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Totally Real Pseudo-Umbilical Submanifolds of a Complex Projective Space

Huafei SUN

Tokyo Metropolitan University
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Abstract. In this paper, we study the totally real pseudo-umbilical submanifolds in the complex projective space, and obtain two integral inequalities and some conditions under which the submanifold has parallel second fundamental form.

1. Introduction.

Let CP^n be the n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4, and M an n-dimensional totally real submanifold in CP^n. Let h be the second fundamental form of the immersion. We denote by S the square of the length of h. Let ζ be the mean curvature vector, and $\langle \cdot , \cdot \rangle$ the scalar product in CP^n. If there exists a function λ on M such that

$$\langle h(X, Y), \zeta \rangle = \lambda \langle X, Y \rangle \tag{1.1}$$

for all tangent vectors X, Y on M, then M is called a *pseudo-umbilical* submanifold of CP^n (cf. [3]). It is clear that $\lambda \geq 0$.

Recently, B. Y. Chen ([4, 5]) studied totally real H-umbilical submanifolds of CP^n, and gave some of examples.

DEFINITION. M is called a totally real H-umbilical submanifold of CP^n, if the second fundamental form of M takes the following simple form:

$$h(e_1, e_1) = \alpha J e_1 , \qquad h(e_2, e_2) = \cdots = h(e_n, e_n) = \beta J e_1 ,$$
$$h(e_1, e_j) = \beta J e_j , \quad h(e_j, e_k) = 0 , \qquad j \neq k , \ j, k = 2, \cdots, n$$

for some suitable functions α and β with respect to some suitable orthonormal local frame field, where J is the complex structure of CP^n.

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It is clear that H-umbilical submanifold is pseudo-umbilical submanifold of CP^n. The following result is well known ([1, 6]).

THEOREM A. *Let M be an n-dimensional compact minimal, totally real submanifold in CP^n. If*

$$S \leq \frac{n+1}{2-1/n},$$

then either M is totally geodesic or $n=2$ and $M=S^1 \times S^1$.

In this paper, we consider the case that M is pseudo-umbilical and extend Theorem A. Our main results are

THEOREM 1. *Let M be an n-dimensional compact totally real pseudo-umbilical submanifold in CP^n. Then*

$$\int_M ((2(n+1)+2nH^2)S-3S^2-4n^2H^2)dM \leq 0 \qquad \text{for } n>1,$$

$$\int_M ((2(n+1)+8nH^2)S-3S^2-5n^2H^4-4n^2H^2)dM \leq 0 \qquad \text{for } n>2,$$

where H and dM denote the mean curvature of M and the volume element of M, respectively.

THEOREM 2. *Let M be an n-dimensional compact totally real pseudo-umbilical submanifold in CP^n. If*

$$2nH\Delta H+(2(n+1)+2nH^2)S-3S^2-4n^2H^2 \geq 0 \qquad \text{for } n>1 \tag{1.2}$$

or

$$2nH^2\Delta H+(2(n+1)+8nH^2)S-3S^2-5n^2H^4-4n^2H^2 \geq 0 \qquad \text{for } n>2, \tag{1.3}$$

then the second fundamental form is parallel. In particular, the equality holds in (1.2) if and only if M is totally geodesic or $n=2$ and $M=S^1 \times S^1$ and the equality holds in (1.3) if and only if M is totally geodesic.

REMARK. If $H \equiv 0$, then by Theorem 2 we may obtain ([7]).

PROPOSITION. *Let M be a compact minimal, totally real submanifold in CP^n. If*

$$S \leq \frac{2(n+1)}{3},$$

then either M is totally geodesic or $n=2$ and $M=S^1 \times S^1$.

2. Local formulas.

Let M be an n-dimensional totally real submanifold in the complex projective space CP^n. We shall make use of the following convention on the ranges of indices:

$$A, B, C, \cdots = 1, \cdots, n, 1^*, \cdots, n^* ; \qquad i, j, k, \cdots = 1, \cdots, n .$$

We choose a local field of orthonormal frames $e_1, \cdots e_n, e_{1^*} = Je_1, \cdots, e_{n^*} = Je_n$ in CP^n (J is the complex structure of CP^n), such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. Let (ω_A) be the field of dual frames. Then the structure equations of CP^n are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B , \qquad \omega_{AB} + \omega_{BA} = 0 . \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D , \tag{2.2}$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD} , \tag{2.3}$$

where $J = \sum_{AB} e_A \otimes \omega_B$ so that

$$(J_{AB}) = \left(\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array} \right) \tag{2.4}$$

with I_n being the identity matrix of order n. Restricting these forms to M, we obtain the following ([1]):

$$\omega_{i^*} = 0 , \quad \omega_{ij} = \omega_{i^*j^*} , \quad \omega_{i^*j} = \omega_{j^*i} , \tag{2.5}$$

$$\omega_{k^*i} = \sum_j h_{ij}^{k^*} \omega_j , \qquad h_{ij}^{k^*} = h_{ji}^{k^*} = h_{jk}^{i^*} = h_{ik}^{j^*} , \tag{2.6}$$

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j , \qquad \omega_{ij} = \omega_{ji} , \tag{2.7}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l , \tag{2.8}$$

where R_{ijkl} denotes the component of the curvature tensor of M.

$$R_{ijkl} = K_{ijkl} + \sum_m (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}) , \tag{2.9}$$

$$d\omega_{i^*j^*} = -\sum_k \omega_{i^*k^*} \wedge \omega_{k^*j^*} + \frac{1}{2} \sum_{kl} R_{i^*j^*kl} \omega_k \wedge \omega_l , \tag{2.10}$$

where $h = \sum_{ijm} h_{ij}^{m^*} \omega_i \omega_j e_{m^*}$ is the second fundamental form of the immersion. We denote the square of the length of h by $S=\sum_{ijm}(h_{ij}^{m^*})^2$. Let $\zeta=\frac{1}{n}\sum_m \operatorname{tr} H_{m^*}e_{m^*}$ and $H=\|\zeta\|$ denote the mean curvature vector and the mean curvature of M, respectively. Here tr denotes the trace of the matrix $H_{m^*}=(h_{ij}^{m^*})$. Now, let e_{n^*} be parallel to ζ. Then we have

$$\operatorname{tr} H_{n^*}=nH\,, \qquad \operatorname{tr} H_{i^*}=0\,,\ i\neq n\,. \tag{2.11}$$

$h_{ijk}^{m^*}$ and $h_{ijkl}^{m^*}$ are defined by

$$\sum_k h_{ijk}^{m^*}\omega_k=dh_{ij}^{m^*}-\sum_k h_{kj}^{m^*}\omega_{ki}-\sum_k h_{ik}^{m^*}\omega_{kj}-\sum_l h_{ij}^{l^*}\omega_{l^*m^*}\,, \tag{2.12}$$

$$\sum_l h_{ijkl}^{m^*}\omega_l=dh_{ijk}^{m^*}-\sum_l h_{ljk}^{m^*}\omega_{li}-\sum_l h_{ilk}^{m^*}\omega_{lj}-\sum_l h_{ijl}^{m^*}\omega_{ik}-\sum_l h_{ijk}^{l^*}\omega_{l^*m^*}\,, \tag{2.13}$$

respectively. The Laplacian $\Delta h_{ij}^{m^*}$ of the second fundamental form $h_{ij}^{m^*}$ is defined by $\Delta h_{ij}^{m^*}=\sum_k h_{ijkk}^{m^*}$. By a simple calculation we have

$$\begin{aligned}\sum_{ijm} h_{ij}^{m^*}\Delta h_{ij}^{m^*}=&\sum_{ijkm} h_{ij}^{m^*}h_{kkij}^{m^*}+\sum_{ijklm} h_{ij}^{m^*}h_{lk}^{m^*}R_{lijk}\\&+\sum_{ijklm} h_{ij}^{m^*}h_{li}^{m^*}R_{lkjk}+\sum_{ijklm} h_{ki}^{l^*}h_{ij}^{m^*}R_{l^*m^*jk}\\=&\sum_{ijkm} h_{ij}^{m^*}h_{kkij}^{m^*}+(n+1)S-\left(\sum_i(\operatorname{tr} H_{i^*})^2+\left(\sum_i \operatorname{tr} H_{i^*}\right)^2\right)\\&+\sum_{ij}\operatorname{tr} H_{j^*}\operatorname{tr}(H_{i^*}H_{j^*}H_{i^*})+\sum_{ij}\operatorname{tr}(H_{i^*}H_{j^*}-H_{j^*}H_{i^*})^2\\&-\sum_{ij}(\operatorname{tr} H_{i^*}H_{j^*})^2\,.\end{aligned} \tag{2.14}$$

□

3. Proofs of theorems.

From (1.1) and (2.11) we can get $\sum_m \operatorname{tr} H_{m^*}h_{ij}^{m^*}=n\lambda\delta_{ij}$, $H^2=\lambda$ and

$$h_{ij}^{n^*}=H\delta_{ij}\,. \tag{3.1}$$

Using (2.11) and (3.1) we have

$$\sum_{ijkm} h_{ij}^{m^*}h_{kkij}^{m^*}=nH\Delta H\,, \tag{3.2}$$

$$\sum_i(\operatorname{tr} H_{i^*})^2=n^2H^2\,, \tag{3.3}$$

$$\left(\sum_i \operatorname{tr} H_{i^*}\right)^2=n^2H^2\,, \tag{3.4}$$

$$\sum_{ij} \operatorname{tr} H_{j*} \operatorname{tr}(H_{i*}H_{j*}H_{i*}) = nH^2 S . \tag{3.5}$$

Substituting (3.2)–(3.5) into (2.14) we obtain

$$\sum_{ijm} h_{ij}^{m*} \Delta h_{ij}^{m*} = nH\Delta H + ((n+1) + nH^2)S - 2n^2H^2 \tag{3.6}$$

$$- \sum_{ij} (\operatorname{tr} H_{i*}H_{j*})^2 + \sum_{ij} \operatorname{tr}(H_{i*}H_{j*} - H_{j*}H_{i*})^2 .$$

In order to prove our theorems, we need the following lemmas.

LEMMA 1 ([8]). *Let H_i $(i \geq 2)$ be symmetric $(n \times n)$-matrices, $S_i = \operatorname{tr} H_i^2$ and $S = \sum_i S_i$. Then*

$$\sum_{ij} \operatorname{tr}(H_iH_j - H_jH_i)^2 - \sum_{ij} (\operatorname{tr} H_iH_j)^2 \geq -\frac{3}{2} S^2 \tag{3.7}$$

and the equality holds if and only if all $H_i = 0$ or there exist two H_i different from zero. Moreover, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$ $(i \neq 1, 2)$, then $S_1 = S_2$ and there exists an orthogonal $(n \times n)$-matrix T such that

$$TH_1{}^tT = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & \\ 0 & & 0 \end{pmatrix}, \qquad TH_2{}^tT = \begin{pmatrix} 0 & a & 0 \\ a & 0 & \\ 0 & & 0 \end{pmatrix},$$

where $a = \sqrt{S_1/2}$.

LEMMA 2. *When $n > 2$,*

$$\sum_{ij} \operatorname{tr}(H_{i*}H_{j*} - H_{j*}H_{i*})^2 - \sum_{ij} \operatorname{tr}(H_{i*}H_{j*})^2 \geq -\frac{3}{2} S^2 + 3n^2HS - \frac{5}{2} n^2H^4 .$$

PROOF. Using (2.11) and (3.1), when $n > 2$ we have

$$\sum_{ij} \operatorname{tr}(H_{i*}H_{j*} - H_{j*}H_{i*})^2 - \sum_{ij} (\operatorname{tr} H_{i*}H_{j*})^2 \tag{3.8}$$

$$= \sum_{ij \neq n} \operatorname{tr}(H_{i*}H_{j*} - H_{j*}H_{i*})^2 - \sum_{ij \neq n} (\operatorname{tr} H_{i*}H_{j*})^2 - (\operatorname{tr} H_{n*}^2)^2 .$$

Applying Lemma 1 to (3.8) we get

$$\sum_{ij} \operatorname{tr}(H_{i*}H_{j*} - H_{j*}H_{i*})^2 - \sum_{ij} (\operatorname{tr} H_{i*}H_{j*})^2$$

$$\geq -\frac{3}{2} \left(\sum_{i \neq n} \operatorname{tr} H_{i*}^2 \right)^2 - (\operatorname{tr} H_{n*}^2)^2 = -\frac{3}{2} (S - \operatorname{tr} H_{n*}^2)^2 - (\operatorname{tr} H_{n*}^2)^2$$

$$= -\frac{3}{2}(S-nH^2)^2 - n^2H^4 = -\frac{3}{2}S^2 + 3nH^2S - \frac{5}{2}n^2H^4 .$$

This completes the proof of Lemma 2.

Using (3.1) we can get

$$\sum_{ijkm} (h_{ijk}^{m^*})^2 \geq \sum_{ik} (h_{iik}^{n^*})^2 = n\sum_i (\nabla_i H)^2 . \tag{3.9}$$

Since

$$\frac{1}{2}\Delta H^2 = H\Delta H + \sum_i (\nabla_i H)^2 , \tag{3.10}$$

using Lemma 1, (3.6), (3.9) and (3.10) we have

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{ijkm} (h_{ijk}^{m^*})^2 + \sum_{ijm} h_{ij}^{m^*} \Delta h_{ij}^{m^*} \\ &\geq \sum_{ijkm} (h_{ijk}^{m^*})^2 + nH\Delta H + ((n+1)+nH^2)S - 2n^2H^2 - \frac{3}{2}S^2 \\ &\geq \sum_{ik} (h_{iik}^{n^*})^2 + nH\Delta H + ((n+1)+nH^2)S - 2n^2H^2 - \frac{3}{2}S^2 \\ &= n\sum_i (\nabla_i H)^2 + nH\Delta H + ((n+1)+nH^2)S - 2n^2H^2 - \frac{3}{2}S^2 \\ &= \frac{1}{2}n\Delta H^2 + ((n+1)+nH^2)S - 2n^2H^2 - \frac{3}{2}S^2 . \end{aligned} \tag{3.11}$$

Because M is compact, (3.11) yields

$$\int_M ((2(n+1)+2nH^2)S - 3S^2 - 4n^2H^2)dM \leq 0 .$$

On the other hand, from the first inequality of (3.11), we know that if

$$2nH\Delta H + (2(n+1)+2nH^2)S - 3S^2 - 4n^2H^2 \geq 0 \tag{3.12}$$

and M is compact, then the second fundamental form $h_{ij}^{m^*}$ is parallel. In particular, when equality of (3.12) holds, then we see that the equality of (3.7) holds. So by Lemma 1, (3.12) implies that all $H_{i^*}=0$ (M is totally geodesic) or there exist two of H_{i^*} different from zero. In the latter case, by Lemma 1 we may therefore assume that

$$H_{1^*} = \begin{pmatrix} f & 0 & 0 \\ 0 & -f & \\ 0 & & 0 \end{pmatrix}, \quad H_{2^*} = \begin{pmatrix} 0 & g & 0 \\ g & 0 & \\ 0 & & 0 \end{pmatrix}, \tag{3.13}$$

where $f, g \neq 0$. Hence we have

$$\operatorname{tr} H_{1*} = \operatorname{tr} H_{2*} = 0 . \tag{3.14}$$

Using (3.14) we find that $\sum_m \operatorname{tr} H_{m*} h_{ij}^{m*} = 0$ and $H = 0$ identically. So by [6, 7] we see that $n = 2$ and $M = S^1 \times S^1$.

On the other hand, when $n > 2$, using Lemma 2, (3.6), (3.9) and (3.10) we have

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{ijkm} (h_{ijk}^{m*})^2 + nH\Delta H + ((n+1) + nH^2)S - 2n^2H^2 \\ &\quad + \sum_{ij} \operatorname{tr}(H_{i*}H_{j*} - H_{j*}H_{i*})^2 - \sum_{ij} (\operatorname{tr} H_{i*}H_{j*})^2 \\ &\geq \sum_{ijkm} (h_{ijk}^{m*})^2 + nH\Delta H + ((n+1) + nH^2)S \\ &\quad - 2n^2H^2 - \frac{3}{2}S^2 + 3nH^2S - \frac{5}{2}n^2H^4 \\ &\geq \sum_{ik} (h_{iik}^{m*})^2 + nH\Delta H + ((n+1) + 4nH^2)S \\ &\quad - \frac{3}{2}S^2 - 2n^2H^2 - \frac{5}{2}n^2H^4 \\ &= \frac{1}{2}n\Delta H^2 + ((n+1) + 4nH^2)S - \frac{3}{2}S^2 - 2n^2H^2 - \frac{5}{2}n^2H^4 . \end{aligned} \tag{3.15}$$

So, when M is compact, (3.15) gives

$$\int_M ((2(n+1) + 8nH^2)S - 3S^2 - 4n^2H^2 - 5n^2H^4)dM \leq 0 .$$

From the first inequality of (3.15), it is easy to see that if

$$2nH\Delta H + (2(n+1) + 8nH^2)S - 3S^2 - 4n^2H^2 - 5n^2H^4 \geq 0 \tag{3.16}$$

and M is compact, then the second fundamental form h_{ij}^{m*} is parallel. In particular, when the equality of (3.16) holds, from (3.15) we see that

$$\sum_{ij \neq n} \operatorname{tr}(H_{i*}H_{j*} - H_{j*}H_{i*})^2 - \sum_{ij \neq n} (\operatorname{tr} H_{i*}H_{j*})^2 = -\frac{3}{2}\Big(\sum_{i \neq n} \operatorname{tr} H_{i*}^2\Big)^2$$

holds. Thus by Lemma 1 we know that $H_{i*} = 0$ (for every $i^* \neq n^*$) (i.e., M is totally umbilical) or there exist two of H_{i*} $(i^* \neq n^*)$ different from zero. Using the fact that there exist no totally umbilical totally real submanifolds in CP^n except the totally geodesic ones ([2]), we see in the former case, that M is totally geodesic. We may conclude that the latter case is not true. In fact, if it were true, using the same method as that of ([9]), we may get $n=2$. This is a contradiction, since $n>2$. This proves Theorem 2.

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Present Address:
DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY,
MINAMI-OHSAWA, HACHIOJI-SHI, TOKYO, 192–03 JAPAN.