TOKYO J. MATH.
VOL. 20, No. 2, 1997

Probabilities of Large Deviations for Sums of Random Number of I.I.D. Random Variables and Its Application to a Compound Poisson Process

Haruyoshi MITA

University of the Sacred Heart

(Communicated by T. Suzuki)

Abstract. Let $X_1, X_2, \cdots$ be a sequence of independent replicates of a random variable X and let $\{N_t\}_{t\geq 0}$ be a non-negative integer valued random process and assume that $\{N_t\}_{t\geq 0}$ and X are independent. Then, under some conditions it is shown that the probability $P(\sum_{i=1}^{N_t} X_i \geq 0)$ decays exponentially fast as $t\to\infty$. Moreover, we consider a testing problem in a compound Poisson process, and we study the exact slope of a test statistic based on the sum of random number of independent and exponentially distributed random variables.

1. Introduction

Probabilities of large deviations for sums of independent identically distributed (i.i.d.) random variables have been studied by many authors. For a brief review, we may refer to Chernoff [11], Bahadur and Ranga Rao [6], Sethuraman [20], Hoeffding [14], Nagaev [17], and Efron and Traux [12]. Bahadur ([2], [3], [4] and [5]) studied the efficiency of tests and estimates using the probabilities of large deviations for sums of i.i.d. random variables. For the testing problems, Bahadur introduced a concept of the exact slope of a sequence of test statistics and built a theory of efficiency. Many studies have been done since Bahadur proposed a concept of efficiency in the testing problems (e.g., Gleser [13], Sievers [21], Raghavachari [18], Kallenberg [15], Koziol [16], Berk and Brown [8], and Rukhin [19]). In this paper, we study the probabilities of large deviations for sums of random number of i.i.d. random variables. Moreover we apply the results to the exact slope of a test statistic in a compound Poisson process. In particular, we obtain a results of the large deviations probability as follows. Let $X_1, X_2, \cdots$ be a sequence of independent replicates of a random variable X and let $\{N_t\}_{t\geq 0}$ be a nonnegative integer valued random process. Suppose that $\{N_t\}_{t\geq 0}$ is independent of X and let $S_n=\sum_{i=1}^{n} X_i$, where n is an integer. Then, under the suitable conditions with respect to the distributions of X and $\{N_t\}_{t\geq 0}$, we obtain

Received January 18, 1996

$$\frac{1}{t}\log P(S_{N_t}\geq 0)=\frac{1}{t}\log \rho_t+o(1) \qquad \text{as} \quad t\to\infty ,$$

where $\rho_t=\inf_s \varphi_{N_t}(\log \varphi_X(s))$, and φ_{N_t} and φ_X denote the moment generating functions (m.g.f.) of the distribution functions of N_t and X, respectively.

In section 2, we will state several conditions which are required in this paper, and in section 3 we will obtain the results of the probabilities of large deviations for sums of random number of i.i.d. random variables. In section 4, we will apply the results to the exact slope of a test statistic in a compound Poisson process.

2. Conditions.

Let X be a random variable and let $X_1, X_2, \cdots$ be a sequence of independent replicates of X. We assume that $\{N_t\}_{t\geq 0}$ is a non-negative integer valued random process, and that $\{N_t\}_{t\geq 0}$ and X are independent, and furthermore assume that $N_t/t\to\lambda$ in probability as $t\to\infty$, where λ is a positive constant. Let $\varphi_Y(s)$ denote the m.g.f. of the distribution function of a random variable Y, i.e.,

$$\varphi_Y(s)=E(e^{sY}) , \qquad -\infty<s<\infty .$$

It is well known that if $P(Y=0)\neq 1$ and $\varphi_Y(a)<\infty$ for some $a>0$ then φ_Y is strictly convex and continuous on $[0, a]$ and has derivatives of all orders on $(0, a)$, and the first derivative $\varphi_Y'(s)$ is strictly increasing on $(0, a)$ (cf., Bahadur [5]).

We assume the following conditions (C1) through (C5):

(C1) There exists an interval I which contains origin and $\varphi_X(s)<\infty$ for each $s\in I$.

(C2) There exists a unique τ, $0<\tau<\alpha$, such that $\varphi_X(\tau)=\inf_s \varphi_X(s)$, where $\alpha=\sup\{s : s\in I\}$.

(C3) For each $s\in \mathbf{R}^1$ and each $t\geq 0$, $\varphi_{N_t}(s)<\infty$.

(C4) There exists a twice differentiable function ψ_1 satisfying

$$|t^{-1}\log \varphi_{N_t}(s)-\psi_1(s)|=o(1/t) \qquad \text{uniformly on } \mathbf{R}^1 \quad \text{as } t\to\infty .$$

(C5) For each $s\in\mathbf{R}^1$, the first derivative of $t^{-1}\log\varphi_{N_t}(s)$ has a positive limit, which is denoted by $\psi_2(s)$, as $t\to\infty$, and for each $s\in R^1$,

$$|(t^{-1}\log\varphi_{N_t}(s))'-\psi_2(s)|=o(1/t) \qquad \text{as} \quad t\to\infty .$$

We note that ψ_1 is a strictly increasing function, because, by Fatou's lemma, it follows that for any s_1, s_2 $(s_1<s_2)$

$$\begin{aligned}\psi_1(s_2)-\psi_1(s_1)&=\lim_{t\to\infty}(t^{-1}\log\varphi_{N_t}(s_2)-t^{-1}\log\varphi_{N_t}(s_1))\\ &=\lim_{t\to\infty}\int_{s_1}^{s_2}(t^{-1}\log\varphi_{N_t}(s))'ds\end{aligned}$$

$$\geq \int_{s_1}^{s_2} \lim_{t \to \infty} (t^{-1} \log \varphi_{N_t}(s))' ds = \int_{s_1}^{s_2} \psi_2(s) ds > 0 .$$

3. Probabilities of large deviations for sums of random number of i.i.d. random variables.

First, we obtain the m.g.f. of the distribution of S_{N_t}.

LEMMA 3.1. *If* $\{N_t\}_{t \geq 0}$ *and* X *are independent then for each* $s \in I$

$$\varphi_{S_{N_t}}(s) = \varphi_{N_t}(\log \varphi_X(s)) , \qquad t \geq 0 .$$

PROOF. Since $\{N_t\}_{t \geq 0}$ and X are independent, we obtain

$$\varphi_{S_{N_t}}(s) = E(e^{sS_{N_t}}) = \sum_{k=1}^{\infty} \int_{\{N_t = k\}} e^{s(X_1 + X_2 + \cdots + X_k)} dP = \sum_{k=1}^{\infty} (\varphi_X(s))^k P(N_t = k)$$

$$= \sum_{k=1}^{\infty} e^{k \log(\varphi_X(s))} P(N_t = k) = \varphi_{N_t}(\log \varphi_X(s)) .$$

This completes the proof. □

Suppose that the distribution function of X satisfies condition (C2). Then τ is the unique solution of $\varphi'_X(s) = 0$. Since $\{N_t\}_{t \geq 0}$ is a positive integer valued random process, $\varphi_{N_t}(s)$ is a strictly increasing function of s. Hence, by Lemma 3.1, if the distribution function of X satisfies condition (C2) then the distribution function of S_{N_t}, $t > 0$, also satisfies condition (C2), and

$$\inf_s \varphi_{S_{N_t}}(s) = \inf_s \varphi_{N_t}(\log \varphi_X(s)) = \varphi_{N_t}\left(\inf_{s \geq 0} \log \varphi_X(s) \right) = \varphi_{N_t}(\log \varphi_X(\tau)) .$$

Using the method of exponential centering, we study the probabilities of large deviations for the random variables S_{N_t} in the following. Before we state some lemmas, we need to introduce some notations. Let F_t be the distribution function of S_{N_t} and let ρ_t be defined by

$$\rho_t = \inf_{s \geq 0} \varphi_{S_{N_t}}(s) = \varphi_{N_t}(\log \varphi_X(\tau)) .$$

Furthermore, we let

$$G_t(y) = \rho_t^{-1} \int_{x < y} e^{\tau x} dF_t(x) , \tag{3.1}$$

$$\sigma_t^2 = \int y^2 dG_t(y) , \tag{3.2}$$

(3.3) $$H_t(z)=G_t(\sigma_t z)\,.$$

Let Y_t and Z_t be the random variables with distribution functions G_t and H_t, respectively.

LEMMA 3.2. *Under conditions* (C1) *through* (C5), *we obtain*

(3.4) $$E(Y_t)=0\,,$$

(3.5) $$\sigma_t^2=a_\tau t+o(1) \qquad as \quad t\to\infty\,,$$

where $a_\tau=(\varphi''_X(\tau)/\varphi_X(\tau))\psi_2(\log\varphi_X(\tau))$.

PROOF. By virtue of (3.1), we obtain

$$\varphi_{Y_t}(s)=\int e^{sy}dG_t(y)=\rho_t^{-1}\int e^{(s+\tau)x}dF_t(x)=\frac{\varphi_{N_t}(\log\varphi_X(s+\tau))}{\varphi_{N_t}(\log\varphi_X(\tau))}\,.$$

Hence, it follows that

(3.6) $$\varphi'_{Y_t}(s)=\frac{\varphi'_{N_t}(\log\varphi_X(s+\tau))}{\varphi_{N_t}(\log\varphi_X(\tau))}\times\frac{\varphi'_X(s+\tau)}{\varphi_X(s+\tau)}\,,$$

(3.7) $$\varphi''_{Y_t}(s)=\frac{1}{\varphi_{N_t}(\log\varphi_X(\tau))}\left\{\varphi''_{N_t}(\log\varphi_X(s+\tau))\times\left(\frac{\varphi'_X(s+\tau)}{\varphi_X(s+\tau)}\right)^2\right.$$
$$\left.+\varphi'_{N_t}(\log\varphi_X(s+\tau))\times\frac{\varphi''_X(s+\tau)\varphi_X(s+\tau)-(\varphi'_X(s+\tau))^2}{(\varphi_X(s+\tau))^2}\right\}.$$

By (3.6), (3.7), and condition (C5), we obtain

$$E(Y_t)=\varphi'_{Y_t}(0)=0\,,$$

$$\sigma_t^2=\varphi''_{Y_t}(0)=\frac{\varphi'_{N_t}(\log\varphi_X(\tau))}{\varphi_{N_t}(\log\varphi_X(\tau))}\times\frac{\varphi''_X(\tau)}{\varphi_X(\tau)}$$
$$=t[\psi_2(\log\varphi_X(\tau))+o(1/t)]\times\frac{\varphi''_X(\tau)}{\varphi_X(\tau)}=\left(\frac{\varphi''_X(\tau)}{\varphi_X(\tau)}\right)\psi_2(\log\varphi_X(\tau))t+o(1)\,.$$

This completes the proof. □

LEMMA 3.3. *Suppose that conditions* (C1) *through* (C5) *are satisfied. Then it follows that*

$$H_t(z)\to\Phi(c_\tau z) \qquad as \quad t\to\infty\,,$$

where Φ *denotes the standard normal distribution function and* c_τ *is a constant.*

PROOF. By (3.1), (3.2), (3.3), and condition (C4), we obtain

$$
\begin{aligned}
\varphi_{Z_t}(s) &= \int e^{sz} dH_t(z) = \int e^{(s/\sigma_t)y} dG_t(y) \\
&= \rho_t^{-1} \int e^{(s/\sigma_t+\tau)x} dF_t(x) = \frac{\varphi_{N_t}(\log \varphi_X(s/\sigma_t+\tau))}{\varphi_{N_t}(\log \varphi_X(\tau))} \\
&= \exp(t[\Psi_1(\log \varphi_X(s/\sigma_t+\tau)) - \psi_1(\log \varphi_X(\tau))]) \times \exp(o(1)) .
\end{aligned} \tag{3.8}
$$

Here, put $\kappa_t(s) = \psi_1(\log \varphi_X(s/\sigma_t+\tau)) - \psi_1(\log \varphi_X(\tau))$. Since Ψ_1 is a strictly increasing function and $\varphi_X(\tau) = \inf_s \varphi_X(s)$, it follows that $\kappa_t(s) \geq 0$ and $\kappa_t(0) = 0$. By Lemma 3.2, for all sufficiently large $t > 0$ there exists an interval $\tilde{I}$, including origin, such that $\sup\{\kappa_t(s) : s \in \tilde{I}\} \leq \sup\{\psi_1(s) : s \in \mathbf{R}^1\}$. Therefore, for all sufficiently large $t > 0$ we have $\{\psi_1(s) : s \in \mathbf{R}^1\} \supseteq \{\kappa_t(s) : s \in \tilde{I}\}$. Hence, for each $s \in \tilde{I}$ and for all sufficiently large $t > 0$ there exists a unique $u = u_t(s)$ such that $\psi_1(u) = \kappa_t(s)$. In view of (3.8), it follows that

$$
\begin{aligned}
\varphi_{Z_t}(s) \exp(t\kappa_t(s)) \exp(o(1)) &= \exp(t\psi_1(\psi_1^{-1}(\kappa_t(s)))) \exp(o(1)) \\
&= \varphi_{N_t}(\log e^{\psi_1^{-1}(\kappa_t(s))}) \exp(o(1)) .
\end{aligned} \tag{3.9}
$$

Now, for each sufficiently large $t > 0$, $\varphi_{N_t}(\log e^{\psi_1^{-1}(\kappa_t(s))})$ is the m.g.f. of $U_{1,t} + U_{2,t} + \cdots + U_{N_t,t}$, where $U_{1,t}$, $U_{2,t}$, $\cdots$ are independent identically distributed random variables with the common m.g.f. $\varphi_{U_{1,t}}(s) = e^{\psi_1^{-1}(\kappa_t(s))}$, and $U_{1,t}$ is independent of N_t. We obtain

$$
\varphi'_{U_{1,t}}(s) = \frac{\kappa'_t(s)}{\psi'_1(\psi_1^{-1}(\kappa_t(s)))} e^{\psi_1^{-1}(\kappa_t(s))} ,
$$

$$
\kappa'_t(s) = \psi'_1(\log \varphi_X(s/\sigma_t+\tau)) \times \frac{\varphi'_X(s/\sigma_t+\tau)}{\varphi_X(s/\sigma_t+\tau)} \times \frac{1}{\sigma_t} ,
$$

$$
\begin{aligned}
\varphi''_{U_{1,t}}(s) = e^{\psi_1^{-1}(\kappa_t(s))} \Bigg\{ & \left(\frac{\kappa'_t(s)}{\psi'_1(\psi_1^{-1}(\kappa_t(s)))} \right)^2 \\
& + \frac{\kappa''_t(s)\psi'_1(\psi_1^{-1}(\kappa_t(s))) - \kappa'_t(s)\psi''_1(\psi_1^{-1}(\kappa_t(s))) \times \dfrac{\kappa'_t(s)}{\psi'_1(\psi_1^{-1}(\kappa_t(s)))}}{(\psi'_1(\psi_1^{-1}(\kappa_t(s))))^2} \Bigg\} ,
\end{aligned}
$$

$$
\begin{aligned}
\kappa''_t(s) = {} & \psi''_1(\log \varphi_X(s/\sigma_t+\tau)) \left(\frac{\varphi'_X(s/\sigma_t+\tau)}{\varphi_X(s/\sigma_t+\tau)} \right)^2 \left(\frac{1}{\sigma_t} \right)^2 + \psi'_1(\log \varphi_X(s/\sigma_t+\tau)) \\
& \times \frac{\varphi''_X(s/\sigma_t+\tau)\varphi_X(s/\sigma_t+\tau) - (\varphi'_X(s/\sigma_t+\tau))^2}{(\varphi_X(s/\sigma_t+\tau))^2} \times \left(\frac{1}{\sigma_t} \right)^2 .
\end{aligned}
$$

Since $\kappa_t(0) = 0$, $\kappa'_t(0) = 0$ and $\psi_1^{-1}(0) = 0$, we obtain

$$
E(U_{1,t}) = 0 ,
$$

$$
\mathrm{Var}(U_{1,t}) = \frac{\kappa''_t(0)\psi'_1(0)}{(\psi'_1(0))^2} = \frac{\psi'_1(\log \varphi_X(\tau))\varphi''_X(\tau)}{\psi'_1(0)\varphi_X(\tau)} \times \frac{1}{\sigma_t^2} = \frac{b_\tau}{\sigma_t^2} ,
$$

where

$$b_\tau=\frac{\psi_1'(\log\varphi_X(\tau))\varphi_X''(\tau)}{\psi_1'(0)\varphi_X(\tau)}.$$

Let $V_{i,t}=\sigma_t U_{i,t}$ $(i=1, 2, \cdots)$. Then we have

$$\begin{aligned}\varphi_{V_{1,t}}(s)=\varphi_{U_{1,t}}(\sigma_t s)&=\exp(\psi_1^{-1}(\kappa_t(\sigma_t s)))\\&=\exp(\psi_1^{-1}(\psi_1(\log\varphi_X(s+\tau))-\psi_1(\log\varphi_X(\tau)))).\end{aligned}$$

Hence, $\varphi_{V_{1,t}}(s)$ is independent of t. Therefore, we put $V_i=V_{i,t}$. Consequently, $\varphi_{N_t}(\log e^{\psi_1^{-1}(\kappa_t(s))})$ is the m.g.f. of the random sums $(V_1+V_2+\cdots+V_{N_t})/\sigma_t$, where $V_1, V_2, \cdots$ are i.i.d. random variables with the m.g.f. $\varphi_{V_1}(s)=\exp(\psi_1^{-1}(\psi_1(\log\varphi_X(s+\tau))-\psi_1(\log\varphi_X(\tau))))$. Since $E(V_1)=0$ and $\mathrm{Var}(V_1)=b_\tau$, by the central limit theorem for sums of i.i.d. random variables it follows that

$$\frac{V_1+V_2+\cdots+V_n}{\sqrt{nb_\tau}}\to N(0, 1)\quad\text{in law}\qquad\text{as}\quad n\to\infty.$$

Since $N_t/t\to\lambda>0$ in probability as $t\to\infty$, by Anscombe [1] (cf., also, Billingsley [9]), we obtain

$$\frac{V_1+V_2+\cdots+V_{N_t}}{\sqrt{N_t b_\tau}}\to N(0, 1)\quad\text{in law}\qquad\text{as}\quad t\to\infty.$$

Hence, it follows that

$$\begin{aligned}\frac{V_1+V_2+\cdots+V_{N_t}}{\sigma_t}&=\frac{V_1+V_2+\cdots+V_{N_t}}{\sqrt{N_t b_\tau}}\times\frac{\sqrt{N_t b_\tau}}{\sigma_t}\\&=\frac{V_1+V_2+\cdots+V_{N_t}}{\sqrt{N_t b_\tau}}\times\frac{\sqrt{N_t b_\tau}}{\sqrt{a_\tau t+o(1)}}\to N\left(0, \frac{\lambda b_\tau}{a_\tau}\right)\quad\text{in law}\qquad\text{as}\quad t\to\infty.\end{aligned}$$

From the continuity theorem for the m.g.f. (cf., Billingsley [10]), $\varphi_{N_t}(\log e^{\psi_1^{-1}(\kappa_t(s))})$ converges to the m.g.f. of $N(0, \lambda b_\tau/a_\tau)$ as $t\to\infty$, and by (3.9), for each $s\in\tilde{I}$, $\varphi_{Z_t}(s)$ converges to the same m.g.f. as $t\to\infty$. Therefore we obtain

$$H_t(z)\to\Phi(c_\tau z)\qquad\text{as}\quad t\to\infty,$$

where $c_\tau=\sqrt{a_\tau/(\lambda b_\tau)}$. This completes the proof. □

THEOREM 3.1. *Suppose that conditions* (C1) *through* (C5) *are satisfied. Then it follows that*

$$\frac{1}{t}\log P(S_{N_t}\geq 0)-\frac{1}{t}\log\rho_t\to 0\qquad as\quad t\to\infty.$$

PROOF. By Markov inequality, for each $s \geq 0$ we have

$$P(S_{N_t} \geq 0) = P(e^{sS_{N_t}} \geq 1) \leq E(e^{sS_{N_t}}) .$$

Thus we obtain

$$P(S_{N_t} \geq 0) \leq \inf_{s \geq 0} \varphi_{S_{N_t}}(s) = \rho_t .$$

Hence, it follows that

$$\limsup_{t \to \infty} (t^{-1} \log P(S_{N_t} \geq 0) - t^{-1} \log \rho_t) \leq 0 . \tag{3.10}$$

Next, we shall consider the lower bound. Let $\varepsilon > 0$. For each $s \geq 0$, by (3.1), (3.2), and (3.3), we have

$$\begin{aligned} P(S_{N_t} \geq 0) &= \int_0^\infty dF_t(x) = \rho_t \int_0^\infty e^{-\tau y} dG_t(y) \geq \rho_t \int_0^{\varepsilon \sigma_t} e^{-\tau y} dG_t(y) \\ &\geq \rho_t e^{-\tau \sigma_t \varepsilon} \int_0^{\varepsilon \sigma_t} dG_t(y) = \rho_t e^{-\tau \sigma_t \varepsilon} (H_t(\varepsilon) - H_t(0)) . \end{aligned} \tag{3.11}$$

By Lemma 3.2 and Lemma 3.3, we have

$$\lim_{t \to \infty} \frac{\sigma_t}{t} = 0 , \qquad \lim_{t \to \infty} \frac{1}{t} \log(H_t(\varepsilon) - H_t(0)) = 0 .$$

In view of (3.11), it follows that

$$\liminf_{t \to \infty} \left(\frac{1}{t} \log P(S_{N_t} \geq 0) - \frac{1}{t} \log \rho_t \right) \geq 0 . \tag{3.12}$$

By (3.10) and (3.12), the proof has been completed. □

Before we state the next theorem, we introduce the following notations. For a constant d, let

$$p_t(d) = P\left(\sum_{i=1}^{N_t} (X_i - d) \geq 0 \right),$$

$$\rho_t(d) = \inf_s \varphi_{S_{N_t} - dN_t}(s) = \inf_s \varphi_{N_t}(\log \varphi_{X-d}(s)) .$$

Next theorem is a generalization of Theorem 3.1.

THEOREM 3.2. *Let $\{d_t\}_{t \geq 0}$ be a random process such that*

$$d_t / N_t \to d \quad \textit{in probability} \qquad \textit{as } t \to \infty ,$$

where $d \neq 0$ is a constant. Suppose that the distributions of $X - d$ and $\{N_t\}_{t \geq 0}$ satisfy *conditions* (C1) *through* (C5) *and* $P(X>d)>0$, *and moreover, for all sufficiently small* $\varepsilon>0$

$$\frac{P(|d_t/N_t-d|>\varepsilon)}{\min\{p_t(d+\varepsilon), p_t(d-\varepsilon)\}}\to 0 \qquad as \quad t\to\infty .$$

Then we obtain

$$\lim_{t\to\infty}\left\{\frac{1}{t}\log P\left(\sum_{i=1}^{N_t} X_i\geq d_t\right)-\frac{1}{t}\log \rho_t(d)\right\}=0 .$$

PROOF. For any $\varepsilon>0$, we have

$$P\left(\sum_{i=1}^{N_t} X_i\geq d_t\right)=P\left(\sum_{i=1}^{N_t} X_i\geq d_t, |d_t/N_t-d|\leq\varepsilon\right)+P\left(\sum_{i=1}^{N_t} X_t\geq d_t, |d_t/N_t-d|>\varepsilon\right)$$
$$=I_1+I_2 ,$$

where

$$I_1=P\left(\sum_{i=1}^{N_t} X_i\geq d_t, |d_t/N_t-d|\leq\varepsilon\right), \qquad I_2=P\left(\sum_{i=1}^{N_t} X_i\geq d_t, |d_t/N_t-d|>\varepsilon\right).$$

It follows that

$$I_1\leq P\left(\sum_{i=1}^{N_t} X_i\geq(d-\varepsilon)N_t\right)=P\left(\sum_{i=1}^{N_t} (X_i-(d-\varepsilon))\geq 0\right), \tag{3.13}$$

$$\begin{aligned} I_1 &\geq P\left(\sum_{i=1}^{N_t} X_i\geq N_t(d+\varepsilon), |d_t/N_t-d|\leq\varepsilon\right) \\ &\geq P\left(\sum_{i=1}^{N_t} X_i\geq N_t(d+\varepsilon)\right)-P(|d_t/N_t-d|>\varepsilon) \\ &= P\left(\sum_{i=1}^{N_t} (X_i-(d+\varepsilon))\geq 0\right)-P(|d_t/N_t-d|>\varepsilon) . \end{aligned} \tag{3.14}$$

It is clear that $0\leq I_2\leq P(|d_t/N_t-d|>\varepsilon)$. By (3.13) and (3.14), we obtain

$$\begin{aligned} &P\left(\sum_{i=1}^{N_t} (X_i-(d+\varepsilon))\geq 0\right)-P(|d_t/N_t-d|>\varepsilon) \\ &\qquad \leq I_1+I_2\leq P\left(\sum_{i=1}^{N_t} (X_i-(d-\varepsilon))\geq 0\right)+P(|d_t/N_t-d|>\varepsilon) . \end{aligned}$$

Hence, we have

$$p_t(d+\varepsilon)\left[1-\frac{P(|d_t/N_t-d|>\varepsilon)}{p_t(d+\varepsilon)}\right]\leq I_1+I_2\leq p_t(d-\varepsilon)\left[1+\frac{P(|d_t/N_t-d|>\varepsilon)}{p_t(d-\varepsilon)}\right].$$

Thus we obtain for all sufficiently large $t>0$,

$$\text{(3.15)} \quad \frac{1}{t}\log p_t(d+\varepsilon)+\frac{1}{t}\log\left[1-\frac{P(|d_t/N_t-d|>\varepsilon)}{p_t(d+\varepsilon)}\right]\leq\frac{1}{t}\log P\left(\sum_{i=1}^{N_t}X_i\geq d_t\right)$$
$$\leq\frac{1}{t}\log p_t(d-\varepsilon)+\frac{1}{t}\log\left[1+\frac{P(|d_t/N_t-d|>\varepsilon)}{p_t(d-\varepsilon)}\right].$$

If the distribution of $X-d$ satisfies conditions (C1) and (C2) then for all sufficiently small $\varepsilon>0$, the distributions of $X-(d+\varepsilon)$ and $X-(d-\varepsilon)$ also satisfy the same conditions. Let $\tau(d)$ denote the unique solution of $\varphi'_{X-d}(s)=0$, i.e., $\varphi_{X-d}(\tau(d))=\inf_{s\geq 0}\varphi_{X-d}(s)$. From the right-hand side inequality in (3.15), using Theorem 3.1 and condition (C4) we have

$$\limsup_{t\to\infty}\left(\frac{1}{t}\log P\left(\sum_{i=1}^{N_t}X_i\geq d_t\right)-\frac{1}{t}\log p_t(d)\right)$$
$$\leq\limsup_{t\to\infty}\left(\frac{1}{t}\log p_t(d-\varepsilon)-\frac{1}{t}\log p_t(d)\right)$$
$$=\limsup_{t\to\infty}\left(\frac{1}{t}\log\rho_t(d-\varepsilon)-\frac{1}{t}\log\rho_t(d)\right)$$
$$=\psi_1(\log\varphi_{X-(d-\varepsilon)}(\tau(d-\varepsilon)))-\psi_1(\log\varphi_{X-d}(\tau(d))).$$

By lemma 3.3 in Bahadur [5], $\log\varphi_{X-d}(\tau(d))$ is continuous in a neighborhood of d. Therefore, by letting $\varepsilon\to 0$, we have

$$\limsup_{t\to\infty}\left(\frac{1}{t}\log P\left(\sum_{i=1}^{N_t}X_i\geq d_t\right)-\frac{1}{t}\log p_t(d)\right)\leq 0.$$

Similarly, from the left-hand side inequality in (3.15), we have

$$\liminf_{t\to\infty}\left(\frac{1}{t}\log P\left(\sum_{i=1}^{N_t}X_i\geq d_t\right)-\frac{1}{t}\log p_t(d)\right)\geq 0.$$

This completes the proof. □

4. Exact slope for a test statistic in a compound Poisson process.

Let X be a random variable with exponential distribution. Its density function is

$$f(x\,;\,\mu)=\mu e^{-\mu x} \qquad (x\geq 0),$$
$$=0 \qquad (x<0),$$

where $\mu>0$. Let $X_1, X_2, \cdots$ be a sequence of independent replicates of X and let $\{N_t\}_{t\geq 0}$ be the Poisson process with parameter $\lambda>0$ starting at 0. We assume that λ is known and that X and $\{N_t\}_{t\geq 0}$ are independent. Let $Z_t=\sum_{i=1}^{N_t} X_i$. Then $\{Z_t\}_{t\geq 0}$ is a compound Poisson process defined on a probability space $(\Omega, \mathbf{F}, P_\mu)$, where the probability measure P_μ depends on an unknown parameter $\mu\in\Theta$ and Θ is an open set in $(0, \infty)$. Let $\mathbf{F}_t$, $t\geq 0$, denote the σ-field generated by the random process $\{Z_s\}_{0\leq s\leq t}$, and let $P_{\mu,t}$ be the restriction of P_μ on $\mathbf{F}_t$. Fix any $\mu_0\in\Theta$. Then for any $\mu\in\Theta$ and all $t\geq 0$, $P_{\mu,t}$ is dominated by $P_{\mu_0,t}$, and the likelihood ratio statistics, denoting it as $\Lambda_t(\mu_0, \mu)$, is given by

$$\Lambda_t(\mu_0, \mu)=\frac{dP_{\mu,t}}{dP_{\mu_0,t}}=\left(\frac{\mu}{\mu_0}\right)^{N_t}\exp(-(\mu-\mu_0)Z_t) \tag{4.1}$$

(cf., Basawa and Prakasa Rao [7]). We consider a simple test: μ_0 against μ $(\mu\neq\mu_0)$. For each $t\geq 0$ let $T_t=T_t(N_t, Z_t)$ be a real valued test statistic based on (N_t, Z_t) and assume that the large value of T_t is significant. Let J_t denote the distribution function of T_t when μ_0 obtains, i.e.,

$$J_t(x)=P_{\mu_0}(T_t<x)\,.$$

We define *the attained level of* T_t to be

$$L_t=1-J_t(T_t)\,.$$

We shall say that the process $\{T_t\}_{t\geq 0}$ has *the exact slope* $c(\mu)$ when μ obtains if

$$\lim_{t\to\infty}\frac{1}{N_t}\log L_t=-\frac{1}{2}c(\mu) \qquad \text{a.s.} \quad P_\mu\,.$$

Following theorem is useful to find the exact slope of $\{T_t\}_{t\geq 0}$. It is analogous to Theorem 7.2 in Bahadur [5] in case when number of random variables is non-random. Its proof may be obtained along Bahadur [5]. Therefore we omit the proof.

THEOREM 4.1. *Suppose that*

$$\lim_{t\to\infty} T_t=b(\mu) \qquad \text{a.s.} \quad P_\mu\,,$$

where $-\infty<b(\mu)<\infty$, *and that*

$$\lim_{t\to\infty}\frac{1}{N_t}\log[1-J_t(x)]=-f(x) \qquad \text{a.s.} \quad P_{\mu_0}$$

for each $x\in I$, *where* I *is an open interval and* f *is a continuous function on* I *and* $b(\mu)\in I$. *Then we obtain* $c(\mu)=2f(b(\mu))$.

Here we let

$$T_t=\frac{1}{N_t}\log\Lambda_t(\mu_0, \mu)\,.$$

We assume that $\mu_0>\mu$. For the testing problem: μ_0 against μ, we find the exact slope of $\{T_t\}_{t\geq 0}$ in the following. Note that

$$\lim_{t\to\infty}\frac{N_t}{t}=\lambda>0 \qquad \text{a.s.} \quad P_\mu\,,$$

$$\varphi_X(s)=\frac{\mu}{\mu-s}\,(s<\mu)\,, \qquad \varphi_{N_t}(s)=e^{\lambda t(e^s-1)}\,,$$

when μ obtains.

We have

$$T_t=\frac{1}{N_t}\log \Lambda_t(\mu_0,\mu)=\log\frac{\mu}{\mu_0}-(\mu-\mu_0)\frac{Z_t}{N_t}=A-B\frac{Z_t}{N_t}\,,$$

where $A=\log(\mu/\mu_0)$, $B=\mu-\mu_0$. Therefore, we have

$$P_{\mu_0}(T_t\geq x)=P_{\mu_0}\left(\sum_{i=1}^{N_t}(A-x-BX_i)\geq 0\right).$$

We suppose that $x>A-B/\mu_0$. It is easy to see that the distribution of random variable $A-x-BX$ satisfies conditions (C1) and (C2), and that the distributions of $\{N_t\}_{t\geq 0}$ satisfy conditions (C3), (C4), and (C5). Under the null hypothesis μ_0 we obtain

$$\varphi_{A-x-BX}(s)=e^{(A-x)s}\varphi_X(-Bs)=e^{(A-x)s}\frac{\mu_0}{\mu_0+Bs} \qquad (-Bs<\mu_0)\,.$$

By a straightforward calculation we obtain

$$\inf_{s\geq 0}\varphi_{A-x-BX}(s)=e^{(A-x)s(x)}\frac{\mu_0}{\mu_0+Bs(x)}\,,$$

where $s(x)=1/(A-x)-\mu_0/B>0$, because $x>A-B/\mu_0$. Hence

$$\log\rho_t=\log\left(\inf_{s\geq 0}\varphi_{N_t}(\log\varphi_{A-x-BX}(s))\right)=\log\varphi_{N_t}(\log\varphi_{A-x-BX}(s(x)))$$
$$=\lambda t\left(\frac{(A-x)\mu_0}{B}e^{1-(A-x)\mu_0/B}-1\right).$$

Since $\lim_{t\to\infty}N_t/t=\lambda$ a.s. P_μ, by Theorem 3.1 we have

$$\lim_{t\to\infty}\frac{1}{N_t}\log(1-F_t(x))=\frac{(A-x)\mu_0}{B}e^{1-(A-x)\mu_0/B}-1 \qquad \text{a.s.} \quad P_{\mu_0}\,,$$

$$\lim_{t\to\infty}T_t=A-\frac{B}{\mu} \qquad \text{a.s.} \quad P_\mu\,.$$

Since $A-B/\mu>A-B/\mu_0$, in view of Theorem 4.1, we obtain

$$c(\mu)=2\left(1-\frac{\mu_0}{\mu}e^{1-\mu_0/\mu}\right).$$

ACKNOWLEDGMENTS. The author is grateful to the referee for his helpful comments and valuable suggestions.

References

[1] F. J. ANSCOMBE, Large sample theory of sequential estimation, Proc. Cambridge Philos. Soc. **48** (1952), 600–607.

[2] R. R. BAHADUR, Asymptotic efficiency of tests and estimates, Sankhyā **22** (1960), 229–252.

[3] R. R. BAHADUR, An optimal property of the likelihood ratio statistic, Proc. Fifth Berkeley Symp. Math. Statist. Prob. **1** (1965), 13–26.

[4] R. R. BAHADUR, Rates of convergence of estimates and test statistics, Ann. Math. Statist. **38** (1967), 303–324.

[5] R. R. BAHADUR, *Some Limit Theorems in Statistics*, SIAM (1971).

[6] R. R. BAHADUR and R. RANGA RAO, On deviations of the sample mean, Ann. Math. Statist. **31** (1960), 1015–1027.

[7] I. V. BASAWA and B. L. S. PRASAKA RAO, *Statistical Inference for Stochastic Processes*, Academic Press (1980).

[8] R. H. BERK and L. D. BROWN, Sequential Bahadur efficiency, Ann. Statist. **6** (1978), 567–581.

[9] P. BILLINGSLEY, *Convergence of Probability Measures*, Wiley (1968).

[10] P. BILLINGSLEY, *Probability and Measure*, Wiley (1986).

[11] H. CHERNOFF, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, Ann. Math. Statist. **23** (1952), 493–507.

[12] B. EFRON and D. TRUAX, Large deviations theory in exponential families, Ann. Math. Statist. **39** (1968), 1402–1424.

[13] L. J. GLESER, On a measure of test efficiency proposed by R. R. Bahadur, Ann. Math. Statist. **35** (1964), 1537–1544.

[14] W. HOEFFDING, On probabilities of large deviations, Proc. Fifth Berkley Symp. Math. Statist. Prob. **1** (1965), 203–219.

[15] W. C. M. KALLENBERG, *Asymptotic Optimality of Likelihood Ratio Tests in Exponential Families*, Mathematisch Centrum (1978).

[16] J. A. KOZIOL, Exact slopes of certain multivariate tests of hypotheses, Ann. Statist. **6** (1978), 546–558.

[17] S. V. NAGAEV, Some limit theorems for large deviations, Theory Probab. Appl. **10** (1965), 214–235.

[18] M. RAGHAVACHARI, On a theorem of Bahadur on the rate of convergence of test statistics, Ann. Math. Statist. **41** (1970), 1695–1699.

[19] A. L. RUKHIN, Bahadur efficiency of tests of separate hypotheses and adaptive test statistics, J. Amer. Statist. Assoc. **88** (1993), 161–165.

[20] J. SETHURAMAN, On the probability of large deviations of families of sample means, Ann. Math. Statist. **35** (1964), 1304–1316.

[21] G. L. SIEVERS, On the probability of large deviations and exact slopes, Ann. Math. Statist. **40** (1969), 1908–1921.

Present Address:
UNIVERSITY OF THE SACRED HEART,
HIROO, SHIBUYA-KU, TOKYO, 150 JAPAN.