde{h}_y(x) = 1$ and $\tilde{h}_y(y) = 0$. In the same way as above there exists a compact subset F_y of $\Phi_{M''}$ which is also an open neighborhood of x and satisfies $y \in \Phi_{M''} \backslash F_y$. Then by the Šilov idempotent theorem there exists $l_y \in M''$ such that

$$\tilde{l}_y = \begin{cases} 1 & \text{on} \quad F_y \\ 0 & \text{on} \quad \Phi_{M''} \backslash F_y \, . \end{cases}$$

Since $K \cap F$ is compact we can choose a finite number of points $y_1, \cdots, y_n \in K \cap F \subset \bigcup_{j=1}^{n} (\Phi_{M''} \backslash F_{y_j})$. Put $l = l_0 \prod_{j=1}^{n} l_{y_j}$. Then we have $\tilde{l}(x) = 1$ and $\tilde{l} = 0$ on K, which shows that M'' is regular. Q.E.D.

PROOF OF THEOREM 2.3. Since M is a semi-simple commutative Banach algebra we see by a theorem of Neumann [12] that $\operatorname{Dec} M \subset \operatorname{Ns}(M)$ since Y is hull-kernel dense in Φ_M, so by a theorem of Neumann [12] we have $\operatorname{Reg} M \subset \operatorname{Dec} M \subset \operatorname{Ns}(M)$. It follows by Lemma 2.4 and Lemma 2.7 that $\operatorname{Reg} M = \operatorname{Dec} M = \operatorname{Ns}(M) = M''$. Q.E.D.

3. Spectra of multipliers.

Let G be a locally compact abelian group and $\hat{G}$ its dual group. Laursen and Neumann [10] investigated the relationship between the greatest regular closed subalgebra of the multiplier algebra $M(A)$ of a commutative Banach algebra A, the decomposability of a multiplier, the hull-kernel continuity of its Gelfand transform on the maximal ideal spaces Φ_A and $\Phi_{M(A)}$, and a natural spectral property. For $B = M(G)$, $M_0(G)$, $M_p(G)$, $C_0M_p(G)$, we will denote $\operatorname{NS}(B) = \{T \in B : \operatorname{sp}(T, B) = \overline{\hat{T}(\hat{G})}\}$, where $\hat{T}$ denotes the Fourier transform of T. As an application of their results they showed that $\operatorname{Reg} M_0(G) = \operatorname{Dec} M_0(G) = M_{00}(G)$, which is a generalization of work of Zafran [18] and Albrecht [1]. Laursen and Neumann also studied the case where Φ_A is scattered. As a consequence of their results we see that for $1 \leq p < \infty$,

$$\operatorname{Reg} C_0M_p(G) = \operatorname{Dec} C_0M_p(G) = C_0M_p(G)'' = \operatorname{NS}(C_0M_p(G))$$

if G is compact (cf. [7], [8], [18]), where $C_0M_p(G)'' = \{T \in C_0M_p(G) : \operatorname{sp}(T, C_0M_p(G))$ is countable$\}$.

In this section we observe that $\operatorname{Reg} C_0M_p(G) = \operatorname{Dec} C_0M_p(G)$ holds and is, in a sense, maximal in $\operatorname{NS}(C_0M_p(G))$ for arbitrary locally compact abelian group G.

COROLLARY 3.1. *Let G be a locally compact abelian group and $1 \leq p < \infty$. Then* $\operatorname{Reg} C_0M_p(G) = \operatorname{Dec} C_0M_p(G) \subset \operatorname{NS}(C_0M_p(G))$. *Suppose that* $\operatorname{Reg} C_0M_p(G) \subset S \subset \operatorname{NS}(C_0M_p(G))$ *and $S + S \subset S$. Then* $\operatorname{Reg} C_0M_p(G) = S$. *We also have* $\operatorname{Reg} C_0M_p(G) = (\operatorname{Reg} M_p(G)) \cap C_0M_p(G)$, $\operatorname{Dec} C_0M_p(G) = (\operatorname{Dec} M_p(G)) \cap C_0M_p(G)$ *and* $\operatorname{NS}(C_0M_p(G)) =$

$\mathrm{NS}(M_p(G)) \cap C_0M_p(G)$.

PROOF. Put $M = \widehat{C_0M_p}(G)$ with the norm induced by $C_0M_p(G)$ and $Y = \hat{G}$. Since $L^1(G) \subset C_0M_p(G)$ and $\widehat{L^1}(G)$ is dense in $C_0(\hat{G})$ we see that M satisfies the conditions in Theorem 2.1. Henceforce $\mathrm{Reg}\, M = \mathrm{Dec}\, M = \hat{S}$. It follows that $\mathrm{Reg}\, C_0M_p(G) = \mathrm{Dec}\, C_0M_p(G) = S$.

It is easy to see that for every $T \in C_0M_p(G)$, $\mathrm{sp}(T, C_0M_p(G))$ coincides with $\mathrm{sp}(T, M_p(G))$. It follows that $\mathrm{NS}(C_0M_p(G)) = \mathrm{NS}(M_p(G)) \cap C_0M_p(G)$. By a theorem of Neumann [12] we see $\mathrm{Reg}\, M_p(G) \subset \mathrm{Dec}\, M_p(G) \subset \mathrm{NS}(M_p(G))$, thus we have

$$\mathrm{Reg}\, C_0M_p(G) \subset (\mathrm{Reg}\, M_p(G)) \cap C_0M_p(G) \subset$$
$$(\mathrm{Dec}\, M_p(G)) \cap C_0M_p(G) \subset \mathrm{NS}(C_0M_p(G)) .$$

It follows that

$$\mathrm{Reg}\, M \subset ((\mathrm{Reg}\, M_p(G)) \cap C_0M_p(G))\hat{} \subset$$
$$((\mathrm{Dec}\, M_p(G)) \cap C_0M_p(G))\hat{} \subset \mathrm{Ns}(M) .$$

Since $((\mathrm{Dec}\, M_p(G)) \cap C_0M_p(G))\hat{}$ is closed under addition we see by the first part of the proof that

$$\mathrm{Reg}\, M = ((\mathrm{Reg}\, M_p(G)) \cap C_0M_p(G))\hat{} = ((\mathrm{Dec}\, M_p(G)) \cap C_0M_p(G))\hat{} .$$

Hence the conclusion holds. Q.E.D.

As a consequence of Theorem 2.3 we see that a theorem of Laursen and Neumann holds, that is,

$$\mathrm{Reg}\, C_0M_p(G) = \mathrm{Dec}\, C_0M_p(G) = C_0M_p(G)'' = \mathrm{NS}(C_0M_p(G))$$

holds if G is compact.

Let T be the circle group and H^p the usual Hardy space for $1 \le p \le \infty$. A bounded function φ defined on the set of all non-negative integers N_0 is an H^1-H^p multiplier if $\varphi \hat{f} \in H^p$ for every $f \in H^1$. We denote the set of all H^1-H^p multipliers by $M(H^1, H^p)$ and $C_0M(H^1, H^p) = C_0(N_0) \cap M(H^1, H^p)$. Then $C_0M(H^1, H^p)$ is a commutative Banach algebra included in $C_0(N_0)$.

COROLLARY 3.2. $\mathrm{Reg}\, C_0M(H^1, H^p) = \mathrm{Dec}\, C_0M(H^1, H^p) = C_0M(H^1, H^p)'' = \mathrm{Ns}(C_0M(H^1, H^p))$.

PROOF. Put $M = C_0M(H^1, H^p)$ and $Y = N_0$. Then the conditions in Theorem 2.3 hold, thus the conclusion holds. Q.E.D.

For certain algebras M, $\mathrm{Reg}\, M$ and $\mathrm{Dec}\, M$ need not be large. As is pointed out by Neumann [12] for the disk algebra $A(D)$, $\mathrm{Reg}\, A(D) = \mathrm{Dec}\, A(D) = C$, the space of constant functions. We show a generalization of the result.

COROLLARY 3.3. *Let X be a connected compact Hausdorff space and B a commutative Banach algebra included in $C_0(X)$ which separates the points in X and contains constant functions. Suppose that $\{z_n\}$ is a discrete sequence of distinct points in X with only one cluster point z_0. Suppose also that the restriction map from B onto $B|\{z_n\}$ is an injection. Then we have* $\mathrm{Reg}B=\mathrm{Dec}B=C$.

PROOF. Put $Y=\{z_n\}\cup\{z_0\}$ and $M=B|Y$. Then M satisfies the condition in Theorem 2.3, hence $\mathrm{Reg}M=\mathrm{Dec}M=M''$. Let $f\in M''$. Then there exists a unique $F\in B$ such that $F|Y=f$. Since the restriction map from B to M is an injection we see that B and $B|Y$ are isomorphic, so $\mathrm{sp}(f, M)\supset F(X)$, thus $F(X)$ is connected and countable. Henceforce F is a constant function, so $\mathrm{Reg}M=\mathrm{Dec}M=C$. It follows that $\mathrm{Reg}B=\mathrm{Dec}B=C$. Q.E.D.

In the same way as in the proof of Corollary 3.3 we see

COROLLARY 3.4. *Let Y be a connected non-compact locally compact Hausdorff space and B a commutative Banach algebra included in $C_0(Y)$ which strongly separates the points in Y. Suppose that $\{z_n\}$ is a discrete sequence of distinct points in Y without cluster points. Suppose also that the restriction map from B onto $B|\{z_n\}$ is an injection. Then we have* $\mathrm{Reg}B=\mathrm{Dec}B=\{0\}$.

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