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Dipolarizations in Compact Lie Algebras and Homogeneous Parakähler Manifolds

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Introduction.

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$ and ρ be an alternating 2-form on $\mathfrak{g}$. Then the triple $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ is called a *weak dipolarization* in $\mathfrak{g}$ if the following conditions are satisfied:

(WD1) $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$,

(WD2) $\rho(\mathfrak{g}^+, \mathfrak{g}^+)=\rho(\mathfrak{g}^-, \mathfrak{g}^-)=0$,

(WD3) $\rho(X, \mathfrak{g})=0$ if and only if $X\in \mathfrak{g}^+ \cap \mathfrak{g}^-$,

(WD4) $\rho([X, Y], Z)+\rho([Y, Z], X)+\rho([Z, X], Y)=0$, $\forall X, Y, Z\in \mathfrak{g}$.

A dipolarization in $\mathfrak{g}$ is a triple $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$, formed by two subalgebras $\mathfrak{g}^{\pm}$ and a linear form f, which satisfies the following conditions:

(D1) $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$,

(D2) $f([\mathfrak{g}^+, \mathfrak{g}^+])=f([\mathfrak{g}^-, \mathfrak{g}^-])=0$,

(D3) $f([X, \mathfrak{g}])=0$ if and only if $X\in \mathfrak{g}^+ \cap \mathfrak{g}^-$.

A dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is itself a weak dipolarization, since df satisfies (WD2)–(WD4). A weak dipolarization is called *symmetric* if $\mathfrak{g}^+$ is Lie-isomorphic to $\mathfrak{g}^-$. Otherwise it is called *nonsymmetric*. A dipolarization (resp. weak dipolarization) is called *trivial*, if $\mathfrak{g}^+=\mathfrak{g}^-=\mathfrak{g}$, and if $f=0$ (resp. $\rho=0$).

The notions of dipolarizations and weak dipolarizations in a Lie algebra were first introduced by Kaneyuki ([6]) to describe a class of homogeneous symplectic manifolds, called *homogeneous parakähler manifolds*. Let us recall the definition of homogeneous parakähler manifolds ([6]). A parakähler manifold M is, by definition, a symplectic manifold which admits a pair of transversal Lagrangian foliations. If a Lie group G acts on M as symplectomorphisms which preserves each of the two foliations, then we say that the parakähler structure is G-invariant. Furthermore, if G acts transitively on M, then M is said to be a *homogeneous parakähler manifold*. It was proved in [6] that a necessary and sufficient condition for the existence of an invariant parakähler structure on $M=G/H$ (H is an isotropy subgroup) is that there exists a weak dipolarization in

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$\mathfrak{g}=\mathrm{Lie}\,G$ such that the intersection of the two polarized subalgebras coincides with $\mathfrak{h}=\mathrm{Lie}\,H$. In [7, 8], a large class of homogeneous parakähler manifolds are obtained. In [1], the authors constructed an example of nonsymmetric dipolarization in a Lie algebra, which indicates that homogeneous parakähler structures are substantially different from homogeneous Kähler structures. In this paper we study dipolarizations and weak dipolarizations of compact Lie algebras to obtain the following results:

THEOREM 1. *Let* $\mathfrak{u}$ *be a compact semisimple Lie algebra. Then there exist no nontrivial dipolarizations in* $\mathfrak{u}$.

THEOREM 2. *Let G be a connected compact Lie group, H be a closed subgroup of G. Suppose that the coset space G/H is effective. Then there exists a G-invariant parakähler structure on G/H, if and only if G/H is an even-dimensional torus.*

NOTATION. $\mathfrak{g}^C$ denotes the complexification of a Lie algebra $\mathfrak{g}$. $\mathfrak{c}_{\mathfrak{g}}(X)$ denotes the centralizer of an element $X\in\mathfrak{g}$ in a Lie algebra $\mathfrak{g}$.

1. Dipolarizations in compact Lie algebras.

1.1. Let $\mathfrak{u}$ be a compact semisimple Lie algebra and $\mathfrak{u}^C$ be its complexification. We denote by B the Killing forms of $\mathfrak{u}$ and of $\mathfrak{u}^C$. Let $\{\mathfrak{u}^+, \mathfrak{u}^-, f\}$ be a dipolarization in $\mathfrak{u}$, and let $Z\in\mathfrak{u}$ be the unique element satisfying

$$B(Z, X)=f(X)\,, \qquad X\in\mathfrak{u}\,. \tag{1.1}$$

Then we have ([6])

$$\mathfrak{u}^+\cap\mathfrak{u}^-=\mathfrak{c}_{\mathfrak{u}}(Z). \tag{1.2}$$

Choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{u}$ such that $Z\in\mathfrak{t}$. The complexification $\mathfrak{t}^C$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{u}^C$. Let Δ be the root system for $(\mathfrak{u}^C, \mathfrak{t}^C)$, and let Δ^+ be the positive root system with respect to an order. Let $\{X_\alpha;\ \alpha\in\Delta\}$ be a Weyl basis of $\mathfrak{u}^C$ mod $\mathfrak{t}^C$ with respect to $\mathfrak{u}$ (see Helgason [4]). Then $\mathfrak{u}$ is written as

$$\mathfrak{u}=\mathfrak{t}+\sum_{\alpha\in\Delta^+}\mathbf{R}(X_\alpha-X_{-\alpha})+\sum_{\alpha\in\Delta^+}\mathbf{R}i(X_\alpha+X_{-\alpha})\,. \tag{1.3}$$

1.2. Let $\mathfrak{v}$ be a subalgebra of $\mathfrak{u}$ containing $\mathfrak{t}$. Then $\mathfrak{v}^C$ is a regular subalgebra of $\mathfrak{u}^C$ in the sense of Dynkin [3]. Therefore there exists a closed subsystem Δ' of Δ such that

$$\mathfrak{v}^C=\mathfrak{t}^C+\sum_{\alpha\in\Delta'}\mathbf{C}X_\alpha\,. \tag{1.4}$$

$\mathfrak{v}^C$ is reductive and we have

$$-\Delta'=\Delta'\,. \tag{1.5}$$

LEMMA 1.1. $\mathfrak{v}$ *is written as*

$$\mathfrak{v}=\mathfrak{t}+\sum_{\alpha\in\Delta'^{+}}\mathbf{R}(X_\alpha-X_{-\alpha})+\sum_{\alpha\in\Delta'^{+}}\mathbf{R}i(X_\alpha+X_{-\alpha}),\tag{1.6}$$

where $\Delta'^{+}=\Delta'\cap\Delta^{+}$.

PROOF. Let $X\in\mathfrak{v}$. Then, by (1.3), X is written as

$$X=H+\sum_{\alpha\in\Delta^{+}}a_\alpha(X_\alpha-X_{-\alpha})+\sum_{\alpha\in\Delta^{+}}ib_\alpha(X_\alpha+X_{-\alpha}),\tag{1.7}$$

where $H\in\mathfrak{t}$ and a_α, $b_\alpha\in\mathbf{R}$. If we put $\omega_\alpha=a_\alpha+ib_\alpha$, then X can be written as

$$X=H+\sum_{\alpha\in\Delta^{+}}(\omega_\alpha X_\alpha-\bar{\omega}_\alpha X_{-\alpha}).\tag{1.8}$$

Since X lies in $\mathfrak{v}^C$, it follows from (1.4) that if $\omega_\alpha\neq 0$, then $\alpha\in\Delta'$. This implies the inclusion $\subset$ in (1.6). The converse inclusion follows from $\mathfrak{v}=\mathfrak{v}^C\cap\mathfrak{u}$ and (1.4). □

LEMMA 1.2. *Let $\alpha\in\Delta^{+}$. Then $X_\alpha-X_{-\alpha}$ lies either in $\mathfrak{u}^{+}$ or in $\mathfrak{u}^{-}$. Then same assertion holds for $i(X_\alpha+X_{-\alpha})$.*

PROOF. By (1.2), $\mathfrak{t}$ is contained in $\mathfrak{u}^{+}\cap\mathfrak{u}^{-}$. Consequently the complexifications $(\mathfrak{u}^{+})^C$ and $(\mathfrak{u}^{-})^C$ are regular subalgebras of $\mathfrak{u}^C$. Hence there exist two closed subsystems Δ' and Δ'' of Δ such that

$$(\mathfrak{u}^{+})^C=\mathfrak{t}^C+\sum_{\alpha\in\Delta'}\mathbf{C}X_\alpha,\tag{1.9}$$

$$(\mathfrak{u}^{-})^C=\mathfrak{t}^C+\sum_{\alpha\in\Delta''}\mathbf{C}X_\alpha,\tag{1.10}$$

$$-\Delta'=\Delta',\qquad -\Delta''=\Delta''.\tag{1.11}$$

By (D1), we have $\mathfrak{u}^C=(\mathfrak{u}^{+})^C+(\mathfrak{u}^{-})^C$, and hence $\Delta=\Delta'\cup\Delta''$. If we put $\Delta'^{+}=\Delta'\cap\Delta^{+}$ and $\Delta''^{+}=\Delta''\cap\Delta^{+}$, then we have

$$\Delta^{+}=\Delta'^{+}\cup\Delta''^{+}.\tag{1.12}$$

This implies that the root $\alpha\in\Delta^{+}$ lies either in Δ'^{+} or in Δ''^{+}. Suppose $\alpha\in\Delta'^{+}$. Then (1.9) shows that $X_{\pm\alpha}\in(\mathfrak{u}^{+})^C$. In view of Lemma 1.1, we have $X_\alpha-X_{-\alpha}\in\mathfrak{u}^{+}$ and $i(X_\alpha+X_{-\alpha})\in\mathfrak{u}^{+}$. Similarly, if $\alpha\in\Delta''^{+}$, then we conclude that $X_\alpha-X_{-\alpha}\in\mathfrak{u}^{-}$ and $i(X_\alpha+X_{-\alpha})\in\mathfrak{u}^{-}$. □

LEMMA 1.3. *Let $\alpha\in\Delta$ be a positive root satisfying $\alpha(Z)\neq 0$. Then $X_\alpha-X_{-\alpha}$ lies in $\mathfrak{u}^{+}$ (resp. $\mathfrak{u}^{-}$) if and only if $i(X_\alpha+X_{-\alpha})$ lies in $\mathfrak{u}^{-}$ (resp. $\mathfrak{u}^{+}$).*

PROOF. Suppose that $X_\alpha-X_{-\alpha}\in\mathfrak{u}^{+}$. Suppose further that $i(X_\alpha+X_{-\alpha})\notin\mathfrak{u}^{-}$. Then by Lemma 1.2 we see that $i(X_\alpha+X_{-\alpha})\in\mathfrak{u}^{+}$. Also we have:

$$(1.13)\qquad \begin{aligned} &f([X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})]) \\ &= B(Z, [X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})]) = iB([Z, X_\alpha - X_{-\alpha}], X_\alpha + X_{-\alpha}) \\ &= i\alpha(Z)B(X_\alpha + X_{-\alpha}, X_\alpha + X_{-\alpha}) = 2i\alpha(Z)B(X_\alpha, X_{-\alpha}) \neq 0\,, \end{aligned}$$

which contradicts (D2). Thus we have proved that $i(X_\alpha + X_{-\alpha}) \in \mathfrak{u}^-$. The other case can be proved analogously. □

1.3. Proof of Theorem 1. Let $\{\mathfrak{u}^+, \mathfrak{u}^-, f\}$ be a dipolarization in $\mathfrak{u}$. First we wish to prove $\mathfrak{u}^+ = \mathfrak{u}$. For this it is sufficient to prove $(\mathfrak{u}^+)^C = \mathfrak{u}^C$. We choose Z, $\mathfrak{t}$, Δ and X_α's as in 1.1. The last condition is equivalent to the condition $\Delta' = \Delta$ (cf. (1.9)). Since $\mathfrak{c}_\mathfrak{u}(Z)$ contains $\mathfrak{t}$, $\mathfrak{c}_\mathfrak{u}(Z)^C$ is given by

$$(1.14)\qquad \mathfrak{c}_\mathfrak{u}(Z)^C = \mathfrak{c}_{\mathfrak{u}^C}(Z) = \mathfrak{t}^C + \sum_{\alpha \in \Delta_0} \mathbf{C}X_\alpha\,,$$

where the closed subsystem Δ_0 is given by

$$(1.15)\qquad \Delta_0 = \{\alpha \in \Delta : \alpha(Z) = 0\}\,.$$

Now let $\alpha \in \Delta^+$. If $\alpha \in \Delta_0$, then by (1.2) α lies in Δ'. Suppose next that $\alpha \in \Delta^+ - \Delta_0$. Then we have $\alpha(Z) \neq 0$. By Lemma 1.2, $X_\alpha - X_{-\alpha}$ lies either in $\mathfrak{u}^+$ or in $\mathfrak{u}^-$. Suppose that $X_\alpha - X_{-\alpha} \in \mathfrak{u}^-$. Then by Lemma 1.3, $i(X_\alpha + X_{-\alpha}) \in \mathfrak{u}^+$. We have

$$(1.16)\qquad [Z, i(X_\alpha + X_{-\alpha})] = i\alpha(Z)(X_\alpha - X_{-\alpha})\,.$$

The left side of (1.16) belongs to $\mathfrak{u}^+$, and hence $X_\alpha - X_{-\alpha} \in \mathfrak{u}^+$. We have thus proved that $X_\alpha - X_{-\alpha} \in \mathfrak{u}^+$ for $\alpha \in \Delta^+ - \Delta_0$. Similarly, again by using Lemma 1.2 and Lemma 1.3, we conclude that $i(X_\alpha + X_{-\alpha}) \in \mathfrak{u}^+$ for $\alpha \in \Delta^+ - \Delta_0$. Therefore, in view of Lemma 1.1, we have $X_\alpha \in (\mathfrak{u}^+)^C$ for $\alpha \in \Delta^+$, and hence $\alpha \in \Delta'^+$. Thus we have proved $\Delta' = \Delta$, or equivalently, $\mathfrak{u}^+ = \mathfrak{u}$. Similarly we have $\mathfrak{u}^- = \mathfrak{u}$. Now it follows from (1.2) that Z is a central element in $\mathfrak{u}$. By the semisimplicity of $\mathfrak{u}$, we have $Z = 0$. Therefore $f = 0$ (cf. (1.1)). □

COROLLARY 1.4. *Let $\mathfrak{u}$ be a compact Lie algebra, and let $\{\mathfrak{u}^+, \mathfrak{u}^-, f\}$ be a dipolarization in $\mathfrak{u}$. Then $\mathfrak{u}^+ = \mathfrak{u}^- = \mathfrak{u}$.*

PROOF. Since $\mathfrak{u}$ is compact, we have $\mathfrak{u} = \mathfrak{c} \oplus \mathfrak{u}'$, where $\mathfrak{c}$ is the center of $\mathfrak{u}$ and $\mathfrak{u}'$ is the commutator subalgebra of $\mathfrak{u}$. By (D3) we have $\mathfrak{c} \subset \mathfrak{u}^+ \cap \mathfrak{u}^-$. If we denote $\mathfrak{u}^\pm \cap \mathfrak{u}'$ by $\mathfrak{u}'^\pm$, then it is obvious that $\{\mathfrak{u}'^+, \mathfrak{u}'^-, f|_{\mathfrak{u}'}\}$ is a dipolarization in $\mathfrak{u}'$. Since $\mathfrak{u}'$ is semisimple, by Theorem 1 we have $\mathfrak{u}'^\pm = \mathfrak{u}'$, and thus $\mathfrak{u}' \subset \mathfrak{u}^\pm$. Hence it follows that $\mathfrak{u}^\pm = \mathfrak{u}$. □

2. Compact homogeneous parakähler manifolds.

2.1. We need the following Matsushima's result (cf. Murakami [10]).

PROPOSITION 2.1 (Matsushima [9]). *Let G be a connected compact Lie group and H be a closed subgroup of G. Suppose that the coset space G/H is effective and that there exists a G-invariant symplectic form ω on G/H. Then the following assertions are valid:* (1) *H is contained in the commutator subgroup G' of G.* (2) *H is connected and is the centralizer of an element $Z' \in \mathfrak{g}' = \mathrm{Lie}\, G'$ in G'.* (3) *Let C be the center of G. Then we have $G = C \times G'$ (direct product).*

Now let G and H be the same as in Proposition 2.1. Suppose further that the coset space $M = G/H$ is an effective and homogeneous parakähler manifold. Let $\hat{I}$ and ω be, respectively, the invariant paracomplex structure and the invariant symplectic form of M associated with the parakähler structure of M. Let o be the origin of G/H. We identify the tangent space T_eG of G at the unit element $e \in G$ with $\mathfrak{g} = \mathrm{Lie}\, G$. Let π be the projection of G onto G/H. As we did in [6], we choose a linear endomorphism I on $\mathfrak{g}$ in such a way that

$$\pi_{*e} I = \hat{I}_o \pi_{*e} . \tag{2.1}$$

Let $\mathfrak{h} = \mathrm{Lie}\, H$ and $\rho = \pi^* \omega$. Then we have a parakähler algebra $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$ ([6]). If we put

$$g(X, Y) = \omega(X, \hat{I} Y) \tag{2.2}$$

for smooth vector fields X and Y on M, then g is a G-invariant parakähler metric on M.

LEMMA 2.2. *Let $\mathfrak{c} = \mathrm{Lie}\, C$. Then we have*

$$I\mathfrak{c} \subset \mathfrak{c} + \mathfrak{h} . \tag{2.3}$$

PROOF. Let $X \in \mathfrak{h}$ and $Y \in \mathfrak{c}$. Then, by the axioms of parakähler algebras ([6]), we have

$$[X, IY] \equiv I[X, Y] = 0 \qquad \mathrm{mod}\, \mathfrak{h} . \tag{2.4}$$

This means that $[\mathfrak{h}, I\mathfrak{c}] \subset \mathfrak{h}$. Consequently $I\mathfrak{c}$ is contained in the normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. As was shown in [9], we have that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h} + \mathfrak{c}$, which implies $I\mathfrak{c} \subset \mathfrak{c} + \mathfrak{h}$. □

LEMMA 2.3. *The subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ is I-stable.*

PROOF. First we note that the equality

$$g_o(\pi_*(\mathfrak{c}), \pi_*(\mathfrak{g}')) = 0 \tag{2.5}$$

is valid. In fact, this can be proved quite analogously as for the equality (4) in Matsushima [9], by using (2.3) and by replacing the complex structures there by the paracomplex structures. (2.5) means that $\pi_*(\mathfrak{g}')$ is the orthogonal complement of $\pi_*(\mathfrak{c})$ in T_oM with respect to g_o. Now let $X \in \mathfrak{c}$ and $Y \in \mathfrak{g}'$. Since T_oM can be identified with $\mathfrak{g}/\mathfrak{h}$, it follows from (2.3) that

$$(2.6)\qquad \pi_*(IX)\in\pi_*(\mathfrak{c})\,.$$

Since g is a parakähler metric, we have from (2.5) and (2.6),

$$(2.7)\qquad g_o(\pi_*X,\pi_*IY)=g_o(\pi_*X,\hat{I}_o\pi_*Y)=-g_o(\hat{I}_o\pi_*X,\pi_*Y)=-g_o(\pi_*IX,\pi_*Y)=0\,,$$

which implies that $\pi_*IY\in\pi_*(\mathfrak{g}')$. Therefore, in view of Proposition 2.1 (1), we have $IY\in\mathfrak{g}'$. □

We define the subalgebras $\mathfrak{g}^{\pm}$ of $\mathfrak{g}$ by (cf. [6])

$$(2.8)\qquad \mathfrak{g}^{\pm}=\{X\in\mathfrak{g}:IX\equiv\pm X \bmod \mathfrak{h}\}\,.$$

Then, as is known in [6], $\{\mathfrak{g}^+,\mathfrak{g}^-,\rho\}$ is a weak dipolarization of $\mathfrak{g}$ satisfying

$$(2.9)\qquad \mathfrak{g}^+\cap\mathfrak{g}^-=\mathfrak{h}\,.$$

Now let

$$(2.10)\qquad \mathfrak{g}'^{\pm}=\mathfrak{g}^{\pm}\cap\mathfrak{g}'\,,\qquad \mathfrak{c}^{\pm}=\mathfrak{g}^{\pm}\cap\mathfrak{c}\,.$$

LEMMA 2.4. *$\mathfrak{g}^{\pm}$ can be written as the direct sums:*

$$(2.11)\qquad \mathfrak{g}^+=\mathfrak{g}'^+\oplus\mathfrak{c}^+\,,\qquad \mathfrak{g}^-=\mathfrak{g}'^-\oplus\mathfrak{c}^-\,.$$

PROOF. Let $X\in\mathfrak{g}^+$. By the Levi decomposition $\mathfrak{g}=\mathfrak{g}'\oplus\mathfrak{c}$, one can write X as $X=X_1+X_2$, $X_1\in\mathfrak{g}'$, $X_2\in\mathfrak{c}$. We have

$$(2.12)\qquad IX=IX_1+IX_2\,.$$

By Lemma 2.3, IX_1 lies in $\mathfrak{g}'$. By (2.8) one has

$$(2.13)\qquad IX=X+h=X_1+X_2+h\,,$$

where $h\in\mathfrak{h}$. By (2.3), one has

$$(2.14)\qquad IX_2=(IX_2)_{\mathfrak{c}}+(IX_2)_{\mathfrak{h}}\,,$$

where $(\ )_{\mathfrak{c}}$ and $(\ )_{\mathfrak{h}}$ denote the $\mathfrak{c}$-component and the $\mathfrak{h}$-component, respectively. Substituting (2.13) and (2.14) into (2.12), and comparing the $\mathfrak{g}'$-component and the $\mathfrak{c}$-component in the both sides of (2.12), we have

$$(2.15)\qquad IX_1+(IX_2)_{\mathfrak{h}}=X_1+h\,,$$

$$(2.16)\qquad (IX_2)_{\mathfrak{c}}=X_2\,.$$

By (2.15), we have $IX_1\equiv X_1 \bmod \mathfrak{h}$, which implies $X_1\in\mathfrak{g}'^+$. From (2.14) and (2.16), it follows that $IX_2=X_2+(IX_2)_{\mathfrak{h}}\equiv X_2 \bmod \mathfrak{h}$. Hence $X_2\in\mathfrak{c}^+$. Thus we have proved $\mathfrak{g}^+=\mathfrak{g}'^++\mathfrak{c}^+$. □

LEMMA 2.5. *$\{\mathfrak{g}'^+,\mathfrak{g}'^-,\rho'\}$ is a weak dipolarization of $\mathfrak{g}'$ satisfying $\mathfrak{g}'^+\cap\mathfrak{g}'^-=\mathfrak{h}$, where $\rho'=\rho|_{\mathfrak{g}'\times\mathfrak{g}'}$.*

PROOF. The equalities $\mathfrak{g}'=\mathfrak{g}'^+ + \mathfrak{g}'^-$ and $\mathfrak{c}=\mathfrak{c}^+ + \mathfrak{c}^-$ follow from the equality $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$, Lemma 2.4 and the Levi decomposition $\mathfrak{g}=\mathfrak{g}'\oplus\mathfrak{c}$. The property $\mathfrak{g}'^+\cap\mathfrak{g}'^-=\mathfrak{h}$ follows from the equality (2.9) and Proposition 2.1 (1). Since $\mathfrak{g}^\pm$ and ρ satisfy (WD2), $\mathfrak{g}'^\pm$ and ρ' also satisfy (WD2). Since ρ satisfies (WD4), we have

$$\rho(\mathfrak{c}, \mathfrak{g}')=\rho(\mathfrak{c}, [\mathfrak{g}, \mathfrak{g}])=\rho([\mathfrak{c}, \mathfrak{g}], \mathfrak{g})=0\,. \tag{2.17}$$

Now let $X\in\mathfrak{g}'$, and suppose that $\rho(X, \mathfrak{g}')=0$. Then, by (2.17), we have $\rho(X, \mathfrak{g})=\rho(X, \mathfrak{g}')+\rho(X, \mathfrak{c})=0$. Hence, by (WD3) for $\mathfrak{g}$, we get $X\in\mathfrak{g}^+\cap\mathfrak{g}^-$. This implies that $X\in\mathfrak{g}'^+\cap\mathfrak{g}'^-$. Conversely, let $X\in\mathfrak{g}'^+\cap\mathfrak{g}'^-$. Then, by (WD3) for $\mathfrak{g}$ and (2.17), we have $0=\rho(X, \mathfrak{g})=\rho'(X, \mathfrak{g}')+\rho(X, \mathfrak{c})=\rho'(X, \mathfrak{g}')$. Thus we have proved that $\{\mathfrak{g}'^+, \mathfrak{g}'^-, \rho'\}$ satisfies (WD3). □

LEMMA 2.6. *$\{\mathfrak{c}^+, \mathfrak{c}^-, \rho''\}$ is a weak dipolarization in $\mathfrak{c}$ satisfying $\mathfrak{c}^+\cap\mathfrak{c}^-=(0)$, where $\rho''=\rho|_{\mathfrak{c}\times\mathfrak{c}}$.*

PROOF. We have seen the equality $\mathfrak{c}=\mathfrak{c}^+ + \mathfrak{c}^-$ in the proof of Lemma 2.5. We have $\mathfrak{c}^+\cap\mathfrak{c}^-\subset\mathfrak{g}^+\cap\mathfrak{g}^-=\mathfrak{h}$. Hence, by Proposition 2.1, (1), we get $\mathfrak{c}^+\cap\mathfrak{c}^-\subset\mathfrak{h}\cap\mathfrak{c}\subset\mathfrak{g}'\cap\mathfrak{c}=(0)$. (WD2) is trivially satisfied by $\mathfrak{c}^\pm$. Now let $X\in\mathfrak{c}$ and suppose $\rho''(X, \mathfrak{c})=0$. Then, by (2.17) we have $\rho(X, \mathfrak{g})=0$. Therefore (WD3) for $\mathfrak{g}^+$ implies that $X\in\mathfrak{g}^+\cap\mathfrak{g}^-\cap\mathfrak{c}=\mathfrak{h}\cap\mathfrak{c}=(0)$, that is, $X=0\in\mathfrak{c}^+\cap\mathfrak{c}^-$. Thus we have proved the lemma. □

REMARK 2.7. Lemmas 2.4, 2.5 and 2.6 imply that the weak dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ in $\mathfrak{g}$ can be expressed as a direct sum of two weak subdipolarizations induced on $\mathfrak{g}'$ and $\mathfrak{c}$.

2.2. Proof of Theorem 2. Suppose that G/H is an effective homogeneous parakähler manifold with G compact connected. Then, by Proposition 2.1, we have

$$G/H=C\times(G'/H)\,. \tag{2.18}$$

Consider the weak dipolarization $\{\mathfrak{g}'^+, \mathfrak{g}'^-, \rho'\}$ in $\mathfrak{g}'$. Since $\mathfrak{g}'$ is semisimple, there exists a linear form f on $\mathfrak{g}'$ such that $\rho'=df$. The triple $\{\mathfrak{g}'^+, \mathfrak{g}'^-, f\}$ is a dipolarization in $\mathfrak{g}'$, which is trivial by Theorem 1. Therefore $\mathfrak{g}'=\mathfrak{g}'^\pm=\mathfrak{g}'^+\cap\mathfrak{g}'^-=\mathfrak{h}$. Since H is connected (Proposition 2.1), we get $G'=H$. Therefore, by (2.18) we have $G/H=C$. Note that $\dim\mathfrak{c}^+=\dim\mathfrak{c}^-$, since $\{\mathfrak{c}^+, \mathfrak{c}^-, \rho''\}$ is a weak dipolarization. Hence $G/H=C$ is an even-dimensional torus. The converse assertion can be easily shown (cf. p. 84 in [5]). □

Appendix.

The following lemma justifies calling a triple $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ satisfying (D1)–(D3) a dipolarization. For the definition of a polarization in a Lie algebra, one should refer to Dixmier [2], for instance.

LEMMA. *Let $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ be a dipolarization in $\mathfrak{g}$. Then $\{\mathfrak{g}^+, f\}$ and $\{\mathfrak{g}^-, f\}$ are polarizations in $\mathfrak{g}$.*

PROOF. Let $\mathfrak{g}'$ be a subspace of $\mathfrak{g}$ which contains $\mathfrak{g}^+$ and satisfies $f([\mathfrak{g}', \mathfrak{g}'])=0$. Choose an element $X=X^+ + X^- \in \mathfrak{g}'$, $X^\pm \in \mathfrak{g}^\pm$. Then $X^- \in \mathfrak{g}'$, and consequently $\mathfrak{g}'=\mathfrak{g}^+ + \mathfrak{g}' \cap \mathfrak{g}^-$. We then have

$$0=f([X, \mathfrak{g}'])=f([X^+ + X^-, \mathfrak{g}^+ + \mathfrak{g}' \cap \mathfrak{g}^-])=f([X^-, \mathfrak{g}^+]) .$$

On the other hand $f([X^-, \mathfrak{g}^-])=0$ is obvious. Therefore $f([X^-, \mathfrak{g}])=0$, which implies that $X^- \in \mathfrak{g}^+ \cap \mathfrak{g}^-$. Thus $X \in \mathfrak{g}^+$, or $\mathfrak{g}'=\mathfrak{g}^+$. This shows that $\{\mathfrak{g}^+, f\}$ is a polarization in $\mathfrak{g}$. □

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