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Tangle Decompositions of Doubled Knots

Hiroshi MATSUDA

University of Tokyo
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Abstract. We show that a decomposing 2-sphere which gives an essential tangle decomposition of a doubled knot gives also an essential tangle decomposition of its companion knot.

1. Introduction.

The notion of doubled knots was introduced by J. H. C. Whitehead in [W]. Recall the definition of the doubled knot K of a knot K_1. Let $V_0 \subset S^3$ be an unknotted solid torus and J a simple closed curve in V_0, as shown in Figure 1. Let V_1 be a tubular neighborhood of K_1 and $h: V_0 \to V_1$ a homeomorphism with $h(m_0) = m_1$ and $h(l_0) = l_1 + q \cdot m_1$, where (m_i, l_i) $(\subset \partial V_i)$ is a meridian-longitude pair for V_i $(i = 0, 1)$ and q is an integer. Then we say $K = h(J)$ the *doubled knot* of K_1. The knot K_1 is said to be the *companion knot* of K constructed in this manner up to knot type.

Let B be a 3-ball and $t = t_1 \cup \cdots \cup t_n$ a union of mutually disjoint n arcs properly embedded in B. Then the pair (B, t) is said to be an *n-string tangle*. We say that an n-string tangle (B, t) is *trivial* if (B, t) is homeomorphic to $(D \times I, \{x_1, \cdots, x_n\} \times I)$ as pairs, where D is a 2-disc and x_i is a point in $\mathrm{Int} D$ $(i = 1, \cdots, n)$. Let $N(t)$ denote a regular neighborhood of t in B. The tangle (B, t) is said to be *essential* if $cl(\partial B - N(t))$ is incompressible in $cl(B - N(t))$ and (B, t) is not a trivial 1-string tangle. If $(S^3, K) = (B_1, t_1) \cup (B_2, t_2)$ and (B_i, t_i) is an essential n-string tangle for $i = 1$ and 2, then it is said that a knot K in S^3 admits an essential n-string tangle decomposition. If a knot K admits no essential k-string tangle decomposition, then K is said to be *k-string prime*.

Then our results are;

THEOREM. *If a doubled knot K admits an essential m-string tangle decomposition, then m is even, and this decomposing 2-sphere gives an essential $m/2$-string tangle decomposition of the companion knot.*

COROLLARY. *A doubled knot whose companion is n-string prime for every $n \geq 1$ is also m-string prime for every $m \geq 1$.*

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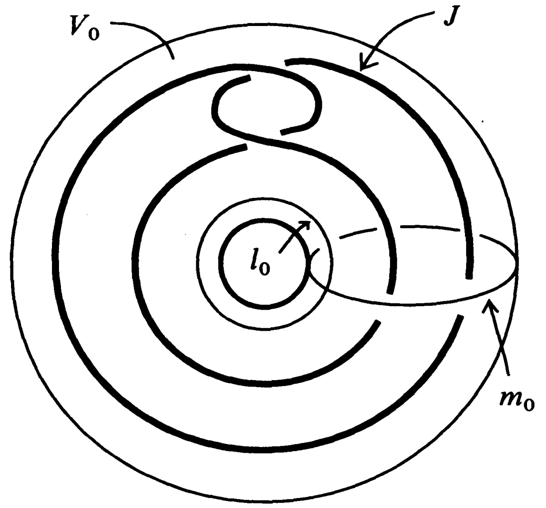


FIGURE 1

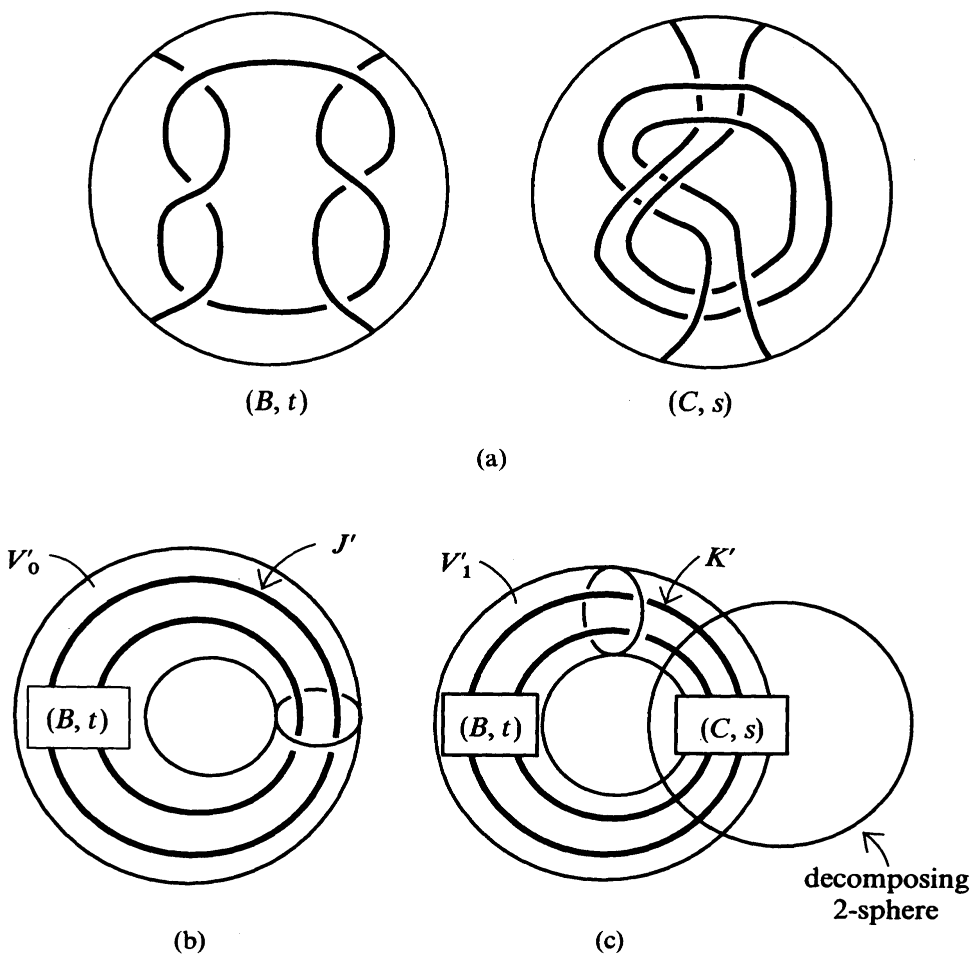


FIGURE 2

PROOF OF COROLLARY. By the above Theorem, a doubled knot is m-string prime for every odd m. The companion knot is n-string prime for every $n \geq 1$, so its doubled knots are m-string prime for every even m. □

We note here that the theorem is not valid for arbitrary satellite knots and links which are 1-string prime.

Let (B, t), (C, s) be essential 2-string tangles, for example, see Figure 2 (a). The diagram (C, s) represents a knotted spanning arc of the ball and another arc running parallel to it. The figure shows the arcs knotted in trefoil knot, but any knot will suffice. For the essentiality of these tangles, see [B] or [L].

Let J' be a knot or link in an unknotted solid torus V'_0 obtained from a tangle (B, t), as shown in Figure 2 (b). Let V'_1 be a tubular neighborhood of a knot K'_1. In the same way as we construct a doubled knot, we construct a *satellite link* $K' = h(J')$. The companion of K' is K'_1. From the way of construction, we assume that $(S^3, K') = (B, t) \cup (C, s)$, as shown in Figure 2 (c).

Then the decomposing 2-sphere which realizes an essential 2-string tangle decomposition of K' gives an inessential 1-string tangle decomposition of the companion. See Figure 2 (c). For the 1-string primeness of the resulting knot and link, see [B] or [L].

2. Reducing the number of intersections.

Let K be a doubled knot in S^3 and D a 2-disc. Then there exists an immersion $f: D \to S^3$ such that $f|_{\partial D}: \partial D \to K$ is a homeomorphism and $f^{-1}(\tilde{\Sigma})$ consists of two arcs σ_1, σ_2 joining a point in ∂D and a point in $\operatorname{Int} D \cap f^{-1}(K)$, where $\tilde{\Sigma} = \{x \in f(D) \mid |f^{-1}(x)| \geq 2\}$.

Put $\tilde{D} = f(D)$, $\Sigma = f^{-1}(\tilde{\Sigma})$ and then $\Sigma = \sigma_1 \cup \sigma_2$. Put $\partial\sigma_i \cap \operatorname{Int} D = \{x_i\}$, $\partial\sigma_i \cap \partial D = \{y_i\}$.

Suppose that K admits an essential m-string tangle decomposition, then there exists a 2-sphere S in S^3 which realizes this decomposition of (S^3, K). We may assume that S intersects $\tilde{D}$ transversely. Put $\tilde{C} = S \cap \tilde{D}$ and $C = f^{-1}(\tilde{C})$, then C consists of m arcs properly embedded in D and simple loops in $\operatorname{Int} D$.

We now introduce a measure of the complexity of the pair $(\tilde{D}, S)$. Let $|f(\sigma_1) \cap S|$ be the number of points in $f(\sigma_1)(= f(\sigma_2)) \cap S$, $|C(l)|$ the number of loops in C. The *complexity* $c(\tilde{D}, S)$ is the pair $(|f(\sigma_1) \cap S|, |C(l)|)$. We assign the standard lexicographic ordering to this complexity function.

We may assume that $c(\tilde{D}, S)$ is minimum among all 2-spheres ambient isotopic in the pair (S^3, K) to S.

We will prove the theorem by showing that C consists of parallel arcs on D each of which separates x_1 from x_2 and is disjoint from σ_1 and σ_2.

LEMMA 1. *A loop component α of C bounds a disc D' in D with $\partial D' = \alpha$, $D' \cap \{x_1, x_2\} \neq \emptyset$.*

Proof. Without loss of generality we assume that α is an innermost loop component of C in D. We assume for a contradiction that $D' \cap x_1 = D' \cap x_2 = \emptyset$. If $\alpha \cap \mathrm{Int}\sigma_i \neq \emptyset$, then there are subarcs γ of σ_i and γ' of $\partial D'$ such that $\partial\gamma' = \partial\gamma$ and γ is outermost in D', that is, the arcs γ and γ' cobound a disc δ in D' such that $(\mathrm{Int}\delta) \cap \sigma_i = \emptyset$. We may then isotope S along δ to reduce the number of intersections of σ_i with S.

Hence we may assume $\alpha \cap \sigma_i = \emptyset$. Let E_1 and E_2 be two discs in S separated by α, and put $S_i = E_i \cup D'$. Let B_i be the 3-ball in S^3 bounded by S_i with $B_1 \cap B_2 = D'$, and $t = (B_1 \cup B_2) \cap K$. If $E_i \cap K \neq \emptyset$ for $i = 1$ and 2, then the disc D' implies that $cl(S - N(t))$ is compressible in $cl(B_1 \cup B_2 - N(t))$, and violates the essentiality of the tangle. If either $E_1 \cap K = \emptyset$ or $E_2 \cap K = \emptyset$, then either $B_1 \cap K$ or $B_2 \cap K$, say $B_2 \cap K$, is an empty set. Then by isotoping S along the ball B_2 we obtain the 2-sphere S_1, and we further isotope S_1 slightly off from D' to reduce the intersection loop α. This is a contradiction. □

Let γ be an outermost arc component of C in D and γ' a subarc in ∂D with $\partial\gamma' = \partial\gamma$ such that the loop $\gamma \cup \gamma'$ bounds a disc δ in D and $(\mathrm{Int}\delta) \cap S$ contains no arc component of C.

Lemma 2. *The disc δ contains x_1 or x_2.*

Proof. Suppose δ contains neither x_1 nor x_2.

We may assume that B_k ($k = 1$ or 2) contains δ. Let $N(\delta; B_k)$ denote a regular neighborhood of δ in B_k, and $t = K \cap B_k$. If δ does not contain a subarc of σ_1, σ_2, then the disc $cl(\partial N(\delta; B_k) - S)$ implies that $cl(\partial B_k - N(t))$ is compressible in $cl(B_k - N(t))$, and violates the essentiality of the tangle.

There exist two types of subarcs of σ_1, σ_2 in δ. One type is a subarc σ^1 whose endpoints are in γ. Without loss of generality we may suppose that σ^1 is outermost in δ, that is, there is a subarc γ^1 of γ with $\partial\sigma^1 = \partial\gamma^1$, and the arcs σ^1 and γ^1 cobound a disc δ^1 in δ such that δ^1 contains no arc component. We may then isotope S along δ^1 to reduce the number of intersections of σ_i with S. This contradicts the minimality of $c(\tilde{D}, S)$. Hence, in fact, there are no arc of this type.

The other type is a subarc σ^2, one of whose endpoints is in γ, the other in γ'. Without loss of generality we may assume that σ^2 is outermost in δ, because there are no subarcs of σ_i whose endopoints are in γ. There is a subarc γ^2 of $\partial\delta$ with $\partial\sigma^2 = \partial\gamma^2$, and the arcs σ^2 and γ^2 cobound a disc δ^2 in D_i such that δ^2 contains no arc component. We may then isotope S along δ^2 to reduce the number of intersections of σ_i with S. This contradicts the minimality of $c(\tilde{D}, S)$. □

By Lemma 2, the arc component of C are mutually parallel. That is, each γ separates x_1 and x_2. Call these arcs $\gamma_1, \gamma_2, \cdots, \gamma_m$. These arcs separate D into $m+1$ discs, say $D_0, D_1, \cdots, D_m$ as shown in Figure 3. The discs D_0, D_m are the outermost discs.

Lemma 3. *$\sigma_i \subset D_0 \cup D_m$ ($i = 1$ and 2).*

Proof. Suppose σ_i meets $\gamma_1 \cup \cdots \cup \gamma_m$. We are in the situation in Figure 3.

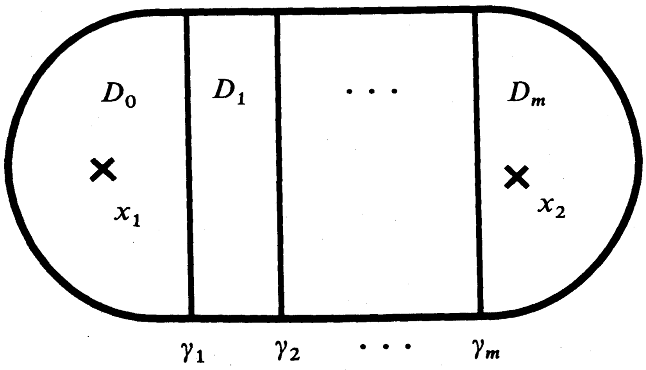


FIGURE 3

If the endpoint y_i of σ_i is in ∂D_0, then a subarc σ^1 of σ_i connects y_i and a point in γ_1 in D_0 because σ_i is not contained in D_0. The arc σ^1 and a subarc of ∂D_0 cobound a disc δ^1 in D_0, which contain neither x_1 nor x_2. Then we can show that we will decrease the complexity $c(\tilde{D}, S)$ in exactly the same way as we did in the proof of Lemma 2.

If y_i is in ∂D_m, then we can show that we will decrease the complexity $c(\tilde{D}, S)$ in exactly the same way as we did in the proof of Lemma 2.

If y_i is in ∂D_j ($j \neq 0, m$), then a subarc σ^2 of σ_i connects y_i and a point in γ_j or γ_{j+1} in D_j. The arc σ^2 and a subarc of ∂D_j which contains one subarc of K cobound a disc δ^2 in D_j. Then we can show that we can again decrease the complexity $c(\tilde{D}, S)$ in exactly the same way as we did in the proof of Lemma 2.

Thus σ_i does not meet $\gamma_1 \cup \cdots \cup \gamma_m$, so we have $\sigma_1 \subset D_0$ and $\sigma_2 \subset D_m$. □

Note that $\gamma_1 \cup \cdots \cup \gamma_m \neq \emptyset$ because $S \cap K \neq \emptyset$. The remaining loops of C are contained in $D_0 \cup D_m$, and each loop meets σ_1 or σ_2. Put $p = |f(\sigma_1) \cap S|$ and $q = |f(\sigma_2) \cap S|$.

LEMMA 4. *Each loop of C meets σ_1 or σ_2 exactly once.*

PROOF. Suppose a loop α_i of C which is contained in D_0 meets σ_1 more than once. Let D_i' be a disc in D with $\partial D_i' = \alpha_i$. The disc D_i' contains a subarc σ^i of σ_1. The arc σ^i and a subarc of α_i cobound a disc δ^i in D_i' which does not contain x_1. Without loss of generality we may assume σ^i is outermost in D_i', that is, $\mathrm{Int}\,\delta^i$ does not contain a subarc of σ_1. If the disc δ^i contains a subarc c_j of α_j, then both endpoints of c_j are in σ^i. Note that α_j meets σ_1 more than once. Hence without loss of generality we may assume $\delta^i \cap \alpha_k = \emptyset$ for $k \neq i$, taking α_i to be an innermost one meeting σ_1 more than once.

Then we isotope S along δ^i to reduce the number of intersections of σ_1 with S. This contradicts the minimality of $c(\tilde{D}, S)$. So we assume that each loop of C in D_0 meets σ_1 exactly once.

In the same way as above, we may assume that each loop of C contained in D_m meets σ_2 exactly once. □

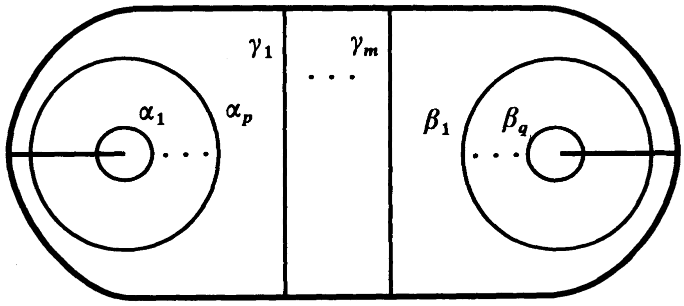
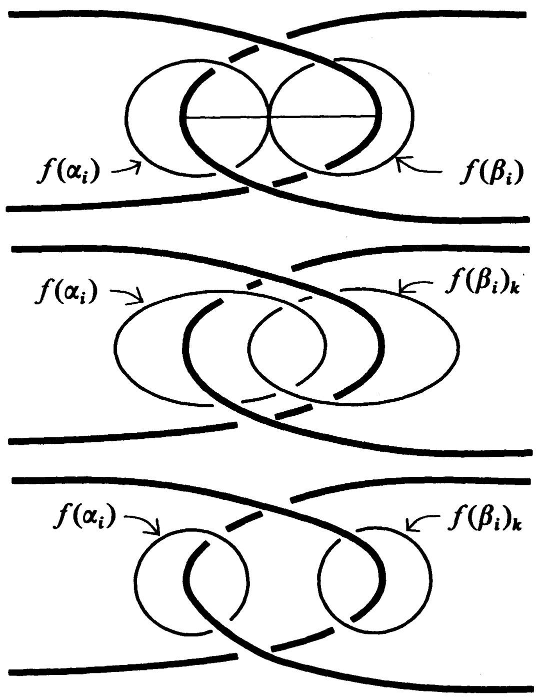


FIGURE 4



($k=a$ or b)

FIGURE 5

We are now in the situation in Figure 4.

We call the loops in D_0 $\alpha_1, \cdots, \alpha_p$ and those in D_m $\beta_1, \cdots, \beta_q$. Note that $p=q$ because $f(\sigma_1)=f(\sigma_2)$.

LEMMA 5. $p=q=0$.

PROOF. Suppose $p\neq 0$. More $f(\beta_i)$ slightly off above (respectively, off below) the 2-sphere S, and call this $f(\beta_i)_a$ (resp. $f(\beta_i)_b$). Then one of $f(\alpha_i)\cup f(\beta_i)_a$ and $f(\alpha_i)\cup f(\beta_i)_b$ forms a Hopf link. See Figure 5.

On the other hand, since both $f(\alpha_i)$ and $f(\beta_i)$ are embedded in a common 2-sphere, both $f(\alpha_i)\cup f(\beta_i)_a$ and $f(\alpha_i)\cup f(\beta_i)_b$ are split links.

Thus we have a contradiction. □

By Lemma 5, we can see that if a doubled knot has an essential tangle decomposition, then this 2-sphere gives a (maybe inessential) tangle decomposition of the companion. In the following, we have to show that m is even, and that this decomposition of the companion is essential.

PROOF OF THEOREM. Suppose m is odd.

Let κ be a subarc of ∂D with $\partial\kappa=\{y_1, y_2\}$. Then we can regard $f(\sigma_1 \cup \kappa)$ as the companion of K, and it is a simple loop in S^3 which intersects S transversely in odd points. This is a contradiction, so m is even.

Suppose that the punctured sphere $cl(S-N(f(D)))$ has a compressing disc Q. Then this disc Q will be a compressing disc for $cl(S-N(K))$, and violates the essentiality of the original tangle.

Suppose that one of the tangles of the decomposition of the companion is a trivial 1-string tangle. Then its original tangle is a trivial 2-string tangle, which contradicts the essentiality of the original tangle. This completes the proof. □

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Present Address:
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO,
KOMABA, MEGURO-KU, TOKYO, 153–0041.