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On Certain Multiple Series with Functional Equation in a Totally Real Number Field II

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1. Introduction.

This is a continuation of the preceding paper [3] with the same title. We shall treat the series with "fractional powers" (see (1.1) below), which is a generalization of the series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n}e^{-2\pi m^k n\tau}\qquad(\tau>0)\,.$$

Let K be a totally real number field of degree n and $K^{(q)}$ $(q=1,\cdots,n)$ be the conjugates of K. Let μ be a number of K. We denote by $\mu^{(q)}$ the conjugates of μ in $K^{(q)}$ $(q=1,\cdots,n)$ and define n-dimensional vector $\mu=(\mu^{(1)},\cdots,\mu^{(n)})$ correspondingly. More generally, we consider n-dimensional complex vectors $\xi=(\xi_1,\cdots,\xi_n)$. For such ξ we put

$$S(\xi)=\sum_{q=1}^{n}\xi_q\,,\qquad N(\xi)=\prod_{q=1}^{n}\xi_q\,.$$

Let U be the unit group of K and $U^k=\{\varepsilon^k\mid\varepsilon\in U\}$ the subgroup of U consisting of the k-th powers of units. Let α, β be the numbers of K. If α/β is an element of U^k, we say that α and β are associated with respect to U^k. Let k and l be coprime positive integers and $\tau_1,\cdots,\tau_n$ be non-zero complex numbers such that

$$|\arg\tau_q|<\frac{\pi}{2k}\qquad(q=1,\cdots,n)\,.$$

Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero fractional ideals of K.

For such numbers $\tau_1,\cdots,\tau_n,k,l$ and ideals $\mathfrak{a}$, $\mathfrak{b}$, we shall define the series as follows:

$$(1.1)\qquad M(\tau;\mathfrak{a},\mathfrak{b};k,l)=M(\tau_1,\cdots,\tau_n;\mathfrak{a},\mathfrak{b};k,l)$$
$$=\sum_{\substack{\mu/U^k\\0\neq\mu\in\mathfrak{a}}}\frac{1}{|N(\mu)|}\sum_{0\neq\nu\in\mathfrak{b}}\exp\{-2\pi S(|\nu|^{k/l}|\mu|\tau^k)\}\,,$$

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where the outer sum is taken over all non-zero numbers of $\mathfrak{a}$, not associated to each other with respect to U^k, and the inner sum is taken over all non-zero numbers of $\mathfrak{b}$. This sum is well-defined. If $\mu_1 \in \mathfrak{a}$ is associated to μ_2 with respect to U^k, then $\mu_1 = \varepsilon^k \mu_2$, where ε is a suitable unit of K. Hence

$$|N(\mu_1)| = |N(\mu_2)|$$

and the inner sum of (1.1) for $\mu = \mu_1$ is equal to that for $\mu = \mu_2$; in fact, we see that

$$\sum_{0 \neq \nu \in \mathfrak{b}} \exp\{-2\pi S(|\nu|^{k/l}|\mu_1|\tau^k)\} = \sum_{0 \neq \nu \in \mathfrak{b}} \exp\{-2\pi S(|\nu\varepsilon^l|^{k/l}|\mu_2|\tau^k)\}$$

$$= \sum_{0 \neq \nu \in \mathfrak{b}} \exp\{-2\pi S(|\nu|^{k/l}|\mu_2|\tau^k)\},$$

which shows that (1.1) is well-defined.

We put

$$\omega_{k,l}(j) = e^{i\pi(l-1-2j)/2kl} \qquad (j = 0, \cdots, l-1)$$

and define the sum, for τ_q's such that $|\arg \tau_q| < \pi/2kl$ $(q = 1, \cdots, n)$,

$$(1.2) \qquad T(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = \sum_{j_1=0}^{l-1} \cdots \sum_{j_n=0}^{l-1} M(\tau_1\omega_{k,l}(j_1), \cdots, \tau_n\omega_{k,l}(j_n); \mathfrak{a}, \mathfrak{b}; k, l).$$

Further we introduce the series

$$\zeta(s, \mathfrak{a}) = \sum_{0 \neq (\mu) \subset \mathfrak{a}} \frac{1}{|N(\mu)|^s} \qquad (s = \sigma + it, \sigma > 1),$$

where the sum is taken over all non-zero principal ideals contained in $\mathfrak{a}$. (See [3, p. 50] or Lemma 2.1 below.)

Let $\mathfrak{d}$ be the different ideal of K, $D = N(\mathfrak{d})$ be the absolute value of the discriminant of K and R be the regulator of K. For any non-zero ideal $\mathfrak{a}$ we denote by $\mathfrak{a}^*$ the ideal $(\mathfrak{a}\mathfrak{d})^{-1}$.

The purpose of this paper is to prove the following

THEOREM. *Suppose that*

$$|\arg \tau_q| < \frac{\pi}{2kl} \qquad (q = 1, \cdots, n).$$

If we put

$$(1.3) \qquad \Phi(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = d_l T(\tau; \mathfrak{a}, \mathfrak{b}; k, l)$$

$$+ 2^n d_k d_l \frac{k^n \Gamma(k/l)^n \zeta(1 + k/l, \mathfrak{b}^*)}{l N(\mathfrak{b}) \sqrt{D} (2\pi)^{nk/l}} \left(\frac{\sin(\pi k/2l)}{\sin(\pi/2l)}\right)^n (\tau_1, \cdots, \tau_n)^k$$

$$-2^{n-2}d_k d_l \frac{k^n l^n R}{N(\mathfrak{a})\sqrt{D}} \log(\tau_1 \cdots \tau_n) + 2^n d_k d_l \frac{k^n l^n}{kn! N(\mathfrak{a})\sqrt{D}} \zeta^{(n)}(0, \mathfrak{a}^*),$$

where $d_k = (2, k)$ *and* $d_l = (2, l)$, *then we have*

$$N(\mathfrak{a}\mathfrak{b})^{1/2} \Phi(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = N(\mathfrak{a}^*\mathfrak{b}^*)^{1/2} \Phi(\tau^{-1}; \mathfrak{b}^*, \mathfrak{a}^*; l, k).$$

The method of the proof is similar to that of [3]. First, in §2, we shall summarize some properties of the zeta functions with Grössencharacters. Then, applying the transformation formula of Hecke-Rademacher, we shall be able to represent $T(\tau; \mathfrak{a}, \mathfrak{b}; k, l)$ as a series of complex integrals:

$$T(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = \sum_{\lambda} \frac{1}{2\pi i} \int_{(l+1/2)} H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) ds.$$

(See (3.13)). In §4, we obtain the functional equation satisfied by $H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)$ and after that, we calculate the residues of $H_1(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)$ and prove our Theorem. In the last section, we observe the case where K is the rational number field. In this case, we can get a functional equation in the form of an infinite product.

2. Zeta functions with Grössencharacters.

Let λ be a Grössencharacter and $\zeta(s, \lambda; \mathfrak{a})$ the series as follows;

$$\zeta(s, \lambda; \mathfrak{a}) = \sum_{0 \neq (\mu) \subset \mathfrak{a}} \frac{\lambda(\mu)}{|N(\mu)|^s} \qquad (s = \sigma + it, \sigma > 1),$$

where

$$\lambda(\mu) = \prod_{q=1}^{n} |\mu^{(q)}|^{-iv_q},$$

and the v_q are linear forms of the rational integers $m_1, \cdots, m_{n-1}$;

$$v_q = 2\pi \sum_{j=1}^{n-1} e_q^{(j)} m_j \qquad (q = 1, \cdots, n). \tag{2.1}$$

We assume that the coefficients $\{e_q^{(j)}\}$ are determined by the fundamental units $\varepsilon_1, \cdots, \varepsilon_{n-1}$ of K, that is, $e_q^{(j)}$ $(q = 1, \cdots, n; j = 1, \cdots, n-1)$ are the numbers satisfying the following conditions:

$$\sum_{q=1}^{n} e_q^{(j)} = 0 \qquad (j = 1, \cdots, n-1),$$

$$\sum_{q=1}^{n} e_q^{(j)} \log|\varepsilon_k^{(q)}| = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad (j, k = 1, \cdots, n-1).$$

(See [3, p. 51]). Here we note that

$$\sum_{q=1}^{n} v_q = 0 .$$

LEMMA 2.1. (1) *$\zeta(s, \lambda; \mathfrak{a})$ has an analytic continuation over the s-plane and satisfies the functional equation as follows;*

$$\Gamma(s; \lambda)\zeta(s, \lambda; \mathfrak{a}) = \frac{\pi^{n(s-1/2)}}{N(\mathfrak{a})\sqrt{D}} \Gamma(1-s; \bar{\lambda})\zeta(1-s, \bar{\lambda}; \mathfrak{a}^*) , \tag{2.2}$$

where

$$\Gamma(s; \lambda) = \prod_{q=1}^{n} \Gamma\left(\frac{s+iv_q}{2}\right) . \tag{2.3}$$

(2) *If $\lambda \neq 1$, then*

$$\Gamma(s; \lambda)\zeta(s, \lambda; \mathfrak{a})$$

is an entire function.

(3) *If $\lambda = 1$, then $\zeta(s, 1; \mathfrak{a}) = \zeta(s, \mathfrak{a})$ and*

$$\Gamma(s/2)^n \zeta(s, \mathfrak{a})$$

is a meromorphic function with only two simple poles at $s=0$ and 1.

(4) *$\zeta(s, \mathfrak{a})$ is regular in the whole s-plane except at $s=1$, where $\zeta(s, \mathfrak{a})$ has a simple pole with the residue $2^{n-1}R/N(\mathfrak{a})\sqrt{D}$.*

PROOF. ([3, Lemma 2.2].) □

LEMMA 2.2. (1) *$\zeta(s, \mathfrak{a})$ has a zero point of order $n-1$ at $s=0$ and we have*

$$\zeta^{(n-1)}(0, \mathfrak{a}) = -(n-1)! \frac{R}{2} .$$

(2) *If we expand $\zeta(1+s, \mathfrak{a})$ for small $|s|$ as follows;*

$$\zeta(1+s, \mathfrak{a}) = \frac{2^{n-1}R}{N(\mathfrak{a})\sqrt{D}} \frac{1}{s} + c(\mathfrak{a}) + O(|s|) ,$$

then

$$c(\mathfrak{a}) = \frac{2^n}{N(\mathfrak{a})\sqrt{D}} \left\{ \frac{nR}{2} (\log(2\pi) + \gamma) + \frac{1}{n!} \zeta^{(n)}(0, \mathfrak{a}^*) \right\} ,$$

where γ is Euler's constant.

PROOF. ([3, Lemma 2.3].) □

LEMMA 2.3. *In the strip $-1/2 \leq \sigma \leq 3/2 + \max(l/k, k/l)$, we have*

(2.4) $$\zeta(s, \lambda; \mathfrak{a})(s-1)^{e(\lambda)} \ll (1+|t|)^{2n},$$

where

$$e(\lambda)=\begin{cases} 1 & \text{if } \lambda=1, \\ 0 & \text{if } \lambda\neq 1 \end{cases}$$

and the constants implied in this estimation (2.4) *depend on* λ *and* $\mathfrak{a}$.

PROOF. If $-1/2 \leq \sigma \leq 3/2$, then (2.4) holds ([4, Hilfssatz 15] or [3]). If $3/2 \leq \sigma \leq 3/2+\max(l/k, k/l)$, then $\zeta(s, \lambda; \mathfrak{a})=O(1)$. Thus the lemma is proved. □

3. Representation by integrals.

Let $\varepsilon_1, \cdots, \varepsilon_{n-1}$ be the fundamental units of K stated above. The elements ε of U^l are of the form

$$\varepsilon=(-1)^{bl}(\varepsilon_1^{a_1}\cdots\varepsilon_{n-1}^{a_{n-1}})^l,$$

where $b=0$ or 1 and $a_1, \cdots, a_{n-1}$ are rational integers. Hence the group index $[U: U^l]$ is $d_l l^{n-1}$ and we can write the inner sum of (1.1) as follows;

(3.1) $$\sum_{0\neq\nu\in\mathfrak{b}} \exp\{-2\pi S(|\nu|^{k/l}|\mu|\tau^k)\} = \sum_{\substack{\nu/U^l \\ 0\neq\nu\in\mathfrak{b}}} \sum_{\varepsilon\in U^l} \exp\{-2\pi S(|\nu\varepsilon|^{k/l}|\mu|\tau^k)\}$$

$$=\frac{2}{d_l}\sum_{\substack{\nu/U^l \\ 0\neq\nu\in\mathfrak{b}}} \sum_{a_1,\cdots,a_{n-1}=-\infty}^{\infty} \exp\{-2\pi S(|\nu|^{k/l}|\mu\varepsilon_1^{ka_1}\cdots\varepsilon_{n-1}^{ka_{n-1}}|\tau^k)\},$$

where the outer sum is taken over all non-zero numbers of $\mathfrak{b}$ not associated to each other with respect to U^l and $a_1, \cdots, a_{n-1}$ run through all rational integers.

Now we quote the transformation formula of Hecke-Rademacher [4]:

LEMMA 3.1. *Let* $W_1, \cdots, W_{n-1}$ *be complex numbers with positive real parts. Then we have*

(3.2) $$\sum_{a_1,\cdots,a_{n-1}=-\infty}^{\infty} \exp\{-2\pi S(W|\varepsilon_1^{ka_1}\cdots\varepsilon_{n-1}^{ka_{n-1}}|)\}$$

$$=\frac{1}{k^n R}\sum_{m_1,\cdots,m_{n-1}=-\infty}^{\infty} \frac{1}{2\pi i}\int_{(l+1/2)} \prod_{q=1}^{n} \frac{\Gamma\left(\dfrac{s+iv_q}{k}\right)}{(2\pi W_q)^{(s+iv_q)/k}}\, ds,$$

where $m_1, \cdots, m_{n-1}$ *run through all rational integers, the* v_q *are the values in* (2.1) *and the integrals in* (3.2) *are the complex integrals taken along the vertical line* $\sigma=l+1/2$.

PROOF. ([4, Hilfssatz 14].) □

Applying this Lemma with $W_q = |\nu^{(q)}|^{k/l} |\mu^{(q)}| \tau_q^k$ $(q=1, \cdots, n)$ to the sum in the right-hand side of (3.1) and putting the result into (1.1), we have

$$(3.3) \quad M(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = \frac{2}{d_l k^n R} \sum_{\substack{\mu/U^k \\ 0 \neq \mu \in \mathfrak{a}}} \frac{1}{|N(\mu)|} \sum_{\substack{\nu/U^l \\ 0 \neq \nu \in \mathfrak{b}}} \sum_{m_1, \cdots, m_{n-1} = -\infty}^{\infty} \frac{1}{2\pi i} \int_{(l+1/2)} \times \prod_{q=1}^{n} \frac{\Gamma\left(\dfrac{s+iv_q}{k}\right)}{(2\pi \tau_q^k)^{(s+iv_q)/k}} \prod_{q=1}^{n} |\nu^{(q)}|^{-iv_q/l} \prod_{q=1}^{n} |\mu^{(q)}|^{-iv_q/k} \times |N(\nu)|^{-s/l} |N(\mu)|^{-s/k} ds .$$

In these integrands, we have, by the well-known estimation of the gamma function,

$$(3.4) \quad \frac{\Gamma\left(\dfrac{s+iv_q}{k}\right)}{(2\pi \tau_q^k)^{(s+iv_q)/k}} \ll (1+|t+v_q|)^{\sigma/k-1/2} \exp(-2\theta |t+v_q|) ,$$

where

$$2\theta = \min_{1 \leq q \leq n} \left(\frac{\pi}{2k} - |\arg \tau_q| \right) .$$

Hence we have the estimation of the right-hand side of (3.3);

$$M(\tau; \mathfrak{a}, \mathfrak{b}; k, l) \ll \sum_{\substack{\mu/U^k \\ 0 \neq \mu \in \mathfrak{a}}} \sum_{\substack{\nu/U^l \\ 0 \neq \nu \in \mathfrak{b}}} \frac{1}{|N(\mu)|^{1+l/k+1/2k}} \frac{1}{|N(\nu)|^{1+1/2l}} \times \sum_{m_1, \cdots, m_{n-1} = -\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\theta \sum_{q=1}^{n} |t+v_q| \right) dt .$$

Since this last sum over $m_1, \cdots, m_{n-1}$ is convergent ([2, p. 206]), we see that the series in the right-hand side of (3.3) is absolutely convergent. Therefore we can change, in (3.3), the order of the summation over $\{\nu\}$, $\{\mu\}$ and $m_1, \cdots, m_{n-1}$ and, moreover, we can invert the order of the summation over $\{\nu\}$, $\{\mu\}$ and the integration. Thus we have

$$(3.5) \quad M(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = \frac{2}{d_l k^n R} \sum_{m_1, \cdots, m_{n-1} = -\infty}^{\infty} \frac{1}{2\pi i} \int_{(l+1/2)} \prod_{q=1}^{n} \frac{\Gamma\left(\dfrac{s+iv_q}{k}\right)}{(2\pi \tau_q^k)^{(s+iv_q)/k}} \times \sum_{\substack{\mu/U^k \\ 0 \neq \mu \in \mathfrak{a}}} \frac{1}{|N(\mu)|^{1+s/k}} \prod_{q=1}^{n} |\mu^{(q)}|^{-iv_q/k} \times \sum_{\substack{\nu/U^l \\ 0 \neq \nu \in \mathfrak{b}}} \frac{1}{|N(\nu)|^{s/l}} \prod_{q=1}^{n} |\nu^{(q)}|^{-iv_q/l} ds .$$

Now we consider the last series over ν in this integrand;

$$(3.6)\qquad \sum_{\substack{\nu/U^l\\ 0\neq\nu\in\mathfrak{b}}} \frac{1}{|N(\nu)|^{s/l}} \prod_{q=1}^{n} |\nu^{(q)}|^{-iv_q/l}$$

$$= \sum_{0\neq(\nu)\subset\mathfrak{b}} \frac{1}{|N(\nu)|^{s/l}} \prod_{q=1}^{n} |\nu^{(q)}|^{-iv_q/l} \sum_{\varepsilon} \prod_{q=1}^{n} |\varepsilon^{(q)}|^{-iv_q/l},$$

where ε runs through the complete system of the representatives of U/U^l. Further we see that

$$(3.7)\qquad \sum_{\varepsilon} \prod_{q=1}^{n} |\varepsilon^{(q)}|^{-iv_q/l} = d_l \sum_{a_1,\cdots,a_{n-1}=0}^{l-1} \prod_{q=1}^{n} |\varepsilon_1^{(q)a_1}\cdots\varepsilon_{n-1}^{(q)a_{n-1}}|^{-iv_q/l}.$$

From (2.1) and the definition of $\{e_q^{(j)}\}$, it follows that

$$\prod_{q=1}^{n} |\varepsilon_1^{(q)a_1}\cdots\varepsilon_{n-1}^{(q)a_{n-1}}|^{-iv_q/l}$$

$$= \exp\left(-\frac{i}{l}\sum_{j=1}^{n-1}\sum_{q=1}^{n} a_j v_q \log|\varepsilon_j^{(q)}|\right) = \exp\left(-\frac{2\pi i}{l}\sum_{j=1}^{n-1} a_j m_j\right).$$

Hence

$$(3.8)\qquad \sum_{a_1,\cdots,a_{n-1}=0}^{l-1} \prod_{q=1}^{n} |\varepsilon_1^{(q)a_1}\cdots\varepsilon_{n-1}^{(q)a_{n-1}}|^{-iv_q/l}$$

$$= \sum_{a_1,\cdots,a_{n-1}=0}^{l-1} \exp\left(-\frac{2\pi i}{l}\sum_{j=1}^{n-1} a_j m_j\right) = \begin{cases} l^{n-1} & \text{if } l\mid m_1,\cdots,l\mid m_{n-1}, \\ 0 & \text{otherwise}. \end{cases}$$

Consequently we have, by (3.6), (3.7), (3.8) and the definition of the zeta functions,

$$(3.9)\qquad \sum_{\substack{\nu/U^l\\ 0\neq\nu\in\mathfrak{b}}} \frac{1}{|N(\nu)|^{s/l}} \prod_{q=1}^{n} |\nu^{(q)}|^{-iv_q/l}$$

$$= \begin{cases} d_l l^{n-1}\zeta(s/l, \lambda^{1/l}; \mathfrak{b}) & \text{if } l\mid m_1,\cdots,l\mid m_{n-1}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\lambda(\nu)=\prod_{q=1}^{n}|\nu^{(q)}|^{-iv_q}$ and $\lambda^{1/l}$ becomes a Grössencharacter.

Similarly we have

$$(3.10)\qquad \sum_{\substack{\mu/U^k\\ 0\neq\mu\in\mathfrak{a}}} \frac{1}{|N(\mu)|^{1+s/k}} \prod_{q=1}^{n} |\mu^{(q)}|^{-iv_q/k}$$

$$= \begin{cases} d_k k^{n-1}\zeta(1+s/k, \lambda^{1/k}; \mathfrak{a}) & \text{if } k\mid m_1,\cdots,k\mid m_{n-1}, \\ 0 & \text{otherwise} \end{cases}$$

and $\lambda^{1/k}$ is also a Grössencharacter.

Since $(k, l)=1$, (3.9) and (3.10) show that

$$(3.11)\qquad \sum_{\substack{\mu/U^k\\ 0\neq\mu\in\mathfrak{a}}} \frac{1}{|N(\mu)|^{1+s/k}} \prod_{q=1}^{n} |\mu^{(q)}|^{-iv_q/k} \sum_{\substack{\nu/U^l\\ 0\neq\nu\in\mathfrak{b}}} \frac{1}{|N(\nu)|^{s/l}} \prod_{q=1}^{n} |\nu^{(q)}|^{-iv_q/l}$$
$$=\begin{cases} d_l l^{n-1} d_k k^{n-1} \zeta(s/l, \lambda^{1/l}; \mathfrak{b})\zeta(1+s/k, \lambda^{1/k}; \mathfrak{a}) & \text{if } kl \mid m_1, \cdots, kl \mid m_{n-1}, \\ 0 & \text{otherwise}. \end{cases}$$

Putting (3.11) into (3.5) and replacing $m_1, \cdots, m_{n-1}$ by $klm_1, \cdots, klm_{n-1}$, we have finally

$$(3.12)\qquad M(\tau; \mathfrak{a}, \mathfrak{b}; k, l)$$
$$=2d_k \frac{l^{n-1}}{kR} \sum_{m_1,\cdots,m_{n-1}=-\infty}^{\infty} \frac{1}{2\pi i} \int_{(l+1/2)} \prod_{q=1}^{n} \frac{\Gamma\left(\dfrac{s+iklv_q}{k}\right)}{(2\pi\tau_q^k)^{(s+iklv_q)/k}}$$
$$\times \zeta(s/l, \lambda^k; \mathfrak{b})\zeta(1+s/k, \lambda^l; \mathfrak{a})ds,$$

where the λ are the Grössencharacters. The sum of the right-hand side of (3.12) over rational integers $m_1, \cdots, m_{n-1}$ can be regarded as the sum Σ_λ over all Grössencharacters λ, so we write

$$M(\tau; \mathfrak{a}, \mathfrak{b}; k, l)=2d_k \frac{l^{n-1}}{kR} \sum_{\lambda} \frac{1}{2\pi i} \int_{(l+1/2)} \prod_{q=1}^{n} \frac{\Gamma(z_q/k)}{(2\pi\tau_q^k)^{z_q/k}}$$
$$\times \zeta(s/l, \lambda^k; \mathfrak{b})\zeta(1+s/k, \lambda^l; \mathfrak{a})ds,$$

where we put $z_q=s+iklv_q$ $(q=1, \cdots, n)$. Moreover, we have, by the definition (1.2),

$$T(\tau; \mathfrak{a}, \mathfrak{b}; k, l)=2d_k \frac{l^{n-1}}{kR} \sum_{\lambda} \frac{1}{2\pi i} \int_{(l+1/2)} \prod_{q=1}^{n} \frac{\Gamma(z_q/k)}{(2\pi\tau_q^k)^{z_q/k}}$$
$$\times \sum_{j_1=0}^{l-1} \cdots \sum_{j_n=0}^{l-1} \prod_{q=1}^{n} (e^{i\pi(l-1-2j_q)/2kl})^{-z_q}$$
$$\times \zeta(s/l, \lambda^k; \mathfrak{b})\zeta(1+s/k, \lambda^l; \mathfrak{a})ds$$
$$=2d_k \frac{l^{n-1}}{kR} \sum_{\lambda} \frac{1}{2\pi i} \int_{(l+1/2)} \prod_{q=1}^{n} \frac{\Gamma(z_q/k)}{(2\pi\tau_q^k)^{z_q/k}}$$
$$\times \prod_{q=1}^{n} \sum_{j=0}^{l-1} e^{i\pi(l-1-2j)z_q/2kl}$$
$$\times \zeta(s/l, \lambda^k; \mathfrak{b})\zeta(1+s/k, \lambda^l; \mathfrak{a})ds$$

$$=2d_k \frac{l^{n-1}}{kR} \sum_\lambda \frac{1}{2\pi i} \int_{(l+1/2)} \prod_{q=1}^{n} \frac{\Gamma(z_q/k)}{(2\pi\tau_q^k)^{z_q/k}} \frac{\sin(\pi z_q/2k)}{\sin(\pi z_q/2kl)}$$
$$\times \zeta(s/l, \lambda^k; \mathfrak{b})\zeta(1+s/k, \lambda^l; \mathfrak{a})ds .$$

We write this last series as follows;

$$T(\tau; \mathfrak{a}, \mathfrak{b}; k, l)=\sum_\lambda \frac{1}{2\pi i} \int_{(l+1/2)} H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)ds , \tag{3.13}$$

where

$$H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)=2d_k \frac{l^{n-1}}{kR} \prod_{q=1}^{n} \frac{\Gamma(z_q/k)}{(2\pi\tau_q^k)^{z_q/k}} \frac{\sin(\pi z_q/2k)}{\sin(\pi z_q/2kl)} \tag{3.14}$$
$$\times \zeta(s/l, \lambda^k; \mathfrak{b})\zeta(1+s/k, \lambda^l; \mathfrak{a}) .$$

In particular we have, for $\lambda=1$,

$$H_1(s, \tau; \mathfrak{a}; \mathfrak{b}; k, l)=2d_k \frac{l^{n-1}\Gamma(s/k)^n}{kR((2\pi)^{n/k}\tau_1 \cdots \tau_n)^s} \left(\frac{\sin(\pi s/2k)}{\sin(\pi s/2kl)}\right)^n \zeta(s/l, \mathfrak{b})\zeta(1+s/k, \mathfrak{a}) . \tag{3.15}$$

4. Properties of integrands.

Now we shall prove the following three lemmas on the properties of $H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)$.

LEMMA 4.1. (1) *If* $\lambda \neq 1$, *then* $H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)$ *is an entire function.*

(2) $H_1(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)$ *is regular in the whole s-plane except at most two simple poles at* $s=l$ *and* $s=-k$ *and a double pole at* $s=0$.

PROOF. Using (2.3), we write (3.14) in the following form;

$$H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)$$
$$=2d_k \frac{l^{n-1}}{kR} \prod_{q=1}^{n} \left\{ \frac{\Gamma(z_q/k)}{(2\pi\tau_q^k)^{z_q/k}} \frac{1}{\Gamma(z_q/2l)\Gamma(1/2+z_q/2k)} \frac{\sin(\pi z_q/2k)}{\sin(\pi z_q/2kl)} \right\}$$
$$\times \Gamma(s/l; \lambda^k)\zeta(s/l, \lambda^k; \mathfrak{b})\Gamma(1+s/k; \lambda^l)\zeta(1+s/k, \lambda^l; \mathfrak{a}) .$$

Since

$$\frac{\Gamma(s)}{\Gamma(1/2+s/2)} \sin(\pi s/2)=\frac{2^{s-1}}{\pi^{1/2}} \Gamma(s/2)\sin(\pi s/2)=\frac{2^{s-1}\pi^{1/2}}{\Gamma(1-s/2)} ,$$

we have

$$(4.1)\qquad H_\lambda(s,\tau;\mathfrak{a},\mathfrak{b};k,l)$$
$$=2d_k\frac{l^{n-1}}{kR}\prod_{q=1}^{n}\left\{\frac{\sqrt{\pi}}{2(\pi\tau_q^k)^{z_q/k}}\frac{1}{\Gamma(z_q/2l)\Gamma(1-z_q/2k)\sin(\pi z_q/2kl)}\right\}$$
$$\times\Gamma(s/l;\lambda^k)\zeta(s/l,\lambda^k;\mathfrak{b})\Gamma(1+s/k;\lambda^l)\zeta(1+s/k,\lambda^l;\mathfrak{a}).$$

Since the product

$$\Gamma(z/2l)\Gamma(1-z/2k)\sin(\pi z/2kl)$$

is a meromorphic function of z having no zero points, the right-hand side of (4.1) is the product of a certain entire function and of

$$\Gamma(s/l;\lambda^k)\zeta(s/l,\lambda^k;\mathfrak{b})\Gamma(1+s/k;\lambda^l)\zeta(1+s/k,\lambda^l;\mathfrak{a}).$$

From this fact and Lemma 2.1, the proof follows at once. □

LEMMA 4.2. *$H_\lambda(s,\tau;\mathfrak{a},\mathfrak{b};k,l)$ satisfies the functional equation as follows;*

$$H_\lambda(s,\tau;\mathfrak{a},\mathfrak{b};k,l)=\frac{d_k}{d_l}(DN(\mathfrak{a}\mathfrak{b}))^{-1}H_{\bar\lambda}(-s,\tau^{-1};\mathfrak{b}^*,\mathfrak{a}^*;l,k).$$

PROOF. We apply the functional equation (2.2) to (4.1). Then we have

$$H_\lambda(s,\tau;\mathfrak{a},\mathfrak{b};k,l)$$
$$=\frac{2d_kl^{n-1}\pi^{ns(1/l+1/k)}}{kRDN(\mathfrak{a}\mathfrak{b})}\prod_{q=1}^{n}\left\{\frac{\sqrt{\pi}}{2(\pi\tau_q^k)^{z_q/k}}\frac{1}{\Gamma(z_q/2l)\Gamma(1-z_q/2k)\sin(\pi z_q/2kl)}\right\}$$
$$\times\Gamma(1-s/l;\bar\lambda^k)\zeta(1-s/l,\bar\lambda^k;\mathfrak{b}^*)\Gamma(-s/k;\bar\lambda^l)\zeta(-s/k,\bar\lambda^l;\mathfrak{a}^*).$$

In the factors of this right-hand side, we use the following substitution

$$\Gamma(z/2l)\Gamma(1-z/2k)=-\frac{l}{k}\Gamma(1+z/2l)\Gamma(-z/2k)$$

and we obtain

$$(4.2)\qquad H_\lambda(s,\tau;\mathfrak{a},\mathfrak{b};k,l)$$
$$=\frac{2d_kk^{n-1}}{lRDN(\mathfrak{a}\mathfrak{b})}\prod_{q=1}^{n}\left\{\frac{\sqrt{\pi}}{2(\pi\tau_q^{-l})^{-z_q/l}}\frac{1}{\Gamma(-z_q/2k)\Gamma(1+z_q/2l)}\frac{-1}{\sin(\pi z_q/2kl)}\right\}$$
$$\times\Gamma(-s/k;\bar\lambda^l)\zeta(-s/k,\bar\lambda^l;\mathfrak{a}^*)\Gamma(1-s/l;\bar\lambda^k)\zeta(1-s/l,\bar\lambda^k;\mathfrak{b}^*).$$

Comparing (4.2) with (4.1), we see that the right-hand side of (4.2) is equal to

$$\frac{d_k}{d_l}(DN(\mathfrak{a}\mathfrak{b}))^{-1}H_{\bar\lambda}(-s,\tau^{-1};\mathfrak{b}^*,\mathfrak{a}^*;l,k).$$ □

LEMMA 4.3. *For $-k-1/2 \leq \sigma \leq l+1/2$, we have*

$$H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) \ll \exp(-\eta |t|),$$

where

$$\eta = \frac{1}{2} \min_{1 \leq q \leq n} \left(\frac{\pi}{2kl} - |\arg \tau_q| \right).$$

The constants implied in this estimate depend on λ, τ, $\mathfrak{a}$, $\mathfrak{b}$, k and l.

PROOF. In view of Lemma 4.2, it is sufficient to prove Lemma under the assumption $0 \leq \sigma \leq l+1/2$. From (3.4), (3.14) and Lemma 2.3, it follows that

$$\begin{aligned} &H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) \\ &\quad \ll (1+|t|)^{4n} \prod_{q=1}^{n} \left[(1+|t+v_q|)^{1/2-\sigma/k} \exp\left\{ -\left(\frac{\pi}{2kl} - |\arg \tau_q| \right) |t+v_q| \right\} \right] \\ &\quad \ll \exp(-\eta |t|). \end{aligned}$$

□

5. Functional equation and residues.

By Lemma 4.3 we see that

$$\int_{l+1/2+iT}^{-k-1/2+iT} H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) ds \to 0 \quad (|T| \to \infty),$$

where the integral is taken along the horizontal line from $l+1/2+iT$ to $-k-1/2+iT$. Therefore by Lemma 4.1 and Cauchy's formula,

$$(5.1) \quad \frac{1}{2\pi i} \int_{(l+1/2)} H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) ds$$
$$= \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{(-k-1/2)} H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) ds, & \text{if } \lambda \neq 1, \\ \dfrac{1}{2\pi i} \displaystyle\int_{(-k-1/2)} H_1(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) ds + R(\tau; \mathfrak{a}, \mathfrak{b}; k, l) & \text{if } \lambda = 1, \end{cases}$$

where

$$R(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = \operatorname*{Res}_{s=l} H_1 + \operatorname*{Res}_{s=0} H_1 + \operatorname*{Res}_{s=-k} H_1$$

is the sum of the residues of $H_1(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)$. Hence we have, by (5.1) and (3.13),

$$(5.2) \quad T(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = \sum_\lambda \frac{1}{2\pi i} \int_{(-k-1/2)} H_\lambda(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l) ds + R(\tau; \mathfrak{a}, \mathfrak{b}; k, l).$$

By Lemma 4.2,

$$\sum_{\lambda}\frac{1}{2\pi i}\int_{(-k-1/2)} H_{\lambda}(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)ds$$
$$=\frac{d_k}{d_l}(DN(\mathfrak{ab}))^{-1}\sum_{\lambda}\frac{1}{2\pi i}\int_{(-k-1/2)} H_{\bar{\lambda}}(-s, \tau^{-1}; \mathfrak{b}^*, \mathfrak{a}^*; l, k)ds$$
$$=\frac{d_k}{d_l}(DN(\mathfrak{ab}))^{-1}\sum_{\lambda}\frac{1}{2\pi i}\int_{(k+1/2)} H_{\lambda}(s, \tau^{-1}; \mathfrak{b}^*, \mathfrak{a}^*; l, k)ds\,.$$

Thus we have, by (5.2),

$$\text{(5.3)}\qquad T(\tau; \mathfrak{a}, \mathfrak{b}; k, l)$$
$$=\frac{d_k}{d_l}(DN(\mathfrak{ab}))^{-1}\sum_{\lambda}\frac{1}{2\pi i}\int_{(k+1/2)} H_{\lambda}(s, \tau^{-1}; \mathfrak{b}^*, \mathfrak{a}^*; l, k)ds + R(\tau; \mathfrak{a}, \mathfrak{b}; k, l)$$
$$=\frac{d_k}{d_l}(DN(\mathfrak{ab}))^{-1}T(\tau^{-1}; \mathfrak{b}^*, \mathfrak{a}^*; l, k) + R(\tau; \mathfrak{a}, \mathfrak{b}; k, l)\,.$$

Now we shall calculate $R(\tau; \mathfrak{a}, \mathfrak{b}; k, l)$. First we easily obtain from (3.15)

$$\text{(5.4)}\qquad \operatorname*{Res}_{s=l} H_1 = \frac{2d_k l^{n-1}\Gamma(l/k)^n}{kR((2\pi)^n\tau_1^k\cdots\tau_n^k)^{l/k}}\zeta(1+l/k, \mathfrak{a})\left(\frac{\sin(\pi l/2k)}{\sin(\pi/2k)}\right)^n \operatorname*{Res}_{s=l}\zeta(s/l, \mathfrak{b})$$
$$=\frac{2^n d_k l^n\Gamma(l/k)^n}{kN(\mathfrak{b})\sqrt{D}\,(2\pi)^{nl/k}}\left(\frac{\sin(\pi l/2k)}{\sin(\pi/2k)}\right)^n\zeta(1+l/k, \mathfrak{a})(\tau_1\cdots\tau_n)^{-1}\,.$$

As for $\operatorname{Res}_{s=-k}H_1$, it follows from Lemma 4.2 and (5.4) that

$$\text{(5.5)}\qquad \operatorname*{Res}_{s=-k} H_1 = \lim_{s\to -k}\{(s+k)H_1(s, \tau; \mathfrak{a}, \mathfrak{b}; k, l)\}$$
$$=-\frac{d_k}{d_l}(DN(\mathfrak{ab}))^{-1}\lim_{s\to k}\{(s-k)H_1(s, \tau^{-1}; \mathfrak{b}^*, \mathfrak{a}^*; l, k)\}$$
$$=-\frac{2^n d_k k^n\Gamma(k/l)^n}{lN(\mathfrak{b})\sqrt{D}\,(2\pi)^{nk/l}}\left(\frac{\sin(\pi k/2l)}{\sin(\pi/2l)}\right)^n\zeta(1+k/l, \mathfrak{b}^*)(\tau_1\cdots\tau_n)^k\,.$$

To calculate $\operatorname{Res}_{s=0}H_1$, we expand the functions involved in (3.15) as follows;

$$\Gamma(s/k)^n = \frac{k^n}{s^n} - \frac{nk^{n-1}\gamma}{s^{n-1}} + \cdots\,,$$

$$((2\pi)^n\tau_1^k\cdots\tau_n^k)^{-n/k} = 1 - \frac{s}{k}\log((2\pi)^n\tau_1^k\cdots\tau_n^k) + \cdots\,,$$

$$\zeta(s/l, \mathfrak{b}) = -\frac{R}{2}\frac{s^{n-1}}{l^{n-1}} + \frac{1}{n!l^n}\zeta^{(n)}(0, \mathfrak{b})s^n + \cdots, \quad \text{(Lemma 2.2, (1))},$$

$$\zeta(1+s/k, \mathfrak{a}) = \frac{k2^{n-1}R}{N(\mathfrak{a})\sqrt{D}}\frac{1}{s} + c(\mathfrak{a}) + \cdots, \quad \text{(Lemma 2.2, (2))},$$

$$\left(\frac{\sin(\pi s/2k)}{\sin(\pi s/2kl)}\right)^n = l^n + O(|s|^2),$$

then we have

$$(5.6)\quad \operatorname*{Res}_{s=0} H_1 = 2d_k\frac{l^{2n-1}}{kR}\left\{\frac{nk^n\gamma R^2 2^{n-1}}{2l^{n-1}N(\mathfrak{a})\sqrt{D}} + \frac{k2^{n-1}R^2}{2l^{n-1}N(\mathfrak{a})\sqrt{D}}\log((2\pi)^n\tau_1^k\cdots\tau_n^k)\right.$$
$$\left. + \frac{k^n}{n!l^n}\zeta^{(n)}(0, \mathfrak{b})\frac{k2^{n-1}R}{N(\mathfrak{a})\sqrt{D}} - c(\mathfrak{a})\frac{k^nR}{2l^{n-1}}\right\}$$
$$= 2d_k\frac{l^{2n-1}}{kR}\left\{\frac{2^{n-1}k^{n+1}R^2}{2l^{n-1}N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n)\right.$$
$$\left. + \frac{k^{n+1}2^{n-1}R}{n!l^nN(\mathfrak{a})\sqrt{D}}\zeta^{(n)}(0, \mathfrak{b}) - \frac{k^n2^{n-1}R}{n!l^{n-1}N(\mathfrak{a})\sqrt{D}}\zeta^{(n)}(0, \mathfrak{a}^*)\right\}.$$

Collecting the values of residues (5.4), (5.5) and (5.6), we have

$$(5.7)\quad R(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = d_k\frac{2^nl^n\Gamma(l/k)^n\zeta(1+l/k, \mathfrak{a})}{kN(\mathfrak{b})\sqrt{D}(2\pi)^{nl/k}}\left(\frac{\sin(\pi l/2k)}{\sin(\pi/2k)}\right)^n(\tau_1\cdots\tau_n)^{-l}$$
$$- d_k\frac{2^nk^n\Gamma(k/l)^n\zeta(1+k/l, \mathfrak{b}^*)}{lN(\mathfrak{b})\sqrt{D}(2\pi)^{nk/l}}\left(\frac{\sin(\pi k/2l)}{\sin(\pi/2l)}\right)^n(\tau_1\cdots\tau_n)^k$$
$$+ d_k\frac{2^{n-1}k^nl^nR}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n)$$
$$+ d_k\frac{2^nk^nl^n}{ln!N(\mathfrak{a})\sqrt{D}}\zeta^{(n)}(0, \mathfrak{b}) - d_k\frac{2^nk^nl^n}{kn!N(\mathfrak{a})\sqrt{D}}\zeta^{(n)}(0, \mathfrak{a}^*).$$

If we put

$$\Phi(\tau; \mathfrak{a}, \mathfrak{b}; k, l) = d_lT(\tau; \mathfrak{a}, \mathfrak{b}; k, l)$$
$$+ 2^nd_kd_l\frac{k^n\Gamma(k/l)^n\zeta(1+k/l, \mathfrak{b}^*)}{lN(\mathfrak{b})\sqrt{D}(2\pi)^{nk/l}}\left(\frac{\sin(\pi k/2l)}{\sin(\pi/2l)}\right)^n(\tau_1\cdots\tau_n)^k$$
$$- 2^{n-2}d_kd_l\frac{k^nl^nR}{N(\mathfrak{a})\sqrt{D}}\log(\tau_1\cdots\tau_n) + 2^nd_kd_l\frac{k^nl^n}{kn!N(\mathfrak{a})\sqrt{D}}\zeta^{(n)}(0, \mathfrak{a}^*),$$

then it follows from (5.3) and (5.7) that

$$\Phi(\tau;\, \mathfrak{a},\, \mathfrak{b};\, k,\, l)=(DN(\mathfrak{a}\mathfrak{b}))^{-1}\Phi(\tau^{-1};\, \mathfrak{b}^*,\, \mathfrak{a}^*;\, l,\, k)\,. \tag{5.8}$$

Since $N(\mathfrak{a}\mathfrak{b})=(D^2N(\mathfrak{a}^*\mathfrak{b}^*))^{-1}$, (5.8) gives (1.3) in Theorem.

Thus the proof of Theorem is completed.

6. A special case.

Suppose that K is the field of rational numbers. Let a, b be positive rational numbers, $(a)=\mathfrak{a}$ and $(b)=\mathfrak{b}$. Then our series M and Φ are defined as follows;

$$\begin{aligned} M(\tau;\, \mathfrak{a},\, \mathfrak{b};\, k,\, l) &= M(\tau;\, a,\, b;\, k,\, l) \\ &= 2d_k a^{-1}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=1}^{\infty}\exp(-2\pi(bn)^{k/l}am\tau^k) \\ &= 2d_k a^{-1}\log\prod_{n=1}^{\infty}(1-\exp(-2\pi(bn)^{k/l}a\tau^k)\}^{-1} \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \Phi(\tau;\, \mathfrak{a},\, \mathfrak{b};\, k,\, l) &= \Phi(\tau;\, a,\, b;\, k,\, l) \\ &= d_l\sum_{j=0}^{l-1}M(\tau e^{i\pi(l-1-2j)/2kl};\, a,\, b;\, k,\, l) \\ &\quad +2d_kd_lb^{-1}(2\pi)^{-k/l}\Gamma(1+k/l)\zeta(1+k/l,\, b^{-1})\frac{\sin(\pi k/2l)}{\sin(\pi/2l)}\tau^k \\ &\quad -2^{-1}d_kd_la^{-1}kl\log\tau+2d_kd_la^{-1}l\zeta'(0,\, a^{-1})\,. \end{aligned} \tag{6.2}$$

Further we see that

$$\zeta(s,\, a)=a^{-s}\zeta(s)\,,$$

where $\zeta(s)$ is the Riemann zeta function. Since

$$\zeta(0)=-\frac{1}{2}\,,\qquad \zeta'(0)=-\frac{1}{2}\log(2\pi)\,,$$

we have

$$\zeta'(0,\, a)=-\zeta(0)\log a+\zeta'(0)=\frac{1}{2}\log\frac{a}{2\pi}\,. \tag{6.3}$$

Now our Theorem gives the equation

$$\Phi(\tau;\, a,\, b;\, k,\, l)=(ab)^{-1}\Phi(\tau^{-1};\, b^{-1},\, a^{-1};\, l,\, k)\,.$$

Hence we have, by (6.1), (6.2) and (6.3),

$$\begin{aligned}
&a^{-1}\sum_{j=0}^{l-1}\log\prod_{n=1}^{\infty}\{1-\exp(-2\pi(bn)^{k/l}a\tau^k e^{i\pi(l-1-2j)/2l})\}^{-1}\\
&\quad+b^{-1}(2\pi)^{-k/l}b^{1+k/l}\Gamma(1+k/l)\zeta(1+k/l)\frac{\sin(\pi k/2l)}{\sin(\pi/2l)}\tau^k\\
&\quad-\frac{kl}{4a}\log\tau+\frac{l}{2a}\log\frac{1}{2\pi a}\\
&=a^{-1}\sum_{h=0}^{k-1}\log\prod_{n=1}^{\infty}\{1-\exp(-2\pi(n/a)^{l/k}b^{-1}\tau^{-l}e^{i\pi(k-1-2h)/2k})\}^{-1}\\
&\quad+b^{-1}(2\pi)^{-l/k}a^{-1-l/k}\Gamma(1+l/k)\zeta(1+l/k)\frac{\sin(\pi l/2k)}{\sin(\pi/2k)}\tau^{-l}\\
&\quad-\frac{kl}{4a}\log\frac{1}{\tau}+\frac{k}{2a}\log\frac{b}{2\pi}\,.
\end{aligned}$$

Therefore, if we put

$$\begin{aligned}
P(\tau;a,b;k,l)=\tau^{-kl/4}(2\pi a)^{-l/2}\exp\left\{a(2\pi)^{-k/l}b^{k/l}\Gamma(1+k/l)\zeta(1+k/l)\frac{\sin(\pi k/2l)}{\sin(\pi/2l)}\tau^k\right\}\\
\times\prod_{j=0}^{l-1}\prod_{n=1}^{\infty}\{1-\exp(-2\pi(bn)^{k/l}a\tau^k e^{i\pi(l-1-2j)/2l})\}^{-1}\,,
\end{aligned}$$

then

$$P(\tau;a,b;k,l)=P(\tau^{-1};b^{-1},a^{-1};l,k)\,.$$

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