TOKYO J. MATH.
VOL. 20, NO. 2, 1997

On Dirichlet Series of a Certain Commutative Matrix Ring

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To Professor Takeshi Hirai on his 60th birthday

1. Introduction.

Let $GL(n, \mathbf{Z})$ denote the modular group of degree n over the ring of integers $\mathbf{Z}$. For a regular element ζ in $GL(n, \mathbf{Z})$, let R denote the ring generated by ζ over $\mathbf{Z}$ and let $f(X)$ be its characteristic polynomial. The purpose of this paper is to show that a special value of certain Dirichlet series $\zeta_R(s)$ of R at $s=1$ gives rise to an ideal regulator-class number formula for R, which is a generalization of the classical regulator-class number formula for the Dedekind zeta functions of a number field. Before stating our results, we need some preparation. An ideal $\mathfrak{a}\subset R$ is said to be *nonsingular* if the index $(R:\mathfrak{a})$ (as group) is finite, in which case the norm of $\mathfrak{a}$, $N\mathfrak{a}$, is defined to be this index. Let $\mathbf{Q}[\zeta]$ be the ring generated by ζ over the field of rationals $\mathbf{Q}$. An R-submodule $\mathfrak{a}$ of $\mathbf{Q}[\zeta]$ is a *fractional ideal* of R if there exists an invertible element α in $\mathbf{Q}[\zeta]$ such that $\alpha\mathfrak{a}$ is a nonsingular ideal of R. The *pseudo inverse ideal* $\check{\mathfrak{a}}$ of $\mathfrak{a}$ is defined by

$$\check{\mathfrak{a}}=\{\mu\in\mathbf{Q}[\zeta] : \mu\mathfrak{a}\subset R\} .$$

Note that $\check{\mathfrak{a}}$ is a fractional ideal of R. Let

$$f(X)=f_1(X)f_2(X)\cdots f_g(X)$$

be the decomposition of $f(X)$ into the irreducible factors over $\mathbf{Q}$. The ring O of the algebraic integers in $\mathbf{Q}[\zeta]$ is isomorphic to $O_1\oplus O_2\oplus\cdots\oplus O_g$ where O_i is the ring of integers of the algebraic number field $k_i=\mathbf{Q}(\zeta_i)$ and ζ_i a root of $f_i(X)$. Let $\mathbf{E}_O$ be the unit group of O. For each nonsingular ideal $\mathfrak{a}$ we define a subgroup $\mathbf{E}_\mathfrak{a}$ of $\mathbf{E}_O$ by

$$\mathbf{E}_\mathfrak{a}=\{\varepsilon\in\mathbf{E}_O : \varepsilon\mathfrak{a}=\mathfrak{a}\} .$$

We shall prove (Lemma 3.6 below) that the index $(\mathbf{E}_O:\mathbf{E}_\mathfrak{a})$ is finite.

Let us define the *ideal class semigroup* $\mathbf{G}$ of the ring R. Two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *equivalent* if there exists an invertible element λ in $\mathbf{Q}[\zeta]$ such that $\lambda\mathfrak{a}=\mathfrak{b}$. We denote by $\mathbf{G}$ the set of all equivalence classes and a class in $\mathbf{G}$ by $C=C(\mathfrak{a})$ with a representative $\mathfrak{a}$. Note that $\mathbf{G}$ is a semigroup under the canonical multiplication.

Received October 27, 1995

We can now state the main results of this paper. Let $\mathbf{C}$ denote the field of complex numbers.

THEOREM I. *Let $C=C(\mathfrak{a})$ be the ideal class of the ring $R=\mathbf{Z}[\zeta]$ represented by an integral ideal $\mathfrak{a}$. Define a Dirichlet series $\zeta_C(s)$ for C by*

$$\zeta_C(s)=\sum_{\substack{\mathfrak{b}\in C(\mathfrak{a})\\ \mathfrak{b}\subset R}}\frac{1}{(N\mathfrak{b})^s}, \qquad s\in\mathbf{C}.$$

Then $\zeta_C(s)$ is holomorphic in the half plane $\Re(s)>1$. Furthermore, we have

$$\lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_C(\sigma)=2^{r+c}\pi^c\frac{(\mathbf{E}_O:\mathbf{E}_{\mathfrak{a}})|R(\mathbf{E}_O)|}{N\check{\mathfrak{a}}N\mathfrak{a}|H_O|\sqrt{|\mathbf{D}|}}.$$

Here $R(\mathbf{E}_O)$ is the regulator of $\mathbf{E}_O$, H_O is the group of all elements in $\mathbf{E}_O$ with finite order, r (resp. $2c$) is the number of all real (resp. complex) roots of the characteristic polynomial $f(X)$ of ζ, g is the number of irreducible factors of $f(X)$ over $\mathbf{Z}$ and $\mathbf{D}=Nf'(\zeta)$ is the discriminant of R.

THEOREM II. *We define a Dirichlet series $\zeta_R(s)$ by*

$$\zeta_R(s)=\sum_{\mathfrak{b}}\frac{a(\mathfrak{b})}{(N\mathfrak{b})^s}$$

where the summation runs over all nonsingular ideals $\mathfrak{b}$ of R and $a(\mathfrak{b})=N\check{\mathfrak{b}}N\mathfrak{b}/(\mathbf{E}_O:\mathbf{E}_{\mathfrak{b}})$. Then we have

$$\lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_R(\sigma)=|\mathbf{G}|2^{r+c}\pi^c\frac{|R(\mathbf{E}_O)|}{|H_O|\sqrt{|\mathbf{D}|}}.$$

We recall that the order of the ideal class semigroup coincides with the number of conjugacy classes $G_Z(f)/GL(n,\mathbf{Z})$ in the sense of Latimer and MacDuffee ([6], [12]): $G_Z(f)$ is the set of elements of $GL(n,\mathbf{Z})$ with the characteristic polynomial $f(X)$, which is decomposed into $GL(n,\mathbf{Z})$-orbits, under the adjoint action of $GL(n,\mathbf{Z})$. $G_Z(f)/GL(n,\mathbf{Z})$ means the orbit space. The finiteness of the space $G_Z(f)/GL(n,\mathbf{Z})$ has been proved by [10], [14]. We remind that the zeta functions of various kinds have been introduced into the study of algebras. Particularly, Solomon's idea in dealing with the group algebras in [9] and its generalization by Bushnell-Reiner [2], [3], concerning the semisimple $\mathbf{Q}$-algebras, have given the suggestions for this paper.

The contents of this paper are as follows. Preparatory facts collected in §2. We reprove in §3 the theorem of Latimer-MacDuffee and the finiteness of the order of $\mathbf{G}$. In §4 we prove a reduction theorem which enables us to calculate the limit: $\lim_{t\to\infty}(\sigma-1)^g\zeta_C(\sigma)$. In §5, we restate briefly the calculation of the density of ideals (due to Dedekind) for an algebraic number field over $\mathbf{Q}$. We calculate in §6 the special value of certain Dirichlet series $\zeta_i(s:\chi)$ at $s=1$. Finally in §7 we shall prove our two main theorems.

ACKNOWLEDGEMENTS. I wish to thank Professor N. Yui for through reading of this article, and for her helpful advice.

2. Preliminaries.

Let $GL(n, \mathbf{C})$ be the group of all $n \times n$ invertible matrices with entries in $\mathbf{C}$. An element A in $GL(n, \mathbf{C})$ is called *regular* if the centralizer T of A in $GL(n, \mathbf{C})$ forms a maximal split torus of the reductive group $GL(n, \mathbf{C})$. Let ζ be a regular element in the modular group $GL(n, \mathbf{Z})$ of degree n and $R=\mathbf{Z}[\zeta]$ the ring generated by ζ over $\mathbf{Z}$. Since ζ is regular, the characteristic polynomial $f(X)$ of ζ has irreducible factors $f_1(X)$, $f_2(X), \cdots, f_g(X)$ with multiplicity one. Let $\mathbf{Q}[X]$ be the polynomial ring in X with coefficients in $\mathbf{Q}$. The ring $\mathbf{Q}[\zeta]$, which is generated by ζ over $\mathbf{Q}$, is decomposed as follows: For $h_i(X)=f(X)/f_i(X)$ there exist $u_1(X), u_2(X), \cdots, u_g(X)$ in $\mathbf{Q}[X]$ such that $1=\sum_{i=1}^{g} u_i(X)h_i(X)$. Put $e_i=u_i(\zeta)h_i(\zeta)$. Then we have

$$1=\sum_{i=1}^{g} e_i, \quad \text{and} \quad e_ie_j=\delta_{i,j}e_i \tag{2.1}$$

where $\delta_{i,j}$ is the Kronecker delta. Let ζ_i be the restriction of $\mathbf{Q}$-linear endomorphism ζ of $\mathbf{Q}[\zeta]$ to $\mathbf{Q}[\zeta]e_i$. Observe that ζ_i is a root of the irreducible polynomial $f_i(X)$, so that $k_i=\mathbf{Q}[\zeta_i]$ is an algebraic number field over $\mathbf{Q}$. The ring $\mathbf{Q}[\zeta]$ is decomposed into a direct sum of k_ie_i's $(1 \leq i \leq g)$:

$$\mathbf{Q}[\zeta]=k_1e_1 \oplus k_2e_2 \oplus \cdots \oplus k_ge_g\,. \tag{2.2}$$

Let O be the ring of algebraic integers in $\mathbf{Q}[\zeta]$. Since e_i is a root of the monic polynomial X^2-X in $\mathbf{Z}[X]$, e_i belongs to O. Let O_i be the ring of integers of k_i. Then we have

$$O=O_1e_1 \oplus O_2e_2 \oplus \cdots \oplus O_ge_g\,. \tag{2.3}$$

We shall define the norm and trace on $\mathbf{Q}[\zeta]$. Since all eigenvalues of ζ are mutually distinct, ζ is diagonalizable. Furthermore, there exist $\zeta=\zeta^{(0)}, \zeta'=\zeta^{(1)}, \cdots, \zeta^{(n-1)}$ in T such that

$$\prod_{0 \leq i<j \leq n-1} (\zeta^{(i)}-\zeta^{(j)}) \in GL(n, \mathbf{C})\,, \qquad f(\zeta^{(j)})=0 \quad (0 \leq j \leq n-1)\,. \tag{2.4}$$

Put $\Omega=\{\zeta, \zeta', \cdots, \zeta^{(n-1)}\}$. Let $\mathbf{Q}[\Omega]$ be the commutative ring generated by Ω over $\mathbf{Q}$. Then

$$f(X)=(X-\zeta)(X-\zeta') \cdots (X-\zeta^{(n-1)}) \qquad \text{in } (\mathbf{Q}[\Omega])[X]\,. \tag{2.5}$$

The norm N and trace Tr in $\mathbf{Q}[\zeta]$ are defined, respectively, by

$$Np(\zeta)=p(\zeta)p(\zeta') \cdots p(\zeta^{(n-1)})\,, \tag{2.6}$$

(2.7) $$Tr(p(\zeta))=p(\zeta)+p(\zeta')+\cdots+p(\zeta^{(n-1)})$$

where $p(X)$ is a polynomial in $\mathbf{Q}[X]$. Since $Np(\zeta)$ and $Tr(p(\zeta))$ are the symmetric polynomials in $\zeta, \zeta', \cdots, \zeta^{(n-1)}$, (2.5) implies that both $Np(\zeta)$ and $Tr(p(\zeta))$ are rational numbers. If α is an element in R, then $N\alpha$ (resp. $Tr(\alpha)$) is a rational integer. Let N_{k_i} be the norm of the algebraic number field k_i.

LEMMA 2.1. *Let* $\alpha=\alpha_1 e_1+\alpha_2 e_2+\cdots+\alpha_g e_g$ *where* $\alpha_i \in k_i$ *for* $i=1, 2, \cdots, g$. *Then we have*

$$N\alpha=N_{k_1}\alpha_1 N_{k_2}\alpha_2 \cdots N_{k_g}\alpha_g \times 1_n .$$

Here 1_n *is the identity matrix of degree n.*

Since ζ is diagonalizable, it is easy to see the assertion of this lemma.

Define the unit group $\mathbf{E}_O$ of $\mathbf{Q}[\zeta]$ by

(2.8) $$\mathbf{E}_O=\{\varepsilon \in O : N\varepsilon=\pm 1\} .$$

LEMMA 2.2. *Let* $\mathbf{E}_i$ *be the unit group of the algebraic number field* k_i. *Then we have*

(2.9) $$\mathbf{E}_O=\mathbf{E}_1 e_1 \oplus \mathbf{E}_2 e_2 \oplus \cdots \oplus \mathbf{E}_g e_g .$$

PROOF. Let ε be an element in $\mathbf{E}_O$. We shall prove that ε belongs to the set in the right hand side of (2.9). For each i, let ε_i be an element in k_i satisfying $\varepsilon e_i=\varepsilon_i e_i$. Since $\varepsilon=\sum_{i=1}^{g} \varepsilon_i e_i$, (2.3) implies that $\varepsilon_i \in O_i$ for every $i=1, 2, \cdots, g$. Hence by Lemma 2.1 we have $N_{k_i}\varepsilon_i=\pm 1$. This gives what we want. The converse inclusion can be proved in a similar manner.

We now turn to the fractional ideals of R. An element α in R is *nonsingular* if the principal ideal (α) of R is nonsingular. Since $N(\alpha)=|N\alpha|$, α is nonsingular if and only if α is invertible in $\mathbf{Q}[\zeta]$.

DEFINITION 2.1. An R-submodule $\mathfrak{a}$ of $\mathbf{Q}[\zeta]$ is called fractional if there exists an invertible element α in R such that $\alpha\mathfrak{a}$ is a nonsingular ideal of R. For a fractional ideal $\mathfrak{a}$ of R, we define the norm $N\mathfrak{a}$ of $\mathfrak{a}$ by $N\mathfrak{a}=(N(\alpha))^{-1}N(\alpha\mathfrak{a})$.

DEFINITION 2.2. Let $\mathfrak{a}$ be a nonsingular ideal of R. The pseudo inverse ideal $\check{\mathfrak{a}}$ of $\mathfrak{a}$ is defined by

$$\check{\mathfrak{a}}=\{\mu \in \mathbf{Q}[\zeta] : \mu\mathfrak{a} \subset R\} .$$

LEMMA 2.3. *The pseudo inverse ideal* $\check{\mathfrak{a}}$ *of a given nonsingular ideal* $\mathfrak{a}$ *is fractional.*

PROOF. Since $\mathfrak{a}$ is nonsingular, $\mathfrak{a}$ is a $\mathbf{Z}$-free module of rank n. Let $O\mathfrak{a}$ be the ideal of O generated by $\mathfrak{a}$. Since $\mathfrak{a}$ and $O\mathfrak{a}$ have the same rank over $\mathbf{Z}$, the index $(O\mathfrak{a} : \mathfrak{a})$ is finite. Hence, from the invertibility of the ideal $O\mathfrak{a}$ of O it follows that $\mathfrak{a}$ has a nonsingular element α. Then the definition of $\check{\mathfrak{a}}$ implies that $\alpha\check{\mathfrak{a}} \subset R$. Furthermore, since $1 \in \check{\mathfrak{a}}$, we have $\alpha\check{\mathfrak{a}}$ is a nonsingular ideal of R.

We remark that if a nonsingular ideal $\mathfrak{a}$ is invertible (i.e. there exists a fractional ideal $\mathfrak{b}$ of R such that $\mathfrak{a}\mathfrak{b}=R$), then $\check{\mathfrak{a}}$ is actually an inverse ideal of $\mathfrak{a}$. In general, however, there is a nonsingular ideal which has no inverse ideal (see Example below).

EXAMPLE. Take

$$\zeta=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \in GL(4, \mathbf{Z}) .$$

Then the characteristic polynomial f of ζ is decomposed into $f(X)=F(X)F(-X)$, $F(X)=X^2+X+1$. $F(X)$ is irreducible modulo 2 and $f(X)-F(X)^2=-2F(X)X$. Let $(2, F(\zeta))$ be the ideal of $R=\mathbf{Z}[\zeta]$ generated by 2 and $F(\zeta)$. Then $(2, F(\zeta))^2=2(2, F(\zeta))$. This implies that the ideal $(2, F(\zeta))$ is not invertible.

3. Theorem of Latimer and MacDuffee.

Let ζ be a regular element in $GL(n, \mathbf{Z})$ with characteristic polynomial f and $R=\mathbf{Z}[\zeta]$ the ring generated by ζ. Note that $\zeta^{-1}\in R$. In fact, since $det\zeta=a_n=\pm 1$, it is easy to see that

$$\zeta^{-1}=-a_n(\zeta^{n-1}+a_{n-1}\zeta^{n-2}+\cdots+a_{n-1})\in R$$

where $f(X)=X^n+a_{n-1}X^{n-1}+\cdots+a_n$. Put

$$G_Z(f)=\{\gamma\in GL(n, \mathbf{Z}) : f(\gamma)=0\} . \tag{3.1}$$

The group $GL(n, \mathbf{Z})$ acts on $G_Z(f)$ by the rule:

$$GL(n, \mathbf{Z})\times G_Z(f)\ni(g, x)\rightarrow gxg^{-1}\in G_Z(f) . \tag{3.2}$$

The $GL(n, \mathbf{Z})$-orbits in $G_Z(f)$ will be called the conjugacy classes of $G_Z(f)$ and denoted by $G_Z(f)/GL(n, \mathbf{Z})$. In this section we shall rediscover the theorem of Latimar and MacDuffee which establishes a bijection between the ideal class semigroup of R and the conjugacy classes of $G_Z(f)$.

Let $\Omega=\{\zeta, \zeta', \cdots, \zeta^{(n-1)}\}$ be the conjugate system of ζ (see (2.5)). Let $\alpha^{(j)}$ be the j-th conjugate of α in $\mathbf{Q}[\zeta]$ which is defined by $\alpha^{(j)}=p(\zeta^{(j)})$ where $\alpha=p(\zeta)$ and $p(X)\in\mathbf{Q}[X]$. Let $GL(n, \mathbf{Q})$ be the group of rational matrices in $GL(n, \mathbf{C})$.

LEMMA 3.1. *Any two matrices in $G_Z(f)$ are $GL(n, \mathbf{Q})$-conjugate.*

PROOF. Let γ be an element in $G_Z(f)$. Since all roots of $f(X)$ are simple, there exists h in $GL(n, \mathbf{Q})$ such that

$$h\gamma h^{-1}=\begin{pmatrix}\gamma_1 & 0 & \cdots & 0\\ 0 & \gamma_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & \gamma_g\end{pmatrix}$$

where γ_i is an integral matrix of degree n_i with characteristic polynomial $f_i(X)$ (cf. Theorem III. 12 and p. 55, Exercise 7, [8]). Consequently it is sufficient to prove this lemma when $f(X)$ is irreducible over $\mathbf{Q}$. Define an integral matrix γ_0 by $\gamma_0\mathbf{x}_0=\zeta\mathbf{x}_0$ where $\mathbf{x}_0={}^t(1, \zeta, \cdots, \zeta^{n-1})$ if t denotes the transpose. Since $f(\gamma_0)\mathbf{x}_0^{(j)}=0$ $(0\leq j\leq n-1)$ and the matrix $(\mathbf{x}_0, \mathbf{x}_0', \cdots, \mathbf{x}_0^{(n-1)})$ is invertible, we have $f(\gamma_0)=0$. Moreover, $\gamma_0\in G_Z(f)$ since $\zeta^{-1}\in R$ and $\gamma_0^{-1}\mathbf{x}_0=\zeta^{-1}\mathbf{x}_0$. Hence the proof is reduced to the following: Each element γ in $G_Z(f)$ is conjugate to γ_0 under the adjoint action of $GL(n, \mathbf{Q})$. Put $B(X)=X1_n-\gamma$, and let $C(X)$ be the adjoint of the matrix $B(X)$. For the first column vector ${}^t(c_1(X), c_2(X), \cdots, c_n(X))$ of $C(X)$, we put $\mathbf{x}={}^t(c_1(\zeta), c_2(\zeta), \cdots, c_n(\zeta))$. Since $B(X)C(X)=f(X)1_n$ and $\deg c_i\leq n-1$ $(1\leq i\leq n)$, it is easy to see that $\gamma\mathbf{x}=\zeta\mathbf{x}$ and $\mathbf{x}\neq 0$. From the irreducibility of the characteristic polynomial $f(X)$ of γ it follows that the matrix $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. Hence $\{c_1(\zeta), c_2(\zeta), \cdots, c_n(\zeta)\}$ is a $\mathbf{Q}$-basis of $\mathbf{Q}[\zeta]$. Define an element h in $GL(n, \mathbf{Q})$ by letting $h\mathbf{x}=\mathbf{x}_0$. Then we have $h\gamma h^{-1}=\gamma_0$ as claimed.

LEMMA 3.2. *Let γ be an element in $G_Z(f)$. Then there exists a vector $\mathbf{x}$ in R^n such that $\gamma\mathbf{x}=\zeta\mathbf{x}$ and $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. Furthermore, $\mathbf{x}$ is uniquely determined up to scalar multiplication by an invertible element in $\mathbf{Q}[\zeta]$.*

PROOF. For $\mathbf{x}_0={}^t(1, \zeta, \cdots, \zeta^{n-1})$, we define a matrix γ_0 in $G_Z(f)$ by $\gamma_0\mathbf{x}_0=\zeta\mathbf{x}_0$. Let γ be any element in $G_Z(f)$. By Lemma 3.1 there exists h in $GL(n, \mathbf{Q})$ such that $\gamma=h\gamma_0h^{-1}$. Put $\mathbf{x}=h\mathbf{x}_0$. Then $\gamma\mathbf{x}=\zeta\mathbf{x}$ and $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. It remains to show the uniqueness of $\mathbf{x}$ up to a scalar multiplication by element in $\mathbf{Q}[\zeta]$. Suppose there exists $\mathbf{y}\in\mathbf{Q}[\zeta]^n$ such that $\gamma\mathbf{y}=\zeta\mathbf{y}$. Put $C=(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$, $C^{-1}=(d_{ij})$. Observe that

$$(det(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)}))^2 d_{j1} \qquad (1\leq j\leq n)$$

is a symmetric polynomial in $\zeta', \zeta^{(2)}, \cdots, \zeta^{(n-1)}$. Then by (2.5), $d_{j1}\in\mathbf{Q}[\zeta]$ for all $j=1, 2, \cdots, n$. Since $CC^{-1}=1_n$, we have

$$\mathbf{e}_i=\sum_{j=1}^{n} d_{ji}\mathbf{x}^{(j-1)} \qquad \text{for } i=1, 2, \cdots, n$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is the canonical basis of $\mathbf{Q}[\zeta]^n$. Therefore $\mathbf{y}$ is written as $\mathbf{y}=\sum_{j=1}^{n} b_j\mathbf{x}^{(j-1)}$ with $b_j\in\mathbf{Q}[\Omega]$. In particular $b_1\in\mathbf{Q}[\zeta]$. Since $\gamma\mathbf{x}^{(j)}=\zeta^{(j)}\mathbf{x}^{(j)}$ $(0\leq j\leq n-1)$, it follows from the invertibility of $\zeta-\zeta^{(j)}$ that $\mathbf{y}=b_1\mathbf{x}$, $b_j=0$ for all $j\neq 1$. This implies that

$$det(\mathbf{y}, \mathbf{y}', \cdots, \mathbf{y}^{(n-1)}) = Nb_1 det(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)}) .$$

Hence we have $Nb_1 \neq 0$. Consequently b_1 is invertible in $\mathbf{Q}[\zeta]$. This gives the uniqueness of $\mathbf{x}$.

Let us define the ideal class semigroup of R. We denote by $\mathbf{A}$ (resp. $\mathbf{A}_O$) the set of all fractional ideals of R (resp. O). $\mathbf{A}$ (resp. $\mathbf{A}_O$) is a semigroup with the canonical multiplication.

DEFINITION 3.1. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbf{A}$ (resp. $\mathbf{A}_O$) are equivalent if there exists an invertible element λ in $\mathbf{Q}[\zeta]$ such that $\lambda\mathfrak{a} = \mathfrak{b}$.

DEFINITION 3.2. The set of all equivalence classes of $\mathbf{A}$ (resp. $\mathbf{A}_O$) will be denoted by $\mathbf{G}$ (resp. $\mathbf{G}_O$). For $\mathfrak{a}$ in $\mathbf{A}$ (resp. $\mathbf{A}_O$), $C(\mathfrak{a})$ denotes the ideal class represented by $\mathfrak{a}$.

$\mathbf{G}$ is called the ideal class semigroup. In view of (2.3), $\mathbf{G}_O$ is a direct product of a finite number of ideal class groups of the algebraic number fields over $\mathbf{Q}$. Therefore $\mathbf{G}_O$ is a finite group and called the ideal class group of O.

Let us now define a map from $\mathbf{G}$ to $G_Z(f)/GL(n, \mathbf{Z})$ as follows (cf. [6] and [12], or p. 53 in [8]): For each element $\mathfrak{a}$ in $\mathbf{A}$ with a $\mathbf{Z}$-basis $\{w_1, w_2, \cdots, w_n\}$, define an integral matrix γ by

(3.3) $$\zeta\mathbf{x} = \gamma\mathbf{x} \qquad \text{where } \mathbf{x} = {}^t(w_1, w_2, \cdots, w_n) .$$

Then $\gamma \in G_Z(f)$. Define a map $\phi : \mathbf{G} \to G_Z(f)/GL(n, \mathbf{Z})$:

(3.4) $$\phi(C(\mathfrak{a})) = C(\gamma)$$

where $C(\gamma)$ is the class in $G_Z(f)/GL(n, \mathbf{Z})$ represented by γ.

THEOREM 3.3 (Latimer-MacDuffee). *The map ϕ is bijective.*

PROOF. Well definedness: Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals belonging to the same class C in $\mathbf{G}$. For a $\mathbf{Z}$-basis $\{w_1, w_2, \cdots, w_n\}$ (resp. $\{v_1, v_2, \cdots, v_n\}$) of $\mathfrak{a}$ (resp. $\mathfrak{b}$), we define two matrices $\gamma, \delta \in G_Z(f)$ by $\gamma\mathbf{x} = \zeta\mathbf{x}$ and $\delta\mathbf{y} = \zeta\mathbf{y}$ where $\mathbf{x} = {}^t(w_1, w_2, \cdots, w_n)$ and $\mathbf{y} = {}^t(v_1, v_2, \cdots, v_n)$. Since $\mathfrak{a}$ is equivalent to $\mathfrak{b}$, there exists an invertible element λ in $\mathbf{Q}[\zeta]$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. Consequently $g(\lambda\mathbf{x}) = \mathbf{y}$ for a suitable g in $GL(n, \mathbf{Z})$. This implies that $\delta = g\gamma g^{-1}$, so $C(\gamma) = C(\delta)$.

Injectivity: Suppose $\phi(C(\mathfrak{a})) = \phi(C(\mathfrak{b})) = C(\gamma)$ for two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbf{A}$ and $\gamma \in G_Z(f)$. Then there are two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\gamma\mathbf{x} = \zeta\mathbf{x}$, $\gamma\mathbf{y} = \zeta\mathbf{y}$ and $\mathfrak{a}$ (resp. $\mathfrak{b}$) is generated by $\mathbf{x}$ (resp. $\mathbf{y}$). By Lemma 3.2, $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent. Hence ϕ is injective.

Surjectivity: Let $C(\gamma)$ be a class in $G_Z(f)/GL(n, \mathbf{Z})$. Again by Lemma 3.2, we can choose a vector $\mathbf{x}$ in R^n such that $\gamma\mathbf{x} = \zeta\mathbf{x}$ and $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. Let $\mathfrak{a}$ be the $\mathbf{Z}$-module generated by the entries of $\mathbf{x}$. Then $\mathfrak{a}$ is nonsingular and $\phi(C(\mathfrak{a})) = C(\gamma)$.

Let us prove the finiteness of the order of $\mathbf{G}$. For this, we need to prove the following two lemmas. Define a map η from $\mathbf{G}$ to $\mathbf{G}_O$ by

$$\eta(C(\mathfrak{a}))=C(O\mathfrak{a})\,, \qquad \mathfrak{a}\in\mathbf{A}\,. \tag{3.5}$$

For a fixed $\mathfrak{a}$ in $\mathbf{A}$, we put

$$M(\mathfrak{a})=\{\mathfrak{b}\in\mathbf{A} : O\mathfrak{a}=O\mathfrak{b}\}\,. \tag{3.6}$$

LEMMA 3.4. *The map η defined by* (3.5) *is surjective, and*

$$\eta^{-1}(C(O\mathfrak{a}))=\{C(\mathfrak{b}) : \mathfrak{b}\in M(\mathfrak{a})\}\,. \tag{3.7}$$

PROOF. A fractional ideal of O is also a fractional ideal of R. Hence the surjectivity of η is obvious. Let us prove (3.7). Let $C(\mathfrak{b})$ be a class in $\eta^{-1}(C(\mathfrak{a}))$. Then there exists α in $\mathbf{Q}[\zeta]$ such that $\alpha O\mathfrak{b}=O\mathfrak{a}$. Put $\mathfrak{q}=\alpha\mathfrak{b}$. Then $C(\mathfrak{q})=C(\mathfrak{b})$ and $O\mathfrak{a}=O\mathfrak{q}$. Consequently $C(\mathfrak{b})$ belongs to the set in the right hand side of (3.7). Conversely suppose $\mathfrak{b}\in M(\mathfrak{a})$. Then we have $\eta(C(\mathfrak{b}))=C(O\mathfrak{a})$. Hence $C(\mathfrak{b})$ belongs to the set $\eta^{-1}(C(\mathfrak{a}))$. This gives the converse inclusion.

LEMMA 3.5. *Let $\mathfrak{a}$ be a fixed fractional ideal of R. Then we have*

$$|M(\mathfrak{a})|\le(O : R)^n\,.$$

PROOF. Put $m=(O : R)$. Since $1\in O$ and $mO\subset R$, we have

$$O\mathfrak{a}=O\mathfrak{b}\supset\mathfrak{b}\supset mO\mathfrak{b}=mO\mathfrak{a} \qquad \text{for all } \mathfrak{b}\in M(\mathfrak{a})\,. \tag{3.8}$$

Let c be the number of all subgroups of the additive group $O\mathfrak{a}/mO\mathfrak{a}$. Then by (3.8), $|M(\mathfrak{a})|\le c$. On the other hand, by the fundamental theory of finitely generated abelian groups, we have

$$O\mathfrak{a}/mO\mathfrak{a}\cong\mathbf{Z}/(p_1^{m_1})\oplus\mathbf{Z}/(p_2^{m_2})\oplus\cdots\oplus\mathbf{Z}/(p_k^{m_k})\,.$$

Here m_i is a positive integer and p_i is a prime number. Therefore

$$\begin{aligned}|M(\mathfrak{a})|\le c&\le(m_1+1)(m_2+1)\cdots(m_k+1)\\&\le(p_1)^{m_1}(p_2)^{m_2}\cdots(p_k)^{m_k}\\&=(O\mathfrak{a} : mO\mathfrak{a})=m^n\,.\end{aligned}$$

Let $\mathfrak{a}$ be a fractional ideal of R. Then the unit group $\mathbf{E}_O$ of O acts on $M(\mathfrak{a})$ by the rule:

$$\mathbf{E}_O\times M(\mathfrak{a})\ni(\varepsilon, \mathfrak{b})\to\varepsilon\mathfrak{b}\in M(\mathfrak{a})\,. \tag{3.9}$$

LEMMA 3.6. *Let $\mathfrak{a}$ be a nonsingular ideal of R. We define the stabilizer group $\mathbf{E}_\mathfrak{a}$ of $\mathfrak{a}$ by*

$$\mathbf{E}_\mathfrak{a}=\{\varepsilon\in\mathbf{E}_O : \varepsilon\mathfrak{a}=\mathfrak{a}\}\,.$$

Then the index $(\mathbf{E}_O : \mathbf{E}_\mathfrak{a})$ as group is finite.

PROOF. By Lemma 3.5, the set $M(\mathfrak{a})$ is finite. So the subset $\mathbf{E}_O\mathfrak{a}$ of $M(\mathfrak{a})$ is also finite. Therefore $(\mathbf{E}_O : \mathbf{E}_\mathfrak{a})$ is finite.

THEOREM 3.7. *The order of* $\mathbf{G}$ *is finite.*

PROOF. Let η be the map from $\mathbf{G}$ to $\mathbf{G}_O$. By Lemma 3.4 we have

$$|\mathbf{G}|=\left|\bigcup_{C(\tilde{\mathfrak{a}})\in\mathbf{G}_O}\eta^{-1}(C(\tilde{\mathfrak{a}}))\right|\leq\sum_{C(\tilde{\mathfrak{a}})\in\mathbf{G}_O}|M(\tilde{\mathfrak{a}})|\,.$$

From Lemma 3.5 it follows that the order of $\mathbf{G}$ is finite.

4. Reduction theorem.

Let ζ be a regular element in $GL(n, \mathbf{Z})$ and $R=\mathbf{Z}[\zeta]$ be the ring generated by ζ over $\mathbf{Z}$. For an ideal class $C=C(\mathfrak{a})$ of R, we define a Dirichlet series $\zeta_C(s)$ by

$$\zeta_C(s)=\sum_{\substack{\mathfrak{b}\in C(\mathfrak{a})\\ \mathfrak{b}\subset R}}\frac{1}{(N\mathfrak{b})^s}\,,\qquad s\in\mathbf{C}\,. \tag{4.1}$$

$\xi_C(s)$ may be called the zeta function of the class C. We shall prove in §7 that $\zeta_C(s)$ is convergent in the complex half plane $\Re(s)>1$. In this section we shall give a reduction theorem which is useful for investigating the analytic properties of $\zeta_C(s)$.

Let $\mathbf{A}_C$ be the set of all integral ideals of $C=C(\mathfrak{a})$ and $\check{\mathfrak{a}}$ the pseudo inverse ideal of $\mathfrak{a}$. The group $\mathbf{E}_\mathfrak{a}$, which is given in Lemma 3.6, stabilizes the set $\check{\mathfrak{a}}^\times$ of all invertible elements in $\check{\mathfrak{a}}$. We classify $\check{\mathfrak{a}}^\times$ by $\mathbf{E}_\mathfrak{a}$-orbits, and denote by $\check{\mathfrak{a}}^\times/\mathbf{E}_\mathfrak{a}$ the set of all $\mathbf{E}_\mathfrak{a}$-orbits in $\check{\mathfrak{a}}^\times$. From this, it follows immediately the following lemma.

LEMMA 4.1. *The set* $\mathbf{A}_C$ *is parameterized by*

$$\mathbf{A}_C=\{\lambda\mathfrak{a} : [\lambda]\in\check{\mathfrak{a}}^\times/\mathbf{E}_\mathfrak{a}\}\,.$$

Let $\mathfrak{a}$ be the representative of the class C. In the following we assume that $\mathfrak{a}$ is integral. Put $\tilde{\mathfrak{a}}=O\mathfrak{a}$. Since $\tilde{\mathfrak{a}}$ is an ideal of O, $\tilde{\mathfrak{a}}$ has the inverse ideal $\tilde{\mathfrak{a}}^{-1}$. It is easy to see that $\check{\mathfrak{a}}\subset\tilde{\mathfrak{a}}^{-1}$ and $\tilde{\mathfrak{a}}^{-1}/\check{\mathfrak{a}}$ is a finite additive group.

DEFINITION 4.1. Let $\check{\mathfrak{a}}$ be the pseudo inverse ideal of the representative $\mathfrak{a}$ of C. Denote by B^* the character group of $B=\tilde{\mathfrak{a}}^{-1}/\check{\mathfrak{a}}$.

Let $\mathbf{E}_O$ be the unit group of O. $\mathbf{E}_O$ is a direct product of a finite group H_O and a finitely generated free group E_O. Put $E_\mathfrak{a}=\mathbf{E}_\mathfrak{a}\bigcap E_O$, $H_\mathfrak{a}=\mathbf{E}_\mathfrak{a}\bigcap H_O$. Let $(\tilde{\mathfrak{a}}^{-1})^\times$ be the set of all invertible elements in $\tilde{\mathfrak{a}}^{-1}$. Then

$$E_O(\tilde{\mathfrak{a}}^{-1})^\times\subset(\tilde{\mathfrak{a}}^{-1})^\times\,. \tag{4.2}$$

DEFFINITION 4.2. Fix a representative λ for each class $[\lambda]$ in $(\tilde{\mathfrak{a}}^{-1})^\times/E_\mathfrak{a}$, and let χ be a character of B. An L-function $L(s:\chi)$ is defined by

$$L(s:\chi)=\sum_{[\lambda]\in(\tilde{\mathfrak{a}}^{-1})^{\times}/E_{\mathfrak{a}}}\frac{\chi(\lambda)}{(N(\lambda\tilde{\mathfrak{a}}))^{s}}$$

where s is a complex number and $N(*)$ is the ideal norm of $(*)$.

We remark that $\chi(\lambda)$ and hence $L(s:\chi)$ depends on the choice of the representatives λ. We choose a representative λ and fix it once and for all.

THEOREM 4.2. *Let $C=C(\mathfrak{a})$ be an ideal class of* **G**. *Then the zeta function $\zeta_C(s)$ is expressed as*

$$\zeta_C(s)=\frac{(N\tilde{\mathfrak{a}})^s}{|H_{\mathfrak{a}}|(\tilde{\mathfrak{a}}^{-1}:\check{\mathfrak{a}})(N\mathfrak{a})^s}\left\{\sum_{\chi\in B^*}L(s:\chi)\right\}.$$

PROOF. We see that

$$\sum_{\chi\in B^*}L(s:\chi)=\sum_{[\lambda]\in(\tilde{\mathfrak{a}}^{-1})^{\times}/E_{\mathfrak{a}}}\frac{\sum_{\chi\in B^*}\chi(\lambda)}{N(\lambda\tilde{\mathfrak{a}})^s}.$$

From the orthogonality relations (see Theorem 7.3, [5]) on the group B:

$$\sum_{\chi\in B^*}\chi(\lambda)=\begin{cases}|B| & \lambda\in\check{\mathfrak{a}}\\ 0 & \text{otherwise}\,,\end{cases}$$

it follows that

$$\sum_{\chi\in B^*}L(s:\chi)=|B|\sum_{[\lambda]\in\check{\mathfrak{a}}^{\times}/E_{\mathfrak{a}}}\frac{1}{N(\lambda\tilde{\mathfrak{a}})^s}=\frac{|B||H_{\mathfrak{a}}|(N\mathfrak{a})^s}{(N\tilde{\mathfrak{a}})^s}\zeta_C(s).$$

Let χ be a character of B. For each $\varepsilon\in\mathbf{E}_O$, define χ_ε by

$$\chi_\varepsilon(\alpha)=\chi(\varepsilon\alpha)\,,\qquad \alpha\in\tilde{\mathfrak{a}}^{-1}. \tag{4.3}$$

χ_ε is well defined and is a character of the finite additive group $\tilde{\mathfrak{a}}^{-1}/\varepsilon^{-1}\check{\mathfrak{a}}$. The orbit classes $(\tilde{\mathfrak{a}}^{-1})^{\times}/E_{\mathfrak{a}}$ is decomposed as follows:

$$(\tilde{\mathfrak{a}}^{-1})^{\times}/E_{\mathfrak{a}}=\bigcup_{[\varepsilon]\in E_O/E_{\mathfrak{a}}}\ \bigcup_{[\alpha]\in(\tilde{\mathfrak{a}}^{-1})^{\times}/E_O}\alpha\in E_{\mathfrak{a}}.$$

Therefore

$$L(s:\chi)=\sum_{[\varepsilon]\in E_O/E_{\mathfrak{a}}}\ \sum_{[\alpha]\in(\tilde{\mathfrak{a}}^{-1})^{\times}/E_O}\frac{\chi_\varepsilon(\alpha)}{(N\alpha\tilde{\mathfrak{a}})^s}. \tag{4.4}$$

Define an ideal $\tilde{\mathfrak{a}}_i$ of O_i $(1\le i\le g)$ by

$$\tilde{\mathfrak{a}}_i e_i=\tilde{\mathfrak{a}}e_i\,. \tag{4.5}$$

Note that $\tilde{\mathfrak{a}}=\tilde{\mathfrak{a}}_1e_1\oplus\tilde{\mathfrak{a}}_2e_2\oplus\cdots\oplus\tilde{\mathfrak{a}}_ge_g$.

DEFINITION 4.3. For χ in B^* and ε in E_O, define the zeta function $\zeta_i(s:\chi_\varepsilon)$ by

$$\zeta_i(s:\chi_\varepsilon)=\sum_{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i}\frac{\chi_\varepsilon(\alpha e_i)}{(N\alpha\tilde{\mathfrak{a}}_i)^s}$$

where $\tilde{\mathfrak{b}}_i=(\tilde{\mathfrak{a}}_i)^{-1}$.

By Lemma 2.1 and (4.4) we have immediately the following lemma.

LEMMA 4.3. *The notations being the same as above, we have*

$$L(s:\chi)=\sum_{[\varepsilon]\in E_O/E_\mathfrak{a}}\prod_{i=1}^{g}\zeta_i(s:\chi_\varepsilon).$$

We now state the Abel's summation formula which is used frequently in the analytic number theory (cf. Theorem 1.6, [5]).

LEMMA 4.4. *Let ψ be a function on the interval $(0,\infty)$ of C^1-class. Then for a finite number of complex sequence $a_1, a_2, \cdots, a_{[t]}$, we have*

$$\sum_{0<m\le t}a_m\psi(m)=A(t)\psi(t)-\int_1^t A(x)\psi'(x)dx\,,\qquad A(x)=\sum_{0<m\le x}a_m\,.$$

LEMMA 4.5. *For each positive real number t and $\varepsilon\in E_O$, let*

$$A_i(t:\chi_\varepsilon)=\sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i\\ N(\alpha\tilde{\mathfrak{a}}_i)\le t}}\chi_\varepsilon(\alpha e_i)\,.$$

Then we have

$$\sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i\\ N(\alpha\tilde{\mathfrak{a}}_i)\le t}}\frac{\chi_\varepsilon(\alpha e_i)}{(N(\alpha\tilde{\mathfrak{a}}_i))^s}=A_i(t:\chi_\varepsilon)t^{-s}+s\int_1^t A_i(x:\chi_\varepsilon)x^{-s-1}dx\,.$$

PROOF. For each positive integer m, put

$$\psi(m)=m^{-s}\,,\qquad a_m=\sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i\\ N\alpha\tilde{\mathfrak{a}}_i=m}}\chi_\varepsilon(\alpha e_i)\,.$$

Then we have

$$A_i(t:\chi_\varepsilon)=\sum_{m\le t}a_m\,,\qquad \sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i\\ N(\alpha\tilde{\mathfrak{a}}_i)\le t}}\frac{\chi_\varepsilon(\alpha e_i)}{(N\alpha\tilde{\mathfrak{a}}_i)^s}=\sum_{m\le t}a_m\psi(m)\,.$$

Hence by Lemma 4.4 we have our conclusion.

5. Density of ideals.

We keep the same notations as in the previous section. Let $C=C(\mathfrak{a})$ be the fixed class of the ideal class semigroup of R. For each i, consider the i-th component $\tilde{\mathfrak{a}}_i$ (resp. $\tilde{\mathfrak{b}}_i$) of $\tilde{\mathfrak{a}}$ (resp. $\tilde{\mathfrak{a}}^{-1}$). Let t be a positive real number. Define a subset $T_i(t)$ of $(\tilde{\mathfrak{b}}_i)^\times/E_i$ by

$$T_i(t)=\left\{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i\ :\ |N\alpha|\leq\frac{t}{N\tilde{\mathfrak{a}}_i}\right\}. \tag{5.1}$$

Observe that

$$|A_i(t:\chi_\varepsilon)|\leq|T_i(t)| \tag{5.2}$$

where χ is a character of $B=\tilde{\mathfrak{a}}^{-1}/\check{\mathfrak{a}}$ and $\varepsilon\in E_O$. Especially if χ_ε is trivial on $\tilde{\mathfrak{b}}_i e_i$, then

$$A_i(t:\chi)=|T_i(t)|. \tag{5.3}$$

We will evaluate the limit:

$$\lim_{t\to\infty}\frac{|T_i(t)|}{t}.$$

Let $k_i^{(1)}, k_i^{(2)}, \cdots, k_i^{(n_i)}$ be all the conjugates of k_i over $\mathbf{Q}$. We assume that

$k_i^{(j)}$ $(1\leq j\leq r_i)$ are real,

$k_i^{(j)}$ $(r_i+1\leq j\leq r_i+c_i)$ are complex and

$k_i^{(r_i+c_i+j)}$ $(1\leq j\leq c_i)$ is the complex conjugate of $k_i^{(r_i+j)}$

where $n_i=r_i+2c_i$. Let $(k_i)^\times$ be the set of all invertible elements in k_i. We define a map ℓ^j from $(k_i)^\times$ to the field of real numbers $\mathbf{R}$ by

$$\ell^j\alpha=\begin{cases}\log|\alpha^{(j)}| & 1\leq j\leq r_i\\ 2\log|\alpha^{(j)}| & r_i+1\leq j\leq r_i+c_i\end{cases} \tag{5.4}$$

where $\alpha^{(j)}$ is the j-th conjugate of $\alpha\in k_i$. Then we have

$$\sum_{j=1}^{r_i+c_i}\ell^j\alpha=\log|N_{k_i}\alpha|. \tag{5.5}$$

The unit group $\mathbf{E}_i$ of the field k_i is decomposed into a product: $\mathbf{E}_i=E_i\times H_i$ where E_i is a free group with rank r_i+c_i-1 and H_i is a finite group (see [1] or [4]). Define the regulator $R(\mathbf{E}_i)$ of $\mathbf{E}_i$ as follows:

$$R(\mathbf{E}_i)=\begin{vmatrix}\ell^1\varepsilon_1 & \ell^2\varepsilon_1 & \cdots & \ell^{r_i+c_i-1}\varepsilon_1\\ \ell^1\varepsilon_2 & \ell^2\varepsilon_2 & \cdots & \ell^{r_i+c_i-1}\varepsilon_2\\ \vdots & \vdots & \ddots & \vdots\\ \ell^1\varepsilon_{r_i+c_i-1} & \ell^2\varepsilon_{r_i+c_i-1} & \cdots & \ell^{r_i+c_i-1}\varepsilon_{r_i+c_i-1}\end{vmatrix}$$

where $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{r_i+c_i-1}\}$ is a generator of E_i.

We shall identify $\tilde{\mathfrak{b}}_i$ with the lattice $\mathbf{Z}^{n_i}$ in $\mathbf{R}^{n_i}$. Let $\{w_1, w_2, \cdots, w_{n_i}\}$ be a $\mathbf{Z}$-basis of $\mathfrak{b}_i$. For each $\mathbf{x}=(x_1, x_2, \cdots, x_{n_i})$ in $\mathbf{R}^{n_i}$, we put

(5.6) $$\alpha(\mathbf{x})=x_1w_1+x_2w_2+\cdots+x_{n_i}w_{n_i}\,.$$

Then the map $\mathbf{x}\rightarrow\alpha(\mathbf{x})$ from $\mathbf{Q}^{n_i}$ to k_i is bijective. Also through this map, $\tilde{\mathfrak{b}}_i$ can be identified with the lattice $\mathbf{Z}^{n_i}$. Let $w_k^{(j)}$, $1\le j\le r_i+2c_i$, be the algebraic conjugates of w_k. For $\mathbf{x}\in\mathbf{R}^{n_i}$, we put

(5.7) $$N\alpha(\mathbf{x})=\prod_{j=1}^{r_i+2c_i}\alpha^{(j)}(\mathbf{x})$$

where $\alpha^{(j)}(\mathbf{x})=x_1w_1^{(j)}+x_2w_2^{(j)}+\cdots+x_{n_i}w_{n_i}^{(j)}$. Note that N is an extension of N_{k_i} to R^{n_i}. Let S be the subset of R^{n_i} defined by

$$S=\{\mathbf{x}\in\mathbf{R}^{n_i} : N\alpha(\mathbf{x})=0\}\,.$$

We can define a map $\Phi : \mathbf{R}^{n_i}\backslash S\rightarrow\mathbf{R}^{r_i+c_i}$:

(5.8) $$\Phi(\mathbf{x})=(\ell^1\alpha(\mathbf{x}), \cdots, \ell^{r_i+c_i}\alpha(\mathbf{x}))\,.$$

Let $\{\eta_1, \eta_2, \cdots, \eta_{r_i+c_i-1}\}$ be a free basis of E_i. To parameterize $(\tilde{\mathfrak{b}}_i)^\times/E_i$ we choose the following basis for $\mathbf{R}^{r_i+c_i}$:

$$\mathbf{u}=\frac{1}{n_i}(\overbrace{1, 1, \cdots, 1}^{r_i\text{-times}}, \overbrace{2, 2, \cdots, 2}^{c_i\text{-times}})\,,$$

$$\mathbf{v}_1=\Phi(\eta_1),\quad \mathbf{v}_2=\Phi(\eta_2)\,,\quad \cdots,\quad \mathbf{v}_{r_i+c_i-1}=\Phi(\eta_{r_i+c_i-1})\,.$$

LEMMA 5.1. *The vectors* $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{r_i+c_i-1}$ *form a basis of* $\mathbf{R}^{r_i+c_i}$.

PROOF. We put $d=det({}^t\mathbf{u}, {}^t\mathbf{v}_1, \cdots, {}^t\mathbf{v}_{r_i+c_i-1})$. Then it is sufficient to prove that $d\ne0$. Recalling that

$$\sum_{j=1}^{r_i+c_i}\ell^j\eta_k=\log|N_{k_i}(\eta_k)|=0\,,$$

we have immediately $d=\pm R((\mathbf{E}_\mathfrak{a})_i)$. This gives $d\ne0$.

Let $V(=V_i)$ be the subset of $R^{r_i+c_i}$ defined by

$$V=\left\{u\mathbf{u}+\sum_{k=1}^{r_i+c_i-1}v_k\mathbf{v}_k : u\in\mathbf{R}, 0\le v_k<1\ (1\le k\le r_i+c_i-1)\right\}.$$

We now put

(5.9) $$P(=P_i)=\Phi^{-1}(V)\,.$$

Let $\mathbf{R}^\times$ be the set of non-zero real numbers. It is easy to see that

(5.10) $$tP \subset P \quad \text{for all } t \in \mathbf{R}^{\times} .$$

The following lemma asserts that the set $(\tilde{\mathfrak{b}}_i)^{\times}/E_i$ is parametrized by $P \cap \mathbf{Z}^{n_i}$.

LEMMA 5.2. *The map* $\mathbf{x} \to \alpha(\mathbf{x})E_i$ *from* $P \cap \mathbf{Z}^{n_i}$ *to* $(\tilde{\mathfrak{b}}_i)^{\times}/E_i$ *is a bijection.*

PROOF. First we shall prove the surjectivity. Let αE_i be an element in $(\tilde{\mathfrak{b}}_i)^{\times}/E_i$. Choose an element $\mathbf{x}$ in $\mathbf{Z}^{n_i}$ satisfying $\alpha(\mathbf{x}) = \alpha$. Put

$$\Phi(\mathbf{x}) = u\mathbf{u} + \sum_{j=1}^{r_i+s_i-1} v_k \mathbf{v}_k , \qquad v_k = [v_k] + \{v_k\} ,$$

where $m_k = [v_k]$ is the Gaussian integral part of v_k and $\{v_k\} = v_k - [v_k]$. Put

$$\eta = \eta_1^{m_1} \eta_2^{m_2} \cdots \eta_{r_i+c_i-1}^{m_{r_i}+c_i-1} .$$

Then $\eta \in E_i$. Denote

$$\beta = \alpha\eta^{-1} , \qquad \beta = \alpha(\mathbf{y}) \text{ and } \eta = \alpha(\mathbf{z}) \quad (\mathbf{y} \in P \cap \mathbf{Z}^{n_i}, \mathbf{z} \in \mathbf{Z}^{n_i}) .$$

Then $\Phi(\mathbf{y}) = \Phi(\mathbf{x}) - \Phi(\mathbf{z}) \in V$. So $\mathbf{y} \in P \cap \mathbf{Z}^{n_i}$. From this follows that the map $\mathbf{x} \to \alpha(\mathbf{x})E_i$ is surjective. Let us prove the injectivity. Suppose $\alpha(\mathbf{y}) = \alpha(\mathbf{x})\alpha(\mathbf{z})$ and $\alpha(\mathbf{z}) \in E_i$ for $\mathbf{x}, \mathbf{y} \in P \cap \mathbf{Z}^{n_i}$ and $\mathbf{z} \in \mathbf{Z}^{n_i}$. Since $\Phi(\mathbf{y}) = \Phi(\mathbf{x}) + \Phi(\mathbf{z})$ and $\alpha(\mathbf{z}) \in E_i$, we have $\alpha(\mathbf{z}) = 1$. Thus $\mathbf{y} = \mathbf{x}$ as claimed.

For each positive real number t, we put

(5.11) $$P(t) = \left\{ \mathbf{x} \in P : |N\alpha(\mathbf{x})| \le \frac{t}{N\tilde{\mathfrak{a}}_i} \right\} , \qquad P^* = \{ \mathbf{x} \in P : |N\alpha(\mathbf{x})| \le 1 \} .$$

LEMMA 5.3. *Let P^* be the same as in* (5.11). *Then we have*

$$\lim_{t \to \infty} \frac{|T_i(t)|}{t} = \frac{1}{N\tilde{\mathfrak{a}}_i} vol(P^*)$$

where $vol(P^*)$ is the volume of P^*.

PROOF. Define a map $\Phi_t : \mathbf{R}^{n_i} \to \mathbf{R}^{n_i} : \Phi_t(\mathbf{x}) = (N\tilde{\mathfrak{a}}_i/t)^{1/n_i}\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^{n_i}$. We have $\Phi_t(P(t)) = P^*$. Therefore the area P^* is meshed by the n_i-dimensional fundamental parallelepipeds with edges in the lattice $\Phi_t(\mathbf{Z}^{n_i})$. On the other hand by Lemma 5.2, $\alpha^{-1}(T_i(t))$ is the set of all lattices in $P(t)$. From the definition of the integration we may conclude that

$$\lim_{t \to \infty} \frac{|T_i(t)|}{t} = \frac{1}{N\tilde{\mathfrak{a}}_i} vol(P^*) .$$

The volume of P^* is given by

$$vol(P^*)=\int_{P^*} d\mathbf{x} \qquad \text{with } d\mathbf{x}=dx_1\wedge dx_2\wedge\cdots\wedge dx_{n_i}\,.$$

We can calculate $vol(P^*)$ explicitly (cf. §1.3, Chapt. 5, [1]). Then we have the following lemma.

LEMMA 5.4. *With notations as above in force, we have*

$$\lim_{t\to\infty}\frac{|T_i(t)|}{t}=\frac{2^{r_i+c_i}(\pi)^{c_i}|R((\mathbf{E}_a)_i)|}{N\tilde{\mathfrak{b}}_i N\tilde{\mathfrak{a}}_i\sqrt{|\mathbf{D}_i|}}$$

where $\mathbf{D}_i$ *is the discriminant of the field* k_i.

For $\mathbf{a}$, $\mathbf{b}$ ($\mathbf{a}\neq 0$) in $\mathbf{R}^{n_i}$, consider a line $\ell(\mathbf{a}:\mathbf{b})=\{x\mathbf{a}+\mathbf{b}:x\in\mathbf{R}\}$. Let $\overline{P(t)}$ be the closure of $P(t)$ in $\mathbf{R}^{n_i}$.

LEMMA 5.5. *Let* $\mathbf{a}$, $\mathbf{b}$ ($\mathbf{a}\notin S$) *be two elements in* $\mathbf{Z}^{n_i}$. *Denote by* $n(\mathbf{a}:\mathbf{b})$ *the number of connected components of* $\overline{P(t)}\cap\ell(\mathbf{a}:\mathbf{b})$. *Then we have* $n(\mathbf{a}:\mathbf{b})<2n_i^2(n_i+1)$.

PROOF. For $\mathbf{x}$ in $\mathbf{R}^{n_i}\backslash S$, put $\Phi(\mathbf{x})=u\mathbf{u}+\sum_{k=1}^{r_i+c_i-1}v_k\mathbf{v}_k$. Then u and v_k are expressed as in the forms:

$$\begin{cases} u=\log|N\alpha(\mathbf{x})|\,,\\ v_k=\sum_{j=1}^{r_i+c_i}c_{jk}\log|\alpha^{(j)}(\mathbf{x})| \qquad (1\le k\le r_i+c_i-1)\,,\quad c_{jk}\in\mathbf{R}\,.\end{cases}$$

Define the functions $F_k(\mathbf{x})$ $(1\le k\le r_i+c_i-1)$ by

$$F_k(\mathbf{x})=\sum_{j=1}^{r_i+c_i-1}\frac{1}{2}\log|\alpha^{(j)}(\mathbf{x})|^2\,.$$

For each nonnegative real number a, we put

$$S_a=\{\mathbf{x}\in\mathbf{R}^{n_i}:|N\alpha(\mathbf{x})|=a\}\,,$$
$$S_{a,k}=\{\mathbf{x}\in\mathbf{R}\backslash S:F_k(\mathbf{x})=a\} \qquad (1\le k\le r_i+c_i-1)\,.$$

Let $\partial P(t)$ be the boundary of $P(t)$. Then

(5.12) $$\partial P(t)\subset S\cup S_{t/(N\mathfrak{a}_i)}\cup\Big(\bigcup_{k=1}^{r_i+c_i-1}(S_{0,k}\cup S_{1,k})\Big)\,.$$

As $\mathbf{a}\notin S$, it follows that

(5.13) $$|S\cap\ell(\mathbf{a}:\mathbf{b})|\le n_i\,,\qquad |S_{t/(N\mathfrak{a}_i)}\cap \mathrm{l}(\mathbf{a}:\mathbf{b})|\le 2n_i\,.$$

Let us prove that for each k, there are the following two cases:

(5.14) *Case* (1): $(\ell(\mathbf{a}:\mathbf{b})\backslash S)\subset(S_{0,k}\cup S_{1,k})$,

Case (2): $|\ell(\mathbf{a}:\mathbf{b})\cap(S_{0,k}\cup S_{1,k})|\le 4n_i(n_i+1)$.

Put $J=\{x\in\mathbf{R}:x\mathbf{a}+\mathbf{b}\notin S\}$. The first derivative of the function $F_k(x\mathbf{a}+\mathbf{b})$ on J is expressed as

$$\frac{d}{dx}F_k(x\mathbf{a}+\mathbf{b})=\frac{g_k(x)}{(N\alpha(x\mathbf{a}+\mathbf{b}))^2}$$

where g_k is a polynomial in x and $\deg g_k \leq 2n_i - 1$. Suppose $g_k \equiv 0$ on J. Then $F_k(x\mathbf{a}+\mathbf{b})$ is a constant function on J. Therefore $(\ell(\mathbf{a}:\mathbf{b})\backslash S)\cap(S_{0,k}\cup S_{1,k})=\varnothing$ or $(\ell(\mathbf{a}:\mathbf{b})\backslash S)\subset(S_{0,k}\cup S_{1,k})$. Suppose that g_k is not identically 0 on J. Then the equation $F_k(x\mathbf{a}+\mathbf{b})=0$ has at most $2n_i$ solutions on each connected component of J. Furthermore, by the first inequality of (5.13), J has at most n_i+1 connected components. Hence, if $g_k \not\equiv 0$, then $\ell(\mathbf{a}:\mathbf{b})$ intersects to $S_{0,k}\cup S_{1,k}$ at most $4n_i(n_i+1)$ times. Let us now prove this lemma. If $\ell(\mathbf{a}:\mathbf{b})$ satisfies (1) in (5.14) for a number k, then $n(\mathbf{a}:\mathbf{b})\leq n_i+1$. Suppose $\ell(\mathbf{a}:\mathbf{b})$ satisfies (2) for all $k=1,2\cdots,r_i+c_i-1$. Then by (5.12) and (5.13),

$$n(\mathbf{a}\ :\mathbf{b})\leq\frac{1}{2}(n_i+2n_i+4(r_i+c_i-1)n_i(n_i+1))<2n_i{}^2(n_i+1)\,.$$

6. Asymptotic formula for $\zeta_i(s:\chi_\varepsilon)$.

Let $C=C(\mathfrak{a})$ be an ideal class of R with a representative $\mathfrak{a}\subset R$. Put $\tilde{\mathfrak{a}}=O\mathfrak{a}$, and denote by $\tilde{\mathfrak{a}}^{-1}$ the inverse ideal of the ideal $\tilde{\mathfrak{a}}$ of O. Let $\tilde{\mathfrak{a}}^{-1}=\tilde{\mathfrak{b}}_1e_1+\tilde{\mathfrak{b}}_2e_2+\cdots+\tilde{\mathfrak{b}}_ge_g$ be the decomposition of $\tilde{\mathfrak{a}}^{-1}$ by the fractional ideals $\tilde{\mathfrak{b}}_i$ of O_i. By Lemma 4.3 and Lemma 5.2, we can consider the L-functions $L(s:\chi)$ $(\chi\in B^*)$ which have the following properties:

$$(6.1)\quad \begin{aligned}&(1):\quad L(s:\chi)=\sum_{[\varepsilon]\in E_O/E_\mathfrak{a}}\prod_{i=1}^{g}\zeta_i(s:\chi_\varepsilon)\,,\\ &(2):\quad \zeta_i(s:\chi_\varepsilon)=\sum_{\mathbf{a}\in P_i\cap\mathbf{Z}^{n_i}}\frac{\chi_\varepsilon(\alpha(\mathbf{a})e_i)}{(N\alpha(\mathbf{a})\tilde{\mathfrak{a}}_i)^s}\qquad(1\leq i\leq g)\,.\end{aligned}$$

In this section we shall prove that the zeta function $\zeta_i(s:\chi_\varepsilon)$ is holomorphic in the half plane where $\Re(s)>1$, and calculate the value:

$$\lim_{\sigma\to1+0}(\sigma-1)\zeta_i(\sigma:\chi_\varepsilon)\,.$$

The following lemma, which is well known (cf. (2.1.2), [13]), plays a crucial role to calculate this value.

LEMMA 6.1. *Suppose ψ is a function on $\mathbf{R}$ of C^1-class. Then for each closed interval $[a,b]\subset\mathbf{R}$, we have*

$$\sum_{a<m\leq b}\psi(m)=\int_a^b\psi(x)dx+\int_a^b\left(x-[x]-\frac{1}{2}\right)\psi'(x)dx$$
$$-(a-[a]-1/2)\psi(a)-(b-[b]-1/2)\psi(b)$$

where $[x]$ is the Gaussian integral part of x.

LEMMA 6.2. *Let p and q be two positive integers satisfying* $q/p<1$. *For the function* $\psi(x)=e^{2\pi\sqrt{-1}(q/p)x}$, *we have*

$$\left|\sum_{a<m\leq b}\psi(m)\right|\leq\frac{q}{\pi p}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2-(q/p)^2}+1+\frac{p}{\pi q}.$$

PROOF. We shall apply Lemma 6.1 to the function ψ. By the Fourier expansion theorem, we have

$$x-[x]-\frac{1}{2}=-\frac{1}{\pi}\sum_{\nu=1}^{\infty}\frac{\sin 2\pi\nu x}{\nu}\qquad\text{for } x\notin\mathbf{Z}.$$

Therefore

$$\int_a^b\left(x-[x]-\frac{1}{2}\right)\psi'(x)dx=-2\sqrt{-1}\frac{q}{p}\int_a^b\sum_{\nu=1}^{\infty}\frac{\sin 2\pi\nu x}{\nu}\psi(x)dx.$$

Since the series $\sum_{\nu=1}^{\infty}\sin 2\pi\nu x/\nu$ is uniformly convergent on each closed interval $I\subset[a,b]\backslash\mathbf{Z}$, the summation and the integration are interchanged. Consequently

$$\int_a^b\left(x-[x]-\frac{1}{2}\right)\psi'(x)dx=-2\sqrt{-1}\frac{q}{p}\sum_{\nu=1}^{\infty}\int_a^b\frac{\sin 2\pi\nu x}{\nu}\psi(x)dx$$
$$=\frac{q}{p}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_a^b(e^{2\pi\sqrt{-1}(q/p-\nu)x}-e^{2\pi\sqrt{-1}(q/p+\nu)x})dx.$$

Therefore

$$\left|\int_a^b\left(x-[x]-\frac{1}{2}\right)\psi'(x)dx\right|\leq\frac{q}{\pi p}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2-(q/p)^2}.$$

Hence by Lemma 6.1 we have our assertion.

DEFINITION 6.1. We say that $\mathbf{a}=(a_1,a_2,\cdots,a_{n_i})$ in $\mathbf{Z}^{n_i}$ is primitive if the greatest common divisor of $a_1,a_2,\cdots,a_{n_i}$ is equal to 1.

LEMMA 6.3. *Let* $V=V_i$ *and* $P=P_i$ *be the same as in* (5.9). *Suppose* χ_ε *is nontrivial on* $\tilde{\mathfrak{b}}_i e_i$. *Then there exists a primitive element* $\mathbf{a}$ *in* $P\cap\mathbf{Z}^{n_i}$ *such that* $\chi_\varepsilon(\alpha(\mathbf{a})e_i)\neq 1$.

PROOF. Denote by V^0 the set of all interior points in V. Put $P^0=\Phi^{-1}(V^0)$. Then $P^0\subset P$ and P^0 is open in $\mathbf{R}^{n_i}$. Therefore $P^0\cap\mathbf{Z}^{n_i}\neq\varnothing$. Let $\mathbf{b}$ be a primitive element in $P^0\cap\mathbf{Z}^{n_i}$. Choose a $\mathbf{Z}$-basis $\{\mathbf{b}_1=\mathbf{b},\mathbf{b}_2,\cdots,\mathbf{b}_{n_i}\}$ of $\mathbf{Z}^{n_i}$. We shall prove that $P^0\cap\mathbf{Z}^{n_i}$ contains a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$. Since P^0 is open in $\mathbf{R}^{n_i}$, there exists a (sufficiently small) positive irrational number δ such that $\mathbf{b}_1+\delta\mathbf{b}_2\in P^0$. Let $U\,(\subset P^0)$ be an n_i-dimensional open ball in $\mathbf{R}^{n_i}$ centered at $\mathbf{b}_1+\delta\mathbf{b}_2$. Put

$$C(U)=\{x\mathbf{x} : x\in\mathbf{R}^{\times}, \mathbf{x}\in U\} .$$

By (5.10), $C(U)$ is an open cone in $\mathbf{R}^{n_i}$. Furthermore, it is easy to see that $C(U)\subset P^0$. On the other hand, by a theorem of continued fractions (cf. Theorem 7.9, [11]), there exists an infinite sequence $(p_m, q_m)\in\mathbf{Z}^2$ $(m=1, 2, \cdots)$ such that

(1) $0\le p_m$, $0<q_m<q_{m+1}$,

(2) $p_{m+1}q_m-q_{m+1}p_m=\pm 1$,

(3) $\lim_{m\to\infty} p_m/q_m=\delta$.

Put $\mathbf{v}_m=q_m\mathbf{b}_1+p_m\mathbf{b}_2$. By (3), the series of angles θ_m between the two vectors $\mathbf{v}_m$ and $\mathbf{b}_1+\delta\mathbf{b}_2$ converges to 0. Consequently, $\mathbf{v}_m, \mathbf{v}_{m+1}\in C(U)\subset P^0$ for a sufficiently large number m. Put $\mathbf{b}_1'=\mathbf{v}_m$, $\mathbf{b}_2'=\mathbf{v}_{m+1}$. Then $\mathbf{b}_j'\in P^0\cap\mathbf{Z}^{n_i}$ for $j=1, 2$. On the other hand, from (2) it follows that $\{\mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_3, \cdots, \mathbf{b}_{n_i}\}$ is a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$. By the same arguments as above, we may conclude that $P^0\cap\mathbf{Z}^{n_i}$ contains a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$. Hence we can choose a primitive element $\mathbf{a}$ in $\mathbf{Z}^{n_i}$ satisfying $\chi_\varepsilon(\alpha(\mathbf{a})e_i)\neq 1$.

LEMMA 6.4. *Let $A_i(t : \chi_\varepsilon)$ be the function given in Lemma 4.5. Suppose χ_ε is nontrivial on $\tilde{\mathfrak{b}}_i e_i$. Then there exists a positive constant K such that*

$$|A_i(t : \chi_\varepsilon)|\le Kt^{(n_i-1)/n_i} \qquad \text{for all } t>0 .$$

PROOF. Since $\chi_\varepsilon\neq 1$ on $\tilde{\mathfrak{b}}_i e_i$, Lemma 6.3 implies that there exists a primitive element $\mathbf{a}\in P(t)\cap\mathbf{Z}^{n_i}$ such that $\chi(\alpha(\mathbf{a})e_i)\neq 1$. Therefore $\chi_\varepsilon(\alpha(\mathbf{a}))=e^{2\pi\sqrt{-1}q/p}$ for two suitable positive integers p, q $(q<p)$. Let $\{\mathbf{a}=\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_{n_i}\}$ be a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$ and W be the n_i-1 dimensional subspace of $\mathbf{R}^{n_i}$ generated by $\{\mathbf{b}_2, \mathbf{b}_3, \cdots, \mathbf{b}_{n_i}\}$. Define a projection map $\varpi : \mathbf{R}^{n_i}\to W$:

$$\varpi(x_1\mathbf{b}_1+x_2\mathbf{b}_2+\cdots+x_{n_i}\mathbf{b}_{n_i})=x_2\mathbf{b}_2+\cdots+x_{n_i}\mathbf{b}_{n_i} .$$

Put $S(t)=\varpi(P(t))$. $S(t)$ is bounded. Let $\mathbf{b}$ be an element in $S(t)\cap\mathbf{Z}^{n_i}$. Then $\varpi^{-1}(\{\mathbf{b}\})$ is a line which is parallel to the vector $\mathbf{a}$. Let $n(\mathbf{b})$ be the number of the connected components of $\varpi^{-1}(\{\mathbf{b}\})\cap P(t)$. By Lemma 5.5 we have

$$n(\mathbf{b})<2n_i^2(n_i+1) . \tag{6.2}$$

Hence there exist a finite number of the intervals $(a_j, b_j]$ $(1\le j\le n(\mathbf{b}))$ such that

$$P(t)\cap\mathbf{Z}^{n_i}\cap\varpi^{-1}(\{\mathbf{b}\})=\bigcup_{j=1}^{n(\mathbf{b})}\{u\mathbf{a}+\mathbf{b} : u\in(a_j, b_j]\cap\mathbf{Z}\} .$$

Since

$$P(t)\cap\mathbf{Z}^{n_i}=\bigcup_{\mathbf{b}\in\varpi(P(t)\cap\mathbf{Z}^{n_i})} P(t)\cap\varpi^{-1}(\{\mathbf{b}\})\cap\mathbf{Z}^{n_i} ,$$

we have

$$A_i(t:\chi_\varepsilon)=\sum_{\mathbf{b}\in\varpi(P(t)\cap\mathbf{Z}^{n_i})}\sum_{j=1}^{n(\mathbf{b})}\sum_{\substack{a_j<m\leq b_j\\ m\in\mathbf{Z}}}\chi_\varepsilon(\alpha(m\mathbf{a})e_i)\chi_\varepsilon(\alpha(\mathbf{b})e_i)\,.$$

Hence, by Lemma 6.2 and (6.2), there exists a positive constant K such that

$$|A_i(t:\chi_\varepsilon)|\leq K|\varpi(P(t)\cap\mathbf{Z}^{n_i})|\qquad\text{for all }\ t>0\,.$$

Since

$$\lim_{t\to\infty}\frac{|\varpi(P(t)\cap\mathbf{Z}^{n_i})|}{t^{(n_i-1)/(n_i)}}=vol(\varpi(P(1)))\,,$$

we have the assertion of this lemma.

LEMMA 6.5. *The zeta function $\zeta_i(s:\chi_\varepsilon)$ is holomorphic in the half plane where $\Re(s)>1$. Furthermore, we have*

$$\lim_{\sigma\to1+0}(\sigma-1)\zeta_i(\sigma:\chi_\varepsilon)=\begin{cases}\kappa_i(C) & \chi_\varepsilon\equiv1\ \textit{on}\ \tilde{\mathfrak{b}}_ie_i\\ 0 & \textit{otherwise}\end{cases}$$

where

$$\kappa_i(C)=\frac{2^{r_i+c_i}\pi^{c_i}|R(E_i)|}{N\tilde{\mathfrak{a}}_iN\tilde{\mathfrak{b}}_i\sqrt{|\mathbf{D}_i|}}\,.$$

PROOF. We first prove that the series $\zeta_i(s:\chi_\varepsilon)$ is holomorphic in the complex half plane $\Re(s)>1$. By Lemma 4.5 we have

$$\sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i\\ N(\alpha\tilde{\mathfrak{a}}_i)\leq t}}\frac{\chi_\varepsilon(\alpha e_i)}{(N(\alpha\tilde{\mathfrak{a}}_i))^s}=A_i(t:\chi_\varepsilon)t^{-s}+s\int_1^t A_i(x:\chi_\varepsilon)x^{-s-1}dx\,.$$

Since $|A_i(t:\chi_\varepsilon)|\leq|T_i(t)|$, Lemma 5.4 implies that $|A_i(t:\chi_\varepsilon)|/t$ is bounded on the half line: $t\geq1$. Therefore $\lim_{t\to\infty}A_i(t:\chi_\varepsilon)t^{-s}=0$. Hence we have

$$\zeta_i(s:\chi_\varepsilon)=s\int_1^\infty A_i(x:\chi_\varepsilon)x^{-s-1}dx\,. \tag{6.3}$$

Let b be the upper bound of $|T_i(t)|/t$ $(1\leq t)$. Then we have

$$\left|\int_1^\infty A_i(x:\chi_\varepsilon)x^{-s-1}dx\right|\leq b\int_1^\infty x^{-\sigma}dx<\infty$$

where $\sigma=\Re(s)$. Consequently by (6.3), $\zeta_i(s:\chi_\varepsilon)$ is a convergent series in the half plane $\Re(s)>1$.

Finally we prove the asymptotic formula of $\zeta_i(s:\chi_\varepsilon)$. Assume that χ_ε is nontrivial on $\tilde{\mathfrak{b}}_ie_i$. By Lemma 6.5 and (6.3) we have

$$|\zeta_i(s:\chi_\varepsilon)|\leq K|s|\int_1^\infty x^{-\sigma-1/n_i}dx\,.$$

Therefore $\lim_{s\to 1}(s-1)\zeta_i(s:\chi_\varepsilon)=0$.

Let us consider the case: χ_ε is trivial on $\tilde{\mathfrak{b}}_i e_i$. From Lemma 5.4 it follows that

$$\lim_{t\to\infty}\frac{|T_i(t)|}{t}=\kappa_i(C)\,.$$

Put

$$\frac{A_i(t:1)}{t}\equiv\frac{|T_i(t)|}{t}=\kappa_i(C)+R(t)\,. \tag{6.4}$$

Then for each positive real number δ, there exists a number N_0 such that

$$|R(t)|\leq\delta \qquad \text{for all } t\geq N_0\,.$$

On the other hand

$$\begin{aligned}&(\sigma-1)\zeta_i(\sigma:1)-\kappa_i(C)\\&=(\sigma-1)\left\{\sigma\int_1^\infty A_i(x:1)x^{-\sigma-1}dx-\int_1^\infty\kappa_i(C)x^{-\sigma}dx\right\}\\&=(\sigma-1)\left\{\kappa_i(C)+\sigma\int_1^\infty R(x)x^{-\sigma}dx\right\}.\end{aligned}$$

Let R_0 be the upper bound of the set: $\{|R(x)|: x\in[1, N_0]\}$. Then we have

$$\begin{aligned}&|(\sigma-1)\zeta_i(\sigma:1)-\kappa_i(C)|\\&\leq(\sigma-1)\kappa_i(C)+\sigma(\sigma-1)\left\{R_0\int_1^{N_0}\frac{1}{x}dx+\delta\int_{N_0}^\infty x^{-\sigma}dx\right\}\\&=(\sigma-1)\{\kappa_i(C)+\sigma R_0\log N_0\}+\sigma\delta(N_0)^{-\sigma+1}\,.\end{aligned}$$

Therefore

$$\overline{\lim_{\sigma\to 1+0}}\,|(\sigma-1)\zeta_i(\sigma:1)-\kappa_i(C)|\leq\delta$$

for all positive real numbers δ. This implies that

$$\lim_{\sigma\to 1+0}(\sigma-1)\zeta_i(\sigma:1)=\kappa_i(C)\,.$$

This completes our proof of the lemma.

7. Main theorems.

Let ζ be a regular element of $GL(n, \mathbf{Z})$ with the characteristic polynomial $f(X)=f_1(X)f_2(X)\cdots f_g(X)$ and $R=\mathbf{Z}[\zeta]$ the ring generated by ζ over $\mathbf{Z}$. Our main results are formulated in Theorem 7.2 and Theorem 7.3. Let $C=C(\mathfrak{a})$ be a class of the ideal class semigroup $\mathbf{G}$ of R. We can assume that the representative $\mathfrak{a}$ is integral. Put $\tilde{\mathfrak{a}}=O\mathfrak{a}$. Then $\check{\mathfrak{a}}\subset\tilde{\mathfrak{a}}^{-1}$ where $\check{\mathfrak{a}}$ is the pseudo-inverse ideal of $\mathfrak{a}$. Define the i-th component $\tilde{\mathfrak{a}}_i$ (resp. $\tilde{\mathfrak{b}}_i$) of $\tilde{\mathfrak{a}}$ (resp. $\tilde{\mathfrak{a}}^{-1}$) by $\tilde{\mathfrak{a}}_i e_i=\tilde{\mathfrak{a}}e_i$ (resp. $\tilde{\mathfrak{b}}_i e_i=\tilde{\mathfrak{a}}^{-1}e_i$). Let ζ_i be a root of $f_i(X)$. Then $k_i=\mathbf{Q}[\zeta_i]$ is an algebraic number field over $\mathbf{Q}$. Denote by $\mathbf{D}_i$ the discriminant of the field k_i. Let $\mathbf{E}_O$ be the unit group of the ring of algebraic integers O of $\mathbf{Q}[\zeta]$. By Lemma 3.6, the index $(\mathbf{E}_O:\mathbf{E}_\mathfrak{a})$ is finite. (Here $\mathbf{E}_\mathfrak{a}$ is the group of stabilizers of $\mathfrak{a}$.) Let $\mathbf{E}_i$ be the unit group of the number field k_i. $\mathbf{E}_i$ is a product of a finite group H_i and a free group E_i. Let B^* be the character group of the finite additive group $B=\tilde{\mathfrak{a}}^{-1}/\check{\mathfrak{a}}$. We consider the L-series $L(s:\chi)$ $(\chi\in B^*)$ given in (6.1). The following lemma is a direct consequence of Lemma 6.5.

LEMMA 7.1. *The function $L(s:\chi)$ is holomorphic in the complex half plane where $\Re(s)>1$. Furthermore, we have*

$$\lim_{\sigma\to 1+0}(\sigma-1)^g L(\sigma:\chi)=\begin{cases}\dfrac{2^{r+c}\pi^c(E_O:E_\mathfrak{a})|R(\mathbf{E}_O)|}{N\tilde{\mathfrak{a}}^{-1}N\tilde{\mathfrak{a}}\prod_{i=1}^g\sqrt{|\mathbf{D}_i|}} & \chi=1\\ 0 & \chi\neq 1\,.\end{cases}$$

Let $f_i(X)$ be an irreducible factor of $f(X)$. Let r_i (resp. $2c_i$) be the number of real (resp. complex) roots of $f_i(X)$. Put $r=\sum_{i=1}^g r_i$ and $c=\sum_{i=1}^g c_i$.

THEOREM 7.2. *Let $\zeta_C(s)$ be the zeta function of an ideal class $C=C(\mathfrak{a})$, which is defined by*

$$\zeta_C(s)=\sum_{\substack{\mathfrak{b}\in C(\mathfrak{a})\\ \mathfrak{b}\subset R}}\frac{1}{(N\mathfrak{b})^s}\,.$$

Then we have

(1) *$\zeta_C(s)$ is holomorphic in the complex half plane $\Re(s)>1$, and*

(2) $\lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_C(\sigma)=2^{r+c}(\pi)^c\dfrac{(\mathbf{E}_O:\mathbf{E}_\mathfrak{a})|R(\mathbf{E}_O)|}{N(\check{\mathfrak{a}})N\mathfrak{a}|H_O|\sqrt{|Nf'(\zeta)|}}$.

Proof. By Theorem 4.2 and Lemma 7.1 the assertion of (1) is obvious. Furthermore,

$$\lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_C(\sigma)=\frac{(\mathbf{E}_O:\mathbf{E}_\mathfrak{a})|R(\mathbf{E}_O)|}{|H_\mathfrak{a}|(\tilde{\mathfrak{a}}^{-1}:\check{\mathfrak{a}})N\mathfrak{a}N\tilde{\mathfrak{a}}^{-1}\prod_{i=1}^g\sqrt{|\mathbf{D}_i|}}.$$

It is easy to see that $Nf'(\zeta)=(O:R)^2\prod_{i=1}^g\mathbf{D}_i$ and $(\tilde{\mathfrak{a}}^{-1}:\check{\mathfrak{a}})N\tilde{\mathfrak{a}}^{-1}=(O:R)N\check{\mathfrak{a}}$.

Hence the theorem follows.

THEOREM 7.3. *We define a Dirichlet series* $\zeta_R(s)$ *by*

$$\zeta_R(s)=\sum_{\mathfrak{b}}\frac{a(\mathfrak{b})}{(N\mathfrak{b})^s}$$

where $\sum_{\mathfrak{b}}$ *runs over all nonsingular ideals of* R *and* $a(\mathfrak{b})$ *the constant defined by* $a(\mathfrak{b})=N\check{\mathfrak{b}}N\mathfrak{b}/(\mathbf{E}_O : \mathbf{E}_{\mathfrak{b}})$. *Then we have*

$$\lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_R(\sigma)=|\mathbf{G}|2^{r+c}(\pi)^c\frac{|R(\mathbf{E}_O)|}{|H_O|\sqrt{|Nf'(\zeta)|}}$$

where $|\mathbf{G}|$ *is the order of the ideal class semigroup* $\mathbf{G}$ *of* R.

PROOF. Let $\mathfrak{b}$ be an integral ideal in $C(\mathfrak{a})$. Then there exists an invertible element λ in $\mathbf{Q}[\zeta]$ such that $\lambda\mathfrak{a}=\mathfrak{b}$. Since $(\lambda\mathfrak{a})^{\vee}=\lambda^{-1}\check{\mathfrak{a}}$, we have $N\check{\mathfrak{b}}N\mathfrak{b}=N\check{\mathfrak{a}}N\mathfrak{a}$. Also since λ is invertible, we have $\mathbf{E}_{\mathfrak{b}}=\mathbf{E}_{\mathfrak{a}}$. Therefore $a(\mathfrak{b})=a(\mathfrak{a})$. From this it follows that $\zeta_R(s)=\sum_{C(\mathfrak{a})\in\mathbf{G}}a(\mathfrak{a})\zeta_C(s)$. Hence by Theorem 7.2 we have our assertion.

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