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Formulations of Elliptic Lie Algebra $\hat{sl}(2)$ and Elliptic Virasoro Algebra by Vertex Operators

Tadayoshi TAKEBAYASHI

Waseda University

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1. Introduction.

In the definition of a vertex operator algebra ([5]), the Virasoro algebra comes out naturally and the Virasoro algebra is realized by Sugawara construction in terms of the affine Lie algebra, so towards a generalization of a vertex operator algebra in the case of elliptic Lie algebras, in this paper we give a formulation of elliptic Lie algebra $\hat{sl}(2)$ and the elliptic Virasoro algebra by vertex operators. Elliptic Lie algebra is a Lie algebra associated with an elliptic root system ([1]), which also called extended affine Lie algebra ([2], [3]), or 2-toroidal Lie algebra ([6]), and the elliptic Virasoro algebra, we call so for the reason of the correspondence with Lie algebra, the 2-dimensional Virasoro-Bott algebra ([8], [9]). A certain formulation and representation of 2-toroidal Lie algebras by vertex operators are already given by [3], [6], [7]. However, we give another formulation of elliptic Lie algebra $\hat{sl}(2)$ and the elliptic Virasoro algebra by vertex operators from the view point of a generalization of vertex operator formalism. At first we recall the definition and formulation of affine Lie algebras and the Virasoro algebra by vertex operators ([5]), and next with similar correspondence, we describe them in elliptic case.

2. Affine Lie algebras and Virasoro algebra.

We recall the definition of the affine Lie algebras and Virasoro algebra and their formulations by vertex operators [5]. We describe explicitly in the case of $sl(2, \mathbf{C})$ for simplicity, however the result is applicable for any finite dimensional simple Lie algebras. We set $\mathfrak{a} = sl(2, \mathbf{C})$, and choose the basis α_1, x_{α_1}, $x_{-\alpha_1}$ of $\mathfrak{a}$, where

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x_{\alpha_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x_{-\alpha_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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These have commutation relations;

$$[\alpha_1, x_{\pm\alpha_1}] = \pm 2x_{\pm\alpha_1} = \langle \alpha_1, \pm\alpha_1 \rangle x_{\pm\alpha_1}, \qquad [x_{\alpha_1}, x_{-\alpha_1}] = \alpha_1 .$$

Here $\langle \cdot, \cdot \rangle$ denotes the nonsingular invariant symmetric bilinear form on $\mathfrak{a}$ given by

$$\langle x, y \rangle = \operatorname{tr} xy \qquad \text{for all} \quad x, y \in \mathfrak{a}$$

so that

$$\langle \alpha_1, \alpha_1 \rangle = 2, \qquad \langle x_{\alpha_1}, x_{-\alpha_1} \rangle = 1,$$

$$\langle \alpha_1, x_{\alpha_1} \rangle = \langle \alpha_1, x_{-\alpha_1} \rangle = \langle x_{\alpha_1}, x_{\alpha_1} \rangle = \langle x_{-\alpha_1}, x_{-\alpha_1} \rangle = 0 .$$

The affine Lie algebra $\widetilde{sl}(2)$ is

$$\tilde{\mathfrak{a}} = \widetilde{sl}(2) = sl(2, \mathbf{C}) \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$

and has basis

$$\{c, d, \alpha_1 \otimes t^m, x_{\pm\alpha_1} \otimes t^n \mid m, n \in \mathbf{Z}\}$$

with brackets

$$[c, \widetilde{sl}(2)] = 0 \tag{1.1.1}$$

$$[\alpha_1 \otimes t^m, \alpha_1 \otimes t^n] = 2m\delta_{m+n,0}c \tag{1.1.2}$$

$$[\alpha_1 \otimes t^m, x_{\pm\alpha_1} \otimes t^n] = 2x_{\pm\alpha_1} \otimes t^{m+n} \tag{1.1.3}$$

$$[x_{\pm\alpha_1} \otimes t^m, x_{\pm\alpha_1} \otimes t^n] = 0 \tag{1.1.4}$$

$$[x_{\alpha_1} \otimes t^m, x_{-\alpha_1} \otimes t^n] = \alpha_1 \otimes t^{m+n} + m\delta_{m+n,0}c \tag{1.1.5}$$

$$[d, \alpha_1 \otimes t^m] = m\alpha_1 \otimes t^m \tag{1.1.6}$$

$$[d, x_{\pm\alpha_1} \otimes t^m] = mx_{\pm\alpha_1} \otimes t^m . \tag{1.1.7}$$

For the purpose of the realization of $\widetilde{sl}(2)$ by vertex operator, we set

$$x_{\pm\alpha_1}(z) = \sum_{n \in \mathbf{Z}} (x_{\pm\alpha_1} \otimes t^n) z^{-n},$$

$$\alpha_1(z) = \sum_{n \in \mathbf{Z}} (\alpha_1 \otimes t^n) z^{-n} .$$

Then the bracket relations (1.1.3)–(1.1.7) can be expressed as follows:

$$[\alpha_1 \otimes t^m, x_{\pm\alpha_1}(z)] = \pm 2z^m x_{\pm\alpha_1}(z) = \langle \alpha_1, \pm\alpha_1 \rangle z^m x_{\pm\alpha_1}(z) \tag{1.2.3}$$

$$[x_{\pm\alpha_1}(z_1), x_{\pm\alpha_1}(z_2)] = 0 \tag{1.2.4}$$

$$[x_{\alpha_1}(z_1), x_{-\alpha_1}(z_2)] = \alpha_1(z_2)\delta(z_1/z_2) - (D_{z_1}\delta)(z_1/z_2)c \tag{1.2.5}$$

$$[d, \alpha_1(z)] = -D_z\alpha_1(z) \tag{1.2.6}$$

$$[d, x_{\pm\alpha_1}(z)] = -D_z x_{\pm\alpha_1}(z) \tag{1.2.7}$$

where $\delta(z)=\sum_{n\in\mathbf{Z}} z^n$ (formally the Fourier expansion of the δ-function at $z=1$), and $D_z = z(d/dz)$.

Let $d_n = -t^{n+1}d/dt$ for $n\in\mathbf{Z}$, then the commutators have the form

$$[d_m, d_n] = (m-n)d_{m+n} \qquad \text{for} \quad m, n\in\mathbf{Z}\,.$$

One-dimensional central extension of the above algebra with basis consisting of a central element c and elements L_n, $n\in\mathbf{Z}$, corresponding to the basis element d_n, with the bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m+n,0}c\,,$$

is called the Virasoro algebra. We set $L(z)=\sum_{n\in\mathbf{Z}} L(n)\otimes z^{-n-2}$ where $L(n)$ is an operator which acts on the vector space V such that

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{m+n,0}(\operatorname{rank} V)\,.$$

Therefore the operator $L(n)$ provides a representation of the Virasoro algebra on V with

$$L_n \mapsto L(n) \qquad \text{for } n\in\mathbf{Z}$$
$$c \mapsto (\operatorname{rank} V)\,.$$

Then the following commutation relations hold ([5]):

$$[L(m), L(z_2)] = \left(z_2^{m+1}\frac{d}{dz_2} + 2(m+1)z_2^m\right)L(z_2) + \frac{1}{12}(m^3-m)z_2^{m-2}c\,,$$

$$[L(z_1), L(z_2)] = \left(\frac{d}{dz_2}L(z_2)\right)z_2^{-1}\delta(z_1/z_2) - 2L(z_2)z_2^{-1}\frac{\partial}{\partial z_1}\delta(z_1/z_2)$$
$$-\frac{1}{12}z_2^{-1}\left(\frac{\partial}{\partial z_1}\right)^3\delta(z_1/z_2)c\,,$$

where we identify $(\operatorname{rank} V)$ with c.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\alpha\in\mathfrak{h}$, we set

$$\alpha(z) = \sum_{n\in\mathbf{Z}} \alpha(n)\otimes z^{-n-1}\,, \qquad X(\alpha, z) = \sum_{n\in\mathbf{Z}} X_\alpha(n)\otimes z^{-n-1}\,.$$

We recall the following representation ([5])

$$[L(m), \alpha(n)] = -n\alpha(m+n)\,,$$

$$[L(m), X_\alpha(n)] = \left(\frac{m}{2}(\alpha \mid \alpha) - m - n\right) X_\alpha(m+n)\,.$$

Further the commutation actions of $L(m)$ on $\alpha(z)$ and $X(\alpha, z)$ are given as follows:

$$[L(m), \alpha(z)] = \left(z^{m+1}\frac{d}{dz} + (m+1)z^m\right)\alpha(z)\,,$$

$$\begin{aligned}[L(z_1), \alpha(z_2)] &= \alpha(z_1) z_1^{-1} \frac{\partial}{\partial z_2}\partial(z_1/z_2) = -\alpha(z_1) z_2^{-1} \frac{\partial}{\partial z_1}\delta(z_1/z_2) \\ &= \left(\frac{d}{dz_2}\alpha(z_2)\right) z_2^{-1}\delta(z_1/z_2) - \alpha(z_2) z_2^{-1}\frac{\partial}{\partial z_1}\delta(z_1/z_2)\,,\end{aligned}$$

$$[L(m), X(\alpha, z)] = \left(z^{m+1}\frac{d}{dz} + \frac{1}{2}(\alpha \mid \alpha)(m+1)z^m\right) X(\alpha, z)\,,$$

$$[L(z_1), X(\alpha, z_2)] = \left(\frac{d}{dz_2} X(\alpha, z_2)\right) z_2^{-1}\delta(z_1/z_2) - \frac{1}{2}(\alpha \mid \alpha) X(\alpha, z_2) z_2^{-1} \frac{\partial}{\partial z_1}\delta(z_1/z_2)\,.$$

In the sequal, we describe these formulas in the case of elliptic Lie algebra $\hat{sl}(2)$ and the elliptic Virasoro algebra.

3. Formulation of elliptic Lie algebra $\hat{sl}(2)$ and elliptic Virasoro algebra.

The elliptic Lie algebra $\hat{\mathfrak{a}}$ is

$$\hat{\mathfrak{a}} = sl(2, \mathbf{C}) \otimes \mathbf{C}[t, t^{-1}, s, s^{-1}] \oplus \mathbf{C}c_1 \oplus \mathbf{C}c_2 \oplus \mathbf{C}d_1 \oplus \mathbf{C}d_2\,,$$

and has the basis

$$\{c_1, c_2, \alpha_1 \otimes t^n s^k, x_{\pm\alpha_1} \otimes t^n s^k, d_1, d_2 \mid n, k \in \mathbf{Z}\}$$

with brackets

$$[c_1, \hat{\mathfrak{a}}] = [c_2, \hat{\mathfrak{a}}] = 0\,,$$

$$[\alpha_1 \otimes t^n s^k, x_{\pm\alpha_1} \otimes t^m s^l] = 2(nc_1 + kc_2)\delta_{m+n,0}\delta_{k+l,0}\,,$$

$$[\alpha_1 \otimes t^n s^k, x_{\pm\alpha_1} \otimes t^m s^l] = \alpha_1 \otimes t^{n+m} s^{k+l} + (nc_1 + kc_2)\delta_{n+m,0}\delta_{k+l,0}\,,$$

$$[d_1, \alpha_1 \otimes t^n s^k] = n\alpha_1 \otimes t^n s^k\,, \qquad [d_1, x_{\pm\alpha_1} \otimes t^n s^k] = n x_{\pm\alpha_1} \otimes t^n s^k\,,$$

$$[d_2, \alpha_1 \otimes t^n s^k] = k\alpha_1 \otimes t^n s^k\,, \qquad [d_2, x_{\pm\alpha_1} \otimes t^n s^k] = k x_{\pm\alpha_1} \otimes t^n s^k\,,$$

$$[d_1, d_2] = 0\,.$$

For $x \in \mathfrak{a}$, we set

$$x(z, w) = \sum_{n,k \in \mathbf{Z}} (x \otimes t^n s^k) z^{-n} w^{-k} .$$

Then the bracket relations can be expressed as follows:

PROPOSITION 3.1. *For* $x, y \in \mathfrak{a}$

$$[x(z_1, w_1), y(z_2, w_2)] = [x, y](z_2, w_2)\delta(z_1/z_2)\delta(w_1/w_2) - \langle x, y\rangle (c_1 D_{z_1} + c_2 D_{w_1})\delta(z_1/z_2)\delta(w_1/w_2) , \tag{3.1.1}$$

$$[c_1, x(z, w)] = 0 , \qquad [c_2, x(z, w)] = 0 ,$$

$$[d_1, x(z, w)] = -D_z x(z, w) , \qquad [d_2, x(z, w)] = -D_w x(z, w) .$$

PROOF. We show only (3.1.1).

$$\begin{aligned}
&[x(z_1, w_1), y(z_2\ w_2)] \\
&= \sum_{n,k,m,l} [x \otimes t^n s^k, y \otimes t^m s^l] z_1^{-n} w_1^{-k} z_2^{-m} w_2^{-l} \\
&= \sum_{n,k,m,l} ([x, y] \otimes t^{n+m} s^{k+l} + \langle x, y\rangle \delta_{n+m,0}\delta_{k+l,0}(nc_1 + kc_2)) z_1^{-n} w_1^{-k} z_2^{-m} w_2^{-1} \\
&= \sum_{n,k,m,l} [x, y] \otimes t^m s^l z_l^{-n} w_1^{-k} z_2^{-m+n} w_2^{-l+k} + \langle x, y\rangle \sum_{n,k} (nc_1 + kc_2) z_1^{-n} w_1^{-k} z_2^{n} w_2^{k} \\
&= [x, y](z_2, w_2)\delta(z_1/z_2)\delta(w_1/w_2) - \langle x, y\rangle (c_1 D_{z_1} + c_2 D_{w_1})\delta(z_1/z_2)\delta(w_1/w_2) .
\end{aligned}$$

□

Especially, for the elements

$$x_{\pm\alpha_1}(z, w) = \sum_{n,k \in \mathbf{Z}} (x_{\pm\alpha_1} \otimes t^n s^k) z^{-n} w^{-k} ,$$

$$\alpha_1(z, w) = \sum_{n,k \in \mathbf{Z}} (\alpha_1 \otimes t^n s^k) z^{-n} w^{-k} ,$$

the formula (3.1.1) can be formulated as follows:

COROLLARY 3.2.

$$\begin{aligned}
[\alpha_1 \otimes t^n s^k, x_{\pm\alpha_1}(z, w)] &= \pm 2 z^n w^k x_{\pm\alpha_1}(z, w) \\
&= \langle \alpha_1, \pm\alpha_1\rangle z^n w^k x_{\pm\alpha_1}(z, w) ,
\end{aligned}$$

$$[x_{\pm\alpha_1}(z_1, w_1), x_{\pm\alpha_1}(z_2, w_2)] = 0 ,$$

$$[x_{\alpha_1}(z_1, w_1), x_{-\alpha_1}(z_2, w_2)] = \alpha_1(z_2, w_2)\delta(z_1/z_2)\delta(w_1/w_2) - (c_1 D_{z_1} + c_2 D_{w_1})\delta(z_1/z_2)\delta(w_1/w_2) .$$

Here we note that in physical field theory, the similar construction by using local current algebra was given in [4].

From now we define the elliptic Virasoro algebra ([9]). We set

$$a_{kl}=-t^{k+1}s^{l}\frac{d}{dt}, \quad b_{kl}=-t^{k}s^{l+1}\frac{d}{ds}, \quad d_{kl}=la_{kl}-kb_{kl} \qquad \text{for } k, l\in\mathbf{Z},$$

$$d_{t}=t\frac{d}{dt}, \qquad d_{s}=s\frac{d}{ds}.$$

Then the commutations have the form:

$$[d_{kl}, d_{nm}]=(km-nl)d_{k+n,m+l} \qquad \text{for} \quad k, l, m, n\in\mathbf{Z},$$

$$[d_{t}, a_{kl}]=ka_{kl}, \qquad [d_{s}, a_{kl}]=la_{kl}, \quad [d_{t}, d_{s}]=0.$$

The central extension of this algebra is given by ([8], [9]):

$$[L_{n,k}, L_{m,l}]=(nl-mk)L_{n+m,k+l}+\delta_{n+m,0}\delta_{k+l,0}(\alpha nc_{1}+\beta kc_{2}),$$

$$[D_{1}, L_{n,k}]=nL_{n,k}, \quad [D_{2}, L_{n,k}]=kL_{n,k}, \quad [D_{1}, D_{2}]=0,$$

where c_1, c_2 are central elements, and α, $\beta\in\mathbf{R}$.

The above algebra is the 2-dimensional Virasoro-Bott algebra, and we call it *elliptic Virasoro algebra.* We recall an action of the Virasoro algebra on the elliptic Lie algebra ([7]):

$$d_{k}\cdot\alpha(n, m)=-n\alpha(n+k, m),$$

$$d_{k}\cdot X_{n,m}(\alpha)=\left(\frac{k}{2}(\alpha\mid\alpha)-(n+k)\right)X_{n+k,m}(\alpha)$$

where

$$\alpha(n, m)=\alpha\otimes t^{n}s^{m}, \quad X_{n,m}(\alpha)=x_{\alpha}\otimes t^{n}s^{m} \qquad \text{for} \quad n, m\in\mathbf{Z}.$$

Then we can obtain the following representation of Virasoro algebra:

$$[L_{k}, \alpha(n, m)]=-n\alpha(n+k, m),$$

$$[L_{k}, X_{n,m}(\alpha)]=\left(\frac{k}{2}(\alpha\mid\alpha)-(n+k)\right)X_{n+k,m}(\alpha).$$

Next we extend naturally this action to an action of the elliptic Virasoro algebra via the following actions:

$$\alpha_{kl}\cdot\alpha(n, m)=-n\alpha(n+k, m+l), \qquad b_{kl}\cdot\alpha(n, m)=-\mathrm{m}\alpha(n+k, m+l),$$

$$a_{kl}\cdot X_{n,m}(\alpha)=\left(\frac{k}{2}(\alpha\mid\alpha)-(n+k)\right)X_{n+k,m+l}(\alpha),$$

$$b_{kl}\cdot X_{n,m}(\alpha)=\left(\frac{l}{2}(\alpha\mid\alpha)-(m+l)\right)X_{n+k,m+l}(\alpha)\,.$$

From the above we get the following representation.

PROPOSITION 3.3.

$$d_{kl}\cdot\alpha(n,m)=(km-nl)\alpha(n+k,m+l)\,,$$

$$d_{kl}\cdot X_{n,m}(\alpha)=(km-nl)X_{n+k,m+l}(\alpha)\,,$$

$$d_t\cdot X_{n,m}(\alpha)=nX_{n,m}(\alpha)\,,\qquad d_t\cdot\alpha(n,m)=n\alpha(n,m)\,,$$

$$d_s\cdot X_{n,m}(\alpha)=mX_{n,m}(\alpha)\,,\qquad d_s\cdot\alpha(n,m)=m\alpha(n,m)\,.$$

PROOF. It is easily checked by direct calculations. □

We define a representation of the elliptic virasoro algebra on the elliptic Lie algebra as follows:

$$[L_{k,l},\alpha(n,m)]=(km-nl)\alpha(n+k,m+l)\,,$$

$$[L_{k,l},X_{n,m}(\alpha)]=(km-nl)X_{n+k,m+l}(\alpha)\,,$$

$$[D_1,\alpha(n,m)]=n\alpha(n,m)\,,\qquad [D_1,X_{n,m}(\alpha)]=nX_{n,m}(\alpha)\,,$$

$$[D_2,\alpha(n,m)]=m\alpha(n,m)\,,\qquad [D_2,X_{n,m}(\alpha)]=mX_{n,m}(\alpha)\,.$$

Further we set

$$L(z,w)=\sum_{n,k\in\mathbf{Z}}L(n,k)\otimes z^{-n-2}w^{-k-2}\,,$$

$$\alpha(z,w)=\sum_{n,k\in\mathbf{Z}}\alpha(n,k)\otimes z^{-n-1}w^{-k-1}\,,$$

$$X(\alpha,z,w)=\sum_{n,k\in\mathbf{Z}}X_{n,k}(\alpha)\otimes z^{-n-1}w^{-k-1}\,,$$

where $L(n,k)$ is an operator which acts on a vector space V such that $[L(n,k),L(m,l)]=(nl-mk)L(n+m,k+l)+\delta_{n+m,0}\delta_{k+l,0}(\alpha n+\beta k)(\mathrm{rank}\,V)$. By the correspondence with a representation of the elliptic Virasoro algebra, we define a commutation relation such that

$$[L(k,l),\alpha(n,m)]=(km-nl)\alpha(n+k,m+l)\,,$$

$$[L(k,l),X_{n,m}(\alpha)]=(km-nl)X_{n+k,m+l}(\alpha)\,.$$

Then we obtain the following results.

PROPOSITION 3.4.

$$[L(n,k),\alpha(z_2,w_2)]=\left(kz_2^{n+1}w_2^k\frac{d}{dz_2}-nz_2^nw_2^{k+1}\frac{d}{dw_2}+(k-n)z_2^nw_2^k\right)\alpha(z_2,w_2),$$

$$[L(n,k),X(\alpha,z,w)]=z^nw^k\left(kz\frac{d}{dz}-nw\frac{d}{dw}\right)X(\alpha,z,w),$$

$$[L(n,k),L(z_2,w_2)]=\left(kz_2^{n+1}w_2^k\frac{d}{dz_2}-nz_2^nw_2^{k+1}\frac{d}{dw_2}+2(k-n)z_2^nw_2^k\right)L(z_2,w_2)$$
$$+(\alpha nc_1+\beta kc_2)z_2^{n-2}w_2^{k-2},$$

$$[L(z_1,w_1),\alpha(z_2,w_2)]=z_1^{-1}w_1^{-1}\left(w_1^{-1}z_2^{-1}\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2}\right.$$
$$\left.-z_1^{-1}w_2^{-1}\frac{\partial}{\partial w_1}\frac{\partial}{\partial z_2}\right)\alpha(z_1,w_1)\delta(z_1/z_2)\delta(w_1/w_2)z_1w_1,$$

$$[L(z_1,w_1),L(z_2,w_2)]=(z_1z_2w_1w_2)^{-2}\left\{\left(z_1w_2\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2}-w_1z_2\frac{\partial}{\partial w_1}\frac{\partial}{\partial z_2}\right)\right.$$
$$\times L(z_2,w_2)z_2^2w_2^2\delta(z_1/z_2)\delta(w_1/w_2)$$
$$\left.-\left(\alpha z_1\frac{\partial}{\partial z_1}+\beta w_1\frac{\partial}{\partial z_1}\right)\delta(z_1/z_2)\delta(w_1/w_2)\right\},$$

$$[L(z_1,w_1),X(\alpha,z_2,w_2)]=z_1^{-1}w_1^{-1}\left(w_1^{-1}z_2^{-1}\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2}\right.$$
$$\left.-z_1^{-1}w_2^{-1}\frac{\partial}{\partial w_1}\frac{\partial}{\partial z_2}\right)X(\alpha,z_1,w_1)\delta(z_1/z_2)\delta(w_1/w_2)z_1w_1.$$

Proof.

$$[L(n,k),L(z_2,w_2)]=\left[L(n,k),\sum_{m,l\in\mathbf{Z}}L(m,l)z_2^{-m-2}w_2^{-l-2}\right]$$
$$=\sum_{m,l\in\mathbf{Z}}((nl-mk)L(n+m,k+l)+(\alpha nc_1+\beta kc_2)\delta_{n+m,0}\delta_{k+l,0})z_2^{-m-2}w_2^{-l-2}$$
$$=\sum_{m,l\in\mathbf{Z}}(nl-mk)L(m,l)z_2^{-m+n+2}w_2^{-l+k-2}+(\alpha nc_1+\beta kc_2)z_2^{n-2}w_2^{k-2}$$
$$=z_2^{n-2}w_2^{k-2}\left(\sum_{m,l\in\mathbf{Z}}\left(kz_2\frac{\partial}{\partial z_2}-nw_2\frac{\partial}{\partial w_2}\right)L(m,l)z_2^{-m}w_2^{-1}+\alpha nc_1+\beta kc_2\right)$$
$$=z_2^{n-2}w_2^{k-2}\left(\left(kz_2\frac{\partial}{\partial z_2}-nw_2\frac{\partial}{\partial w_2}\right)z_2^2w_2^2\mathrm{L}(z_2,w_2)+\alpha nc_1+\beta kc_2\right)$$
$$=z_2^nw_2^k\left(kz_2\frac{\partial}{\partial z_2}-nw_2\frac{\partial}{\partial w_2}+2(k-n)\right)L(z_2,w_2)+z_2^{n-2}w_2^{k-2}(\alpha nc_1+\beta kc_2)$$

$$[L(z_1, w_1), \alpha(z_2, w_2)] = \sum_{n,k,m,l\in\mathbf{Z}} (nl-mk)\alpha(n+m, k+l) z_1^{-n-2} z_2^{-m-1} w_1^{-k-2} w_2^{-l-1}$$

$$= \sum_{n,k,m,l\in\mathbf{Z}} (nl-mk)\alpha(n, k) z_1^{-n-2} w_1^{-k-2} (z_1/z_2)^m (w_1/w_2)^l z_2^{-1} w_2^{-1}$$

$$= \sum_{n,k,m,l\in\mathbf{Z}} \left(w_1^{-2} z_1^{-1} z_2^{-1} \frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2} - z_1^{-2} w_1^{-1} w_2^{-1} \frac{\partial}{\partial w_1}\frac{\partial}{\partial z_2} \right)$$

$$\times \alpha(n, k) z_1^{-n} w_1^{-k} \delta(z_1/z_2)\delta(w_1/w_2)$$

$$= \left(w_1^{-2} z_1^{-1} z_2^{-1} \frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2} - z_1^{-2} w_1^{-1} w_2^{-1} \frac{\partial}{\partial w_1}\frac{\partial}{\partial z_2} \right)$$

$$\times \alpha(z_1, w_1)\delta(z_1/z_2)\delta(w_1/w_2) z_1 w_1$$

$$= z_1^{-1} w_1^{-1} \left(w_1^{-1} z_2^{-1} \frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2} - z_1^{-1} w_2^{-1} \frac{\partial}{\partial w_1}\frac{\partial}{\partial z_2} \right) \delta(z_1/z_2)\delta(w_1/w_2) z_1 w_1 \,.$$

The others are similarly shown. □

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Present Address:
Department of Mathematics, School of Science and Engineering, Waseda University,
Okubo, Shinjuku-ku, Tokyo, 169–50 Japan.