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On Minimal Surfaces in the Real Special Linear Group $SL(2, \mathbf{R})$

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1. Introduction.

Minimal surface theory is a field of study which has been investigated flourishingly for a long time, and is still developing. In particular, many examples of minimal surfaces are known in spaces of constant curvature. However few examples seem to be known even though the ambient space is a 3-dimensional Riemannian homogeneous space (cf. surfaces of constant mean curvature in the Heisenberg group are studied by Tomter [T]). So, it is one of the subjects of interest for us to obtain such an example in an explicit form.

In this paper we study minimal surfaces in $SL(2, \mathbf{R})$ with a canonical left-invariant metric. We define a *rotational surface* and a *conoid* in $SL(2, \mathbf{R})$ analogous to those of Euclidean 3-space, and classify rotational surfaces of constant mean curvature and minimal conoids. We also investigate differential geometric properties (completeness, stability, etc.) of them.

We refer to [O], [L], [C-W] and the like about fundamental notions, definitions and facts.

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2. Preliminaries.

Let G denote the 2×2 real special linear group, which is a non-compact connected Lie group of dimension 3, i.e., $G=SL(2, \mathbf{R})=\{g\in GL(2, \mathbf{R});\ \det g=1\}$. It is well-known that any element g of G can be written uniquely as

$$g=\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.1}$$

for some $x\in\mathbf{R}$, $y\in\mathbf{R}^+$ and $\theta\in\mathbf{S}^1$ (cf. [H]). Thus G is homeomorphic to $\mathbf{R}\times\mathbf{R}^+\times\mathbf{S}^1$,

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hence $\mathbf{R}^3-\mathbf{R}$, and we may consider (x, y, θ) as a local coordinate system of G.

Let $\mathfrak{g}$ denote the Lie algebra of G, i.e., $\mathfrak{g}=\mathfrak{sl}(2, \mathbf{R})=\{X\in \mathfrak{gl}(2, \mathbf{R})\,;\, \operatorname{tr} X=0\}$. We define an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by $2\langle X, Y\rangle=\operatorname{tr}({}^tXY)$ where ${}^tX$ denotes the transposed matrix of X, in other words, take the following e_1, e_2, e_3 to be an orthonormal basis:

$$e_1=\begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \quad e_2=\begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}, \quad e_3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We identify $\mathfrak{g}$ with the tangent space T_eG at the identity element $e\in G$ and define a metric on G by the left translation in the usual manner. We denote this left-invariant metric by ds_G^2. Using the coordinate (x, y, θ), the metric ds_G^2 is written as

$$ds_G^2=\left(\frac{dx}{2y}\right)^2+\left(\frac{dy}{2y}\right)^2+\left(\frac{dx}{2y}+d\theta\right)^2. \tag{2.2}$$

We say this metric to be *canonical* in the following sense: G has a compact, connected subgroup $H=SO(2)$. The homogeneous space G/H is diffeomorphic to the upper half-plane $\mathbf{H}^2$. Then the metric ds_G^2 is $Ad(H)$-invariant on $\mathfrak{g}$ so H acts isometrically on G also on the right, and the metric on G/H associated with ds_G^2 (cf. [K-N], [H-L]) is exactly the Poincaré metric of constant Gauss curvature -4.

Let $\{\omega^1, \omega^2, \omega^3\}$ be an orthonormal coframe field defined by

$$\omega^1=\frac{1}{2y}dx\,, \quad \omega^2=\frac{1}{2y}dy\,, \quad \omega^3=\frac{1}{2y}dx+d\theta\,, \tag{2.3}$$

and $\{e_1, e_2, e_3\}$ be the orthonormal frame field dual to $\{\omega^1, \omega^2, \omega^3\}$. $\{e_j\}$ are given by

$$e_1=2y\frac{\partial}{\partial x}-\frac{\partial}{\partial \theta}\,, \quad e_2=2y\frac{\partial}{\partial y}\,, \quad e_3=\frac{\partial}{\partial \theta}\,. \tag{2.4}$$

Note that $\{\omega^j\}$ and $\{e_j\}$ are globally defined on G.

By differentiating the equations (2.3), we have $d\omega^1=d\omega^3=2\omega^1\wedge\omega^2$, $d\omega^2=0$. Hence we have the structure equations

$$d\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}=-\begin{bmatrix} 0 & -2\omega^1-\omega^3 & -\omega^2 \\ 2\omega^1+\omega^3 & 0 & \omega^1 \\ \omega^2 & -\omega^1 & 0 \end{bmatrix}\wedge\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}. \tag{2.5}$$

So

$$\omega=\begin{bmatrix} 0 & -2\omega^1-\omega^3 & -\omega^2 \\ 2\omega^1+\omega^3 & 0 & \omega^1 \\ \omega^2 & -\omega^1 & 0 \end{bmatrix} \tag{2.6}$$

is the connection matrix for the Levi-Civita connection of (G, ds_G^2).

Let $\pi : G\rightarrow \mathbf{H}^2$ be the natural projection, i.e.,

$$\pi:\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{a\sqrt{-1}+b}{c\sqrt{-1}+d}.$$

It is easy to see from the above computations that π is a Riemannian submersion with totally geodesic $\mathbf{S}^1$-fibre.

The curvature matrix Ω, which is $d\omega+\omega\wedge\omega$ by definition, is computed to be

$$\Omega=\begin{bmatrix} 0 & -7\omega^1\wedge\omega^2 & \omega^1\wedge\omega^3 \\ 7\omega^1\wedge\omega^2 & 0 & \omega^2\wedge\omega^3 \\ -\omega^1\wedge\omega^3 & -\omega^2\wedge\omega^3 & 0 \end{bmatrix}. \tag{2.7}$$

The Ricci tensor *Ric* and the scalar curvature S are computed to be

$$Ric=-6(\omega^1)^2-6(\omega^2)^2+2(\omega^3)^2, \qquad S=-10. \tag{2.8}$$

3. Rotational surfaces.

DEFINITION 3.1. An immersed surface $f: M\to G$ is said to be *rotational* if it is invariant under the right translation of the subgroup $SO(2)$ of G, i.e., $f(M)SO(2)\subset f(M)$.

Obviously a rotational surface has the following parametrization:

$$f:(t,\theta)\mapsto\begin{bmatrix} 1 & x(t) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y(t)^{1/2} & 0 \\ 0 & y(t)^{-1/2} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

where $(t,\theta)\in I\times\mathbf{S}^1$. ($I$ denotes an interval in $\mathbf{R}$.) The condition $(\dot{x})^2+(\dot{y})^2\neq 0$ is necessary for f to be an immersion. We call the curve $(x(t), y(t))$ on $S_0=\{\theta=0\}$ the *generating curve* of f.

The induced metric $f^*ds_G^2$ is given by

$$\begin{aligned} f^*ds_G^2&=\left(\frac{\dot{x}}{2y}dt\right)^2+\left(\frac{\dot{y}}{2y}dt\right)^2+\left(\frac{\dot{x}}{2y}dt+d\theta\right)^2 \\ &=\left(\frac{\sqrt{(\dot{x})^2+(\dot{y})^2}}{2y}dt\right)^2+\left(\frac{\dot{x}}{2y}dt+d\theta\right)^2 \end{aligned} \tag{3.1}$$

where $\dot{}$ means differentiation with respect to t.

So we define an orthonormal coframe field $\{\eta^1,\eta^2\}$ by

$$\eta^1=\frac{\sqrt{(\dot{x})^2+(\dot{y})^2}}{2y}dt, \qquad \eta^2=\frac{\dot{x}}{2y}dt+d\theta \tag{3.2}$$

and the dual frame field $\{v_1, v_2\}$ by

$$v_1=\frac{2y}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\frac{\partial}{\partial t}-\frac{\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\frac{\partial}{\partial\theta}, \qquad v_2=\frac{\partial}{\partial\theta}. \tag{3.3}$$

The pull-back of ω^i's are computed to be

$$f^*\omega^1=\frac{\dot{x}}{2y}dt=\frac{\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\eta^1\,,\qquad f^*\omega^2=\frac{\dot{y}}{2y}dt=\frac{\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}\eta^1\,,$$
$$f^*\omega^3=\frac{\dot{x}}{2y}dt+d\theta=\eta^2\,, \tag{3.4}$$

and the tangent vectors f_*v_1 and f_*v_2 are computed to be

$$f_*v_1=\frac{\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}e_1+\frac{\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}e_2\,,\qquad f_*v_2=e_3 \tag{3.5}$$

by (2.4) and (3.3).

So we can take the unit normal field N so that

$$N=-\frac{\dot{y}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}e_1+\frac{\dot{x}}{\sqrt{(\dot{x})^2+(\dot{y})^2}}e_2\,. \tag{3.6}$$

We immediately have

PROPOSITION 3.2. *The induced metric of a rotational surface is flat, that is, the Gaussian curvature is identically zero.*

PROOF. From (3.2), $d\eta^1=d\eta^2=0$. Thus the connection matrix with respect to $\{\eta^1,\eta^2\}$ is zero. □

LEMMA 3.3. *A rotational surface is complete if and only if the generating curve is closed or has infinite length toward both ends.*

PROOF. Let M be a rotational surface whose generating curve is γ. If γ is closed then M is homeomorphic to a torus. Therefore M is complete since it is compact.

Assume that γ is not closed and is defined on $I=(a, b)$ and has infinite length toward both ends. To show the completeness it suffices to prove that any divergent path in M has infinite length (cf. [O]). Let $c(s)=(t(s), \theta(s))$ $(s\in[o,\alpha))$ be a divergent path, i.e., $t(s)\to a$ or b as $s\to\alpha$. Without loss of generality we assume $t(s)\to b$ and $t'>0$ where $'$ means differentiation with respect to s. We denote the line element of c by ds_c, which is computed to be

$$\begin{aligned}(ds_c)^2&=\left\{\frac{\sqrt{(\dot{x})^2+(\dot{y})^2}}{2y}t'ds\right\}^2+\left\{\left(\frac{\dot{x}}{2y}t'+\theta'\right)ds\right\}^2\\&=\frac{1}{2}\left\{\frac{2(\dot{x})^2+(\dot{y})^2}{(2y)^2}(t')^2ds^2+\frac{(\dot{y})^2}{(2y)^2}(t')^2ds^2\right\}+\left\{\left(\frac{\dot{x}}{2y}t'+\theta'\right)ds\right\}^2\\&\geq\frac{1}{2}\left\{\frac{2(\dot{x})^2+(\dot{y})^2}{(2y)^2}(t')^2ds^2\right\}.\end{aligned}$$

Hence,

$$L(c)=\int_0^{\alpha} ds_c \geq \frac{1}{\sqrt{2}}\int_0^{\alpha}\frac{\sqrt{2(\dot{x})^2+(\dot{y})^2}}{2y}t'ds=\frac{1}{\sqrt{2}}\int_{t(0)}^{b}\frac{\sqrt{2(\dot{x})^2+(\dot{y})^2}}{2y}dt\,.$$

And the right-hand side is the length of the generating curve restricted to $[t(0), b)$, so it is infinity. Therefore the divergent path c has infinite length.

Conversely, if M is complete then the two ends of γ have infinite length because they are also divergent paths. □

LEMMA 3.4. *Let $\sum_{i,j=1,2} h_{ij}\eta^i\eta^j$ be the second fundamental form of f. Then the coefficients h_{ij}'s are computed as follows*:

$$h_{11}=\frac{2}{((\dot{x})^2+(\dot{y})^2)^{3/2}}\{y(\ddot{y}\dot{x}-\dot{y}\ddot{x})+\dot{x}((\dot{x})^2+(\dot{y})^2)\}\,,$$

$$h_{12}=h_{21}=1\,,\qquad h_{22}=0\,.$$

PROOF. Under a change of the frames

$$(e_1, e_2, e_3)\mapsto(f_*v_1, f_*v_2, N)=(e_1, e_2, e_3)A$$

where A is a $SO(3)$-valued function determined by (3.5) and (3.6), the connection matrix $f^*\omega$ is transformed to $\tilde{\omega}=A^{-1}dA+A^{-1}(f^*\omega)A$. A straightforward calculation proves that

$$\tilde{\omega}_1^3=\frac{2}{((\dot{x})^2+(\dot{y})^2)^{3/2}}\{y(\ddot{y}\dot{x}-\dot{y}\ddot{x})+\dot{x}((\dot{x})^2+(\dot{y})^2)\}\eta^1+\eta^2\,,\qquad \tilde{\omega}_2^3=\eta^1\,.$$

On the other hand, the second fundamental form h is $\tilde{\omega}_1^3\eta^1+\tilde{\omega}_2^3\eta^2$ by definition. □

4. Rotational surfaces of constant mean curvature.

PROPOSITION 4.1. *A rotational surface f has constant mean curvature H if and only if the curve $\pi\circ f$ in $\mathbf{H}^2$ has constant curvature H.*

REMARK. In case $H=0$, this is a corollary of Proposition 1.10 of [H-L].

PROOF. For a rotational surface

$$f(t,\theta)=\begin{bmatrix}1 & x(t)\\ 0 & 1\end{bmatrix}\begin{bmatrix}y(t)^{1/2} & 0\\ 0 & y(t)^{-1/2}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix},$$

put $\gamma(t)=\pi\circ f(t,\theta)$ then $\gamma(t)=(x(t), y(t))$. Let $T=\dot{\gamma}/|\dot{\gamma}|$ and N be the unit normal field in the appropriate direction. By definition, the curvature of γ is given by $k=\langle\nabla_T T, N\rangle$ where ∇ is the Levi-Civita connection of $(\mathbf{H}^2, (dx^2+dy^2)/4y^2)$. The procedure similar to sections 2 and 3 leads us to the following:

$$k=\frac{1}{((\dot{x})^2+(\dot{y})^2)^{3/2}}\{y(\ddot{y}\dot{x}-\dot{y}\ddot{x})+\dot{x}((\dot{x})^2+(\dot{y})^2)\}\,.$$

Hence Lemma 3.4 completes the proof. □

So we deal with the ordinary differential equation

$$y(\ddot{y}\dot{x}-\dot{y}\ddot{x})+\dot{x}((\dot{x})^2+(\dot{y})^2)=H((\dot{x})^2+(\dot{y})^2)^{3/2} \tag{4.1}$$

with conditions $((\dot{x})^2+(\dot{y})^2)\neq 0$ and $y>0$. We may assume that $H\geq 0$, for choosing the unit normal field to be of opposite direction.

It is well-known in case of $H=0$ that the solution to (4.1) is $(x-C)^2+y^2=D^2$ where C is a constant and D is a positive constant, or y is an arbitrary function with $\dot{y}\neq 0$, which are geodesics in $\mathbf{H}^2$.

Without loss of generality, we may assume that t is the arc-length parameter under the metric in $\mathbf{H}^2$, i.e., assume that $(\dot{x})^2+(\dot{y})^2=4y^2$.

The differential equation (4.1) reduces to

$$\begin{cases} y(\ddot{y}\dot{x}-\dot{y}\ddot{x})+4\dot{x}y^2=8Hy^3 \\ (\dot{x})^2+(\dot{y})^2=4y^2 \\ y>0\,. \end{cases} \tag{4.2}$$

Furthermore we introduce a function $\xi(t)$ by

$$\dot{x}=2y\cos\xi\,,\qquad \dot{y}=2y\sin\xi\,.$$

Then the differential equation (4.2) reduces to

$$\begin{cases} \dot{\xi}=2(H-\cos\xi) \\ \dot{x}=2y\cos\xi\,,\quad \dot{y}=2y\sin\xi \\ y>0\,. \end{cases} \tag{4.3}$$

Hence we have

PROPOSITION 4.2. *A rotational surface f has constant mean curvature H if and only if its generating curve is a solution to the differential equation* (4.3). *Namely, up to congruence, a rotational surface of constant mean curvature H can be written as follows*:

$$f(t,\theta)=\begin{bmatrix} 1 & 2\int_0^t\{\cos\xi\exp(2\int_0^t\sin\xi dt)\}dt \\ 0 & 1 \end{bmatrix}$$
$$\times\begin{bmatrix} \exp(\int_0^t\sin\xi dt) & 0 \\ 0 & \exp(-\int_0^t\sin\xi dt) \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

where $\xi=\xi(t)$ is a solution to $\dot{\xi}=2(H-\cos\xi)$.

Furthermore if t takes values all over $\mathbf{R}$, *it is complete.*

PROOF. It remains to show the latter half. Let $(\tilde{x},\tilde{y})$ be a solution for another initial condition. Then it holds that $\tilde{x}=Cx+D$, $\tilde{y}=Cy$ for some constants $C>0$, D. And it is easy to see that

$$\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{y}^{1/2} & 0 \\ 0 & \tilde{y}^{-1/2} \end{bmatrix}=\begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C^{1/2} & 0 \\ 0 & C^{-1/2} \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix}.$$

Hence the surface corresponding to $(\tilde{x}, \tilde{y})$ is congruent to the one corresponding to (x, y).

It can be thought that the first equation of (4.3) defines a vector field on $\mathbf{S}^1$. Since $\mathbf{S}^1$ is compact, it is a complete vector field. Hence the first equation of (4.3) has a solution on whole $\mathbf{R}$ for every initial condition. Therefore the solution $x(t)$ and $y(t)$ to (4.3) can be also defined on whole $\mathbf{R}$. Since we have chosen t as the arclength parameter, the solution curve $\gamma(t)$ can be extended to have infinite length in $\mathbf{H}^2$, and it is easy to see that the rotational surface of constant mean curvature H in G corresponding to γ is complete. □

Next we classify rotational surfaces of constant mean curvature in the explicit form. We solve (4.3) in the following three cases: (1) $H>1$, (2) $H=1$, (3) $0<H<1$.

(1) Under the initial condition $\xi(0)=0$, the solution to the first equation is

$$\tan\frac{\xi}{2}=\sqrt{\frac{H-1}{H+1}}\tan(\sqrt{H^2-1}\,t),$$

and it is equivalent to

$$\begin{cases} \cos\xi=\dfrac{1+H\cos(2\sqrt{H^2-1}\,t)}{H+\cos(2\sqrt{H^2-1}\,t)} \\ \sin\xi=\dfrac{\sqrt{H^2-1}\sin(2\sqrt{H^2-1}\,t)}{H+\cos(2\sqrt{H^2-1}\,t)} \end{cases}.$$

Integrating $\dot{y}=2y\sin\xi$,

$$y=\frac{1}{H+\cos(2\sqrt{H^2-1}\,t)},$$

and integrating $\dot{x}=2y\cos\xi$,

$$x=\frac{\sin(2\sqrt{H^2-1}\,t)}{\sqrt{H^2-1}(H+\cos(2\sqrt{H^2-1}\,t))}.$$

The obtained curve (x, y) is

$$x^2+\left(y-\frac{H}{H^2-1}\right)^2=\left(\frac{1}{H^2-1}\right)^2.$$

Under the congruent transformation by dilation, it is congruent to $x^2+(y-H)^2=1$.

In case of (2) and (3), the differential equation (4.3) can be solved in the similar way to (1), hence we provide result without proof.

PROPOSITION 4.3. *A rotational surface has constant mean curvature H if and only if the generating curve (x, y) is locally congruent to the following*:

$$\begin{cases} x^2+(y-H)^2=1\,, & \text{if } H>1\,,\\ x^2+(y-1)^2=1 \text{ or } y=\text{constant}\,, & \text{if } H=1\,,\\ x^2+(y\pm H)^2=1 \text{ or } y=\pm(\sqrt{1-H^2}/H)x\,, & \text{if } 0<H<1\,,\\ x^2+y^2=1 \text{ or } x=\text{constant}\,, & \text{if } H=0\,. \end{cases}$$

COROLLARY 4.4. (1) *A rotational surface of constant mean curvature $H>1$ is also periodic with respect to t. Hence it induces an embedded torus of constant mean curvature.*

(2) *A rotational surface of constant mean curvature $0\le H\le 1$, whose generating curve has infinite length towards both ends, is an embedded, complete cylinder.*

5. Conoids.

In $\mathbf{R}^3$, the *right conoid* is defined to be a surface given by $(u\cos v, u\sin v, \phi(v))$, and it is shown that the right helicoid is the only minimal surface among the right conoid. We define a conoid and a helicoid in $SL(2, \mathbf{R})$ analogous to those in $\mathbf{R}^3$ and obtain a similar result.

DEFINITION 5.1. We call a surface f given by the following form a *conoid*.

$$f:(t, s)\mapsto\begin{bmatrix} 1 & \phi(t)\\ 0 & 1\end{bmatrix}\begin{bmatrix} s^{1/2} & 0\\ 0 & s^{-1/2}\end{bmatrix}\begin{bmatrix} \cos t & \sin t\\ -\sin t & \cos t\end{bmatrix},$$

where $s\in\mathbf{R}^+$ and $\phi(t)$ is a smooth function on an interval of $\mathbf{R}$.

We calculate the first and the second fundamental forms of a conoid in a similar way to those of a rotational surface, and determine the minimal conoid.

Let $\dot{}$ denote differentiation with respect to t. The induced metric $f^*ds_G^2$ is given by

$$\begin{aligned} f^*ds_G^2&=\left(\frac{\dot\phi}{2s}dt\right)^2+\left(\frac{ds}{2s}\right)^2+\left(\frac{\dot\phi}{2s}dt+dt\right)^2\\ &=\left(\frac{(\dot\phi)^2}{2s^2}+\frac{\dot\phi}{s}+1\right)(dt)^2+\left(\frac{ds}{2s}\right)^2. \end{aligned}\tag{5.1}$$

So we have an orthonormal coframe $\{\rho^1, \rho^2\}$ and the dual frame $\{\varepsilon_1, \varepsilon_2\}$ given by

$$\rho^1=\Phi dt\,,\quad \rho^2=\frac{ds}{2s}\,,\quad \varepsilon_1=\frac{1}{\Phi}\frac{\partial}{\partial t}\,,\quad \varepsilon_2=2s\frac{\partial}{\partial s}\,,\tag{5.2}$$

where

$$\Phi=\Phi(t,s)=\left(\frac{(\dot\phi)^2}{2s^2}+\frac{\dot\phi}{s}+1\right)^{1/2}. \tag{5.3}$$

The pull-back of ω^i's are computed to be

$$f^*\omega^1=\frac{\dot\phi}{2s}dt=\frac{\dot\phi}{2s\Phi}\rho^1, \qquad f^*\omega^2=\frac{ds}{2s}=\rho^2, \tag{5.4}$$
$$f^*\omega^3=\frac{\dot\phi}{2s}dt+dt=\left(\frac{\dot\phi}{2s\Phi}+\frac{1}{\Phi}\right)\rho^1.$$

The tangent vectors $f_*\varepsilon_1$ and $f_*\varepsilon_2$ are computed to be

$$f_*\varepsilon_1=\frac{\dot\phi}{2s\Phi}e_1+\left(\frac{\dot\phi}{2s\Phi}+\frac{1}{\Phi}\right)e_3, \qquad f_*\varepsilon_2=e_2. \tag{5.5}$$

Thus we can take the unit normal field N so that

$$N=\left(\frac{\dot\phi}{2s\Phi}+\frac{1}{\Phi}\right)e_1-\frac{\dot\phi}{2s\Phi}e_3. \tag{5.6}$$

The second fundamental form $\sum_{i,j=1,2}h_{ij}\rho^i\rho^j$ can be computed as follows:

$$h_{11}=\frac{\ddot\phi}{2s\Phi^3}, \quad h_{12}=h_{21}=-\frac{s\Phi^2+\dot\phi}{s\Phi^2}, \quad h_{22}=0. \tag{5.7}$$

Therefore a conoid f is minimal if and only if $\ddot\phi=0$, that is, $\phi(t)=Ct+D$ where C, D are constants.

THEOREM 5.2. *A surface of the form*

$$\begin{bmatrix}1 & D\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & Ct\\ 0 & 1\end{bmatrix}\begin{bmatrix}s^{1/2} & 0\\ 0 & s^{-1/2}\end{bmatrix}\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

is the only minimal conoid, in particular, it is embedded. Furthermore if t takes values all over $\mathbf{R}$, it is complete.

PROOF. We have already shown the first half. It remains to prove the latter half. Under the change of coordinates $(\tilde t,\tilde s)=(t,e^s)$, the metric $f^*ds_G^2$ can be written as

$$f^*ds_G^2=\left\{\left(\frac{C}{2e^{\tilde s}}\right)^2+\left(\frac{C}{2e^{\tilde s}}+1\right)^2\right\}d\tilde t^2+\frac{1}{4}d\tilde s^2.$$

On the other hand,

$$\left(\frac{C}{2e^{\tilde s}}\right)^2+\left(\frac{C}{2e^{\tilde s}}+1\right)^2\ge\begin{cases}1/2, & \text{if } C<0\\ 1, & \text{if } C\ge 0.\end{cases}$$

Thus $f^*ds_G^2 \geq d\tilde{t}^2/2 + d\tilde{s}^2/4$. Since the right-hand side is a complete metric, so is the left-hand side. □

REMARK 5.3. We consider the following isometry σ_τ^C of (G, ds_G^2).

$$\sigma_\tau^C : g \mapsto \begin{bmatrix} 1 & C\tau \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}$$

where $\tau \in \mathbf{R}$ and C is a constant. $\{\sigma_\tau^C\}_{\tau\in\mathbf{R}}$ form a one-parameter group. We call this a one-parameter group of the *helicoidal motions of pitch C*. We call a surface invariant under such motions a *helicoidal surface of pitch C*. The minimal surface given in Theorem 5.2 is helicoidal.

6. Stability.

A minimal surface is characterized as a solution of the Euler-Lagrange equation of a variational problem. To say more precisely, each domain K with compact closure of a minimal surface has the critical value of area among surfaces having the same boundary ∂K. Accordingly, a minimal surface is said to be *stable* if each domain K with compact closure has the local minimum value. It is known that the stability condition for our case is described as follows:

A minimal immersion $f: M \to (G, ds_G^2)$ is stable if and only if

$$Q(u, u) := \int_K \{u\Delta_M u - (Ric(N, N) + \|h\|^2)u^2\}\, dV_M \geq 0 \tag{6.1}$$

holds for any smooth function u on K vanishing on ∂K, on each domain K with compact closure. Here, Δ_M and dV_M denote the Laplace-Beltrami operator and the volume element on (M, f^*ds^2) respectively, and $\|\ \|^2$ denotes the squared norm on $T_p^*M \otimes T_p^*M$ induced by f^*ds^2.

Similarly, a surface of constant mean curvature is characterized as a solution to the Euler-Lagrange equation of a variational problem, and its stability condition is also given by (6.1) (cf. [BdCE]).

PROPOSITION 6.1. (1) *A rotational minimal surface is stable.*

(2) *A rotational surface of constant mean curvature H is stable if and only if* $|H| \leq 1$.

(3) *A minimal conoid is stable if the pitch* $C \geq 0$.

PROOF. (1) and (2): Using the results obtained in the previous sections, we can compute that $Ric(N, N) = -6$ and $\|h\|^2 = 4H^2 + 2$ for a rotational surface of constant mean curvature H. Therefore

$$Q(u, u) = \int \{u\Delta_M u + (4 - 4H^2)u^2\}\, dV_M\,.$$

Hence it is obvious that $Q(u, u) \geq 0$ if $|H| \leq 1$.

Conversely, suppose that $|H| > 1$. It suffices to show that $Q(u, u) < 0$ for some $u \in W^{1,2}(M)$. For instance, consider a function $u_\varepsilon(t, \theta)$ defined by

$$u_\varepsilon(t, \theta) = u_\varepsilon(t) = \begin{cases} t+\varepsilon , & \text{if } \ -\varepsilon \leq t \leq 0 , \\ -t+\varepsilon , & \text{if } \ 0 \leq t \leq \varepsilon , \\ 0 , & \text{otherwise} . \end{cases}$$

Then, calculating Δ_M and dV_M in use of the results in sections 3, 4, we have

$$\begin{aligned} Q(u_\varepsilon, u_\varepsilon) &= \int \{ -u_\varepsilon \ddot{u}_\varepsilon + 4(H^2-1)u_\varepsilon^2 \} \, dt d\theta \\ &= \int \{4(H^2-1)u_\varepsilon^2\} \, dt d\theta \leq 0 . \end{aligned}$$

(3) Similar computations as (1) and (2) show that

$$Q(u, u) = \int u \Delta_M u + \frac{16s^3(s+C)}{(C^2+2Cs+2s^2)^2} u^2 \, dV_M .$$

If $C \geq 0$, $16s^3(s+C)/(C^2+2Cs+2s^2)^2$ is a positive function on M. □

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